Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Sub-Optimal Control in Presence of Unobservable Disturbances

Marc Quincampoix and Vladimir M. Veliov

Abstract— **We investigate the problem of optimization of a terminal cost function for a system depending on a control, and on two disturbances for which a priori set membership is known. The disturbances are of different natures: One becomes known to the controller at the current time (we called it observable) while the other remains unknown. The state of the system is not measured, therefore not exactly known due to the presence of an unobservable disturbance. We reformulate the problem through a set-valued dynamics describing the evolution of the current set estimation of the state. To reduce the complexity of the problem, we pass to a suboptimal problem where the evolution of the state estimation is restricted to a prescribed collection of sets. The main result of the paper is a characterization of the value function of this problem through a Hamilton–Jacobi inequality in terms of Dini derivatives, which implies a convergent scheme for numerical computations. As necessary auxiliary tools, we provide new results on evolution and viability of tubes in a given collection of sets.**

I. Introduction

We consider the system

$$\dot{x} = f(x, u, y, v), \qquad x(0) = e \in E_0, \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in U$ is the control, and $y \in Y$ and $v \in V(y)$ are disturbances (U, Y, and $V(y)$ are given subsets of finite dimensional spaces, $E_0 \subset \mathbf{R}^n$). We consider the problem

$$\min_u g(T, x(T)) \tag{2}$$

against the worst case of disturbances y and v and initial state $e \in E_0$. We distinguish two types of disturbance:

- *observable uncertainty* y, for which the current realization $y(t) \in Y$ becomes known to the controller;
- *unobservable uncertainty* $v \in V(y)$, for which the realization of $v(t) \in V(y(t))$ remains unknown.

Thus we consider a min-max problem or a differential game where the second player wants to maximize (by choosing e, y, and v) the cost (2) while the first player—the controller—wants to minimize it. The information available to the controller implies that the control u should be considered in a feedback form which may depend on the current and the past values of y, but not on v.

Marc Quincampoix is with Laboratoire de Mathématiques, Unité CNRS UMR 6205, Université de Bretagne Occidentale, 6 Avenue Victor Le Gorgeu, 29200 Brest, France, Marc.Quincampoix@univ-brest.fr

Vladimir M. Veliov is with Institute Mathematical Methods in Economics, Vienna University of Technology, Argentinierstrasse 8, A-1040 Vienna, Austria (vveliov@eos.tuwien.ac.at), and with the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria, vveliov@server.eos.tuwien.ac.at

For every given open-loop control $u(\cdot)$ and observable uncertainty $y(\cdot)$, the unobservable uncertainty $v(\cdot)$ gives rise to a differential inclusion

$$\dot{x} \in f(x, u(t), y(t), V(y(t))), \qquad x(0) \in E_0, \tag{3}$$

whose solution is a time-dependent tube providing the deterministic estimation of the trajectory.

Let us formulate the problem in a more precise way. Let $\mathcal{U}_{[t,\theta]}$ be the set of all open-loop admissible controls on the interval $[t, \theta]$, that is, the measurable functions with values in U. Similarly, $\mathcal{Y}_{[t,\theta]}$ denotes the set of all measurable selections of Y on $[t, \theta]$, and $\mathcal{V}_{[t,\theta]}(y(\cdot))$ denotes the set of all measurable selections of the mapping $V(y(\cdot))$ (for a given $y(\cdot)$) on the same interval. The suppositions formulated in section IV will imply that for any $t \in [0, T)$, $e \in \mathbf{R}^n$, $u \in \mathcal{U}_{[t,T]}$, $y \in \mathcal{Y}_{[t,T]}$, and $v \in \mathcal{V}_{[t,T]}(y)$ system (1) has a unique solution on $[t, T]$ starting from e, denoted by $x[t, e; u, y, v](\cdot)$.

To make use of the dynamic programming principle, we consider problem (1), (2) also for an arbitrary initial time t and initial set E, instead of the fixed $t = 0$ and $E = E_0$. The optimal control for initial time t and initial compact set E is sought as a *nonanticipative strategy* ([12]; cf. the definition in section IV), $\alpha : \mathcal{Y}_{[t,T]} \mapsto \mathcal{U}_{[t,T]}$. The guaranteed result obtained by using the strategy α for initial data (t, E) is

$$\begin{aligned} I(t, E; \alpha) \quad &\overset{def}{=} \quad \sup\{g(T, x[t, e; \alpha(y), y, v](T)); \\ &\qquad e \in E,\ y \in \mathcal{Y}_{[t,T]},\ v \in \mathcal{V}_{[t,T]}(y)\}. \end{aligned}$$

Then

$$I(t, E) \overset{def}{=} \inf_{\alpha} I(t, E; \alpha) \tag{4}$$

is the minimal guaranteed (lower) value that can be achieved starting from the set E at time t.

As previous works indicate (see [5], [6], [4]), the optimal control problem in the case of incomplete/inexact information is qualitatively different compared with the perfect information case, since the corresponding Hamilton–Jacobi–Isaacs (HJI) equation for the value function becomes, essentially, infinite dimensional. In the present paper we reformulate the problem as such with complete information, but for a dynamic system in the *estimation space*, which, in principle, is also infinite dimensional, but in some cases can be equivalently replaced by a finite dimensional one. In this case the corresponding HJI equation is finite dimensional. (A finite dimensional HJI equation was derived in [25] for a specific game on the plane where the state information is incomplete by employing the certainty equivalence principle [5].) If this is not the case, one can still formulate a "suboptimal" version of the problem by restricting the consideration to an estimation space, which is a finitely parametrized collection of sets. To develop the technique for passing to such a "suboptimal" problem and its analysis is one of our main goals.

To pass to a finitely parametrized estimation we extend in section II the concept of solution tubes in a given collection of compact sets developed in [23]. For $y(\cdot)$ fixed, (3) can be considered as a controlled differential inclusion whose solution tube takes values in the given collection. Therefore the next step is to develop the viability theory for collections of sets and controlled differential inclusions, which is done in section III.

Our main result extends that of [7] to the incomplete information case. We prove that the value function of the problem is the unique minimal Dini supersolution of the respective HJI equation. The epigraph of the value function is characterized also as the maximal set that for every $y \in Y$ is a viability domain (in a specified collection of sets) for an auxiliary controlled inclusion depending on y. This makes it possible to apply an appropriate modification of the viability kernel algorithm (see [26], [8]) for numerical calculation. All proofs are presented in [24].

II. Solution tubes in a collection of sets

The reachable set of a differential inclusion (the latter interpreted as an uncertain system, as in (3)) is the minimal guaranteed estimation of the current state. Therefore, to calculate reachable sets is a cornerstone of the deterministic estimation and control of uncertain systems and a lot of work has been done toward developing numerical approximation methods (see, e.g., the survey [18]). Since the geometry of the reachable sets could be rather complicated, specific subclasses of sets are usually used as approximation tools: boxes, polyhedral sets, ellipsoids (see [9], [16], [10], [17]), box or polyhedral complexes (see [26], [14], [8], [15]), etc. In problems of control of uncertain systems and differential games, where the state estimation is just an auxiliary tool, one has to employ only fairly simple sets. Thus the issue of *approximation* is not that relevant. A different problem arises: to obtain inclusions of the reachable set in sets from a prescribed collection $\mathcal{E}$, that is, to replace the solution tube $X(t)$ of the differential inclusion by a tube $E(t)$ with values in $\mathcal{E}$. In doing this, one has to ensure at least $X(t) \subset E(t)$, but two more properties are also desirable: (i) the Markov property of the evolution of $E(\cdot)$, which, together with $X(t) \subset E(t)$, requires invariance of the tube $E(t)$ with respect to the differential inclusion, and (ii) minimality.

In this section we modify some results from our paper [23] and establish some new ones needed in the subsequent sections.

A. Definitions and main suppositions

We shall use the following notation: $\mathcal{B}$ is the Euclidean unit ball in $\mathbf{R}^n$, $comp(\mathbf{R}^n)$ is the set of all nonempty compact subsets of $\mathbf{R}^n$, $dist(X,Y) \stackrel{def}{=} \sup_{x\in X} \inf_{y \in Y} |x-y|$ is the distance from $X \in comp(\mathbf{R}^n)$ to $Y \in comp(\mathbf{R}^n)$, $|X| = dist(X,\{0\})$, and $H(X,Y) = \max\{dist(X,Y), dist(Y,X)\}$ is the Hausdorff distance between X and Y. Multiplication of a set with a scalar and summation of sets are understood in the usual (Minkowski) sense. For a set X, $f(X)$ stays for $\{f(x);\ x \in X\}$. For a given closed $X \subset \mathbf{R}^n$, the set of all Hausdorff continuous mappings $X \mapsto comp(\mathbf{R}^n)$ is a complete metric cone (with respect to the Minkowski operations), which will be denoted by $C(X; comp(\mathbf{R}^n))$.

Definition. A set-valued map $E(\cdot) : [0,T] \Rightarrow \mathbf{R}^n$ with nonempty compact values and closed graph will be called a *tube*. *Lipschitz continuity* of a tube is understood with respect to the Hausdorff metric.

We consider a differential inclusion

$$\dot{x} \in F(x,t), \quad x \in \mathbf{R}^n,\ t \in [0,T], \tag{5}$$

supposing the following.

Condition A: $F : \mathbf{R}^n \times [0,T] \Rightarrow \mathbf{R}^n$ is a set-valued mapping with nonempty convex, compact values, measurable in t for every fixed x, and locally Lipschitz continuous in x uniformly with respect to t. Moreover, F satisfies the linear growth condition

$$|F(x,t)| \le a(1+|x|) \quad \forall x \in \mathbf{R}^n,\ t \in [0,T].$$

As usual, a solution to (5) is any absolutely continuous function that satisfies (5) for a.e. t. Given a set of initial states $E_0 \subset \mathbf{R}^n$, the solution tube of (5) on $[0,T]$ is defined as

$$X(s) \stackrel{def}{=} X[0,E_0](s) \stackrel{def}{=} \{x(s);\ x(\cdot) \text{ – solution of (5) on } [0,s] \text{ with } x(0) \in E_0\}.$$

This is the unique *minimal* tube that starts from E_0 at $t=0$ and is *invariant* with respect to (5).

Here and below, "minimal", when applied to sets, means "minimal with respect to inclusion".

Definition. Let $\mathcal{E}$ be a given collection of compact sets in $\mathbf{R}^n$. The tube $E(\cdot) : [0,T] \mapsto comp(\mathbf{R}^n)$ is called the *solution tube of* (5) *in the collection* $\mathcal{E}$ if and only if $E(\cdot)$ is a minimal invariant tube with values in $\mathcal{E}$.

The above definition meets the requirements (i) and (ii) formulated in the preamble of this section and extends the usual concept of a solution tube. Clearly, $X[0,E_0](\cdot)$ is the unique solution tube in the collection $\mathcal{E} = comp(\mathbf{R}^n)$, starting from E_0. To ensure the existence of a solution tube in a more general collection $\mathcal{E}$ we introduce the following conditions for $\mathcal{E}$.

Condition B.2: The collection $\mathcal{E}$ consists of nonempty compact sets and is closed in the Hausdorff metric. For every compact Z there is some $E \in \mathcal{E}$ containing Z.

Condition B.3: There exists a constant $L_{\mathcal{E}}$ such that for each $\varepsilon > 0$ and each $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}$ for which $E + \varepsilon\mathcal{B} \subset E' \subset E + \varepsilon L_{\mathcal{E}} \mathcal{B}$.

Obviously Conditions B.2 and B.3, together with the Zorn lemma, imply that for every $Z \in comp(\mathbf{R}^n)$ there exists a minimal element of $\mathcal{E}$ containing Z.

Condition B.4: For every $Z \in comp(\mathbf{R}^n)$ there is a unique minimal element of $\mathcal{E}$ containing Z.

B. Existence and continuity of the solution tubes

Theorem 2.1: Suppose that Conditions A and B.1–B.3 are fulfilled. Then for every $E_0 \in \mathcal{E}$ inclusion (5) has a unique solution tube on $[0, +\infty)$ in the collection $\mathcal{E}$ starting from E_0 and it is Lipschitz continuous.

In the next sections we shall need the following continuity property of the solution tubes. Let us consider a sequence of differential inclusions of the form

$$\dot{x} \in F_0^k(x, s) + B^k(x, s)u_k(s), \tag{6}$$

where $F_0^k : \mathbf{R}^n \times [t_k, T] \Rightarrow \mathbf{R}^n$, $u_k : [t_k, T] \mapsto U$, U is a subset of a finite dimensional space, and B^k is a matrix function with appropriate size.

Proposition 2.2: Let the following conditions be fulfilled:
(i) For every $k = 0, 1, \ldots$, the mapping $F^k(x, s) \stackrel{def}{=} F_0^k(x, s) + B^k(x, s)u_k(s)$ satisfies Condition A with the growth constant a independent of k; B^k are measurable in s, locally Lipschitz continuous in x, uniformly in s; U is compact; the functions $u_k(\cdot)$ are measurable; and the collection $\mathcal{E}$ satisfies Condition B.
(ii) $t_k \in [0, T]$ and $E_0^k \in \mathcal{E}$, $k = 0, 1, \ldots$, are such that $\lim_{k \to +\infty} t_k = t_0$, $\liminf_{k \to +\infty} dist(E_0^0, E_0^k) = 0$ and $u_k(\cdot)$ converges L_1-weakly to $u_0(\cdot)$; for every compact set Z

$$\liminf_{k \to +\infty} \int_0^T \sup_{x \in Z} [dist(F_0^0(x, s), F_0^k(x, s)) + |B^k(x, s) - B^0(x, s)|] \, \mathrm{d}s = 0.$$

Let $E_k(\cdot) : [t_k, T] \mapsto \mathcal{E}$ be the solution tube in $\mathcal{E}$ of (6), $k = 0, 1, \ldots$, starting from E_0^k at time t_k. Then there is a subsequence of $\{E_k(\cdot)\}$ that converges uniformly to some tube $E(\cdot) : [t_0, T] \mapsto \mathcal{E}$. Every such limit tube $E(\cdot)$ is Lipschitz continuous, and $E_0(t) \subset E(t)$ for all $t \in [t_0, T]$.

Remark 2.3: In the case of single-valued F_0^k and E_0, the above theorem holds without the supposition that the second summand in (6) is linear in u. It could be shown by examples that Proposition 2.2 is false for nonaffine inclusions, even with single-valued F_0^k and convex compact-valued $f_1^k(x, s, U)$.

Below we give two examples of easily implementable collections, which satisfy Conditions B.1–B.3, referring to [23], [24] for more examples.

(a) Let us fix a finite or countable subset $L = \{l_1, l_2, \ldots\}$ of the unit sphere $\partial\mathcal{B} \subset \mathbf{R}^n$ such that the convex cone spanned by L coincides with $\mathbf{R}^n$. With every $Z \in comp(\mathbf{R}^n)$ we associate the sequence of numbers $p(Z) = (p_1, p_2, \ldots)$, where $p_i = \max_{z \in Z} \langle l_i, z \rangle$. Denote $P \stackrel{def}{=} \{p(Z);\ Z \in comp(\mathbf{R}^n)\}$ and consider the collection $\mathcal{E}_L \stackrel{def}{=} \{E(p);\ p \in P\}$. It may consist of all convex compact subsets of $\mathbf{R}^n$ (if L is dense in $\partial\mathcal{B}$), of all "boxes" (if $L = \{\pm e_i\}_{i=1}^n$ with $\{e_i\}_i$—an orthogonal basis in $\mathbf{R}^n$), and of all polyhedrons with given normal vectors to the faces.

(b) Let us fix the points $z_1, \ldots, z_N \in \mathbf{R}^n$, $N \geq 1$, and define

$$\mathcal{E} \stackrel{def}{=} \left\{ \bigcap_{i=1,\ldots,N} (z_i + s_i\mathcal{B}),\ s = (s_1, \ldots, s_N) \in S \right\},$$

where S is the set of those s for which the intersection is nonempty. For every $Z \in comp(\mathbf{R}^n)$, the unique minimal element from $\mathcal{E}$ that contains Z has

$$s_i = dist(Z, z_i).$$

III. Viability theory for collections of sets and solution tubes

The theory developed in the previous section extends the usual notion of trajectory to the notion of a solution tube in a collection of sets. Then the problem arises to develop the viability theory for solution tubes similarly to the way it has been done for usual trajectories [1].

General theories for set-valued dynamical systems (in metric spaces) were developed in [20], [2]. In particular, the viability theory was extended to more general dynamics in metric spaces in [11], [13], [2]. The dynamics of the solution tubes in a given collection of sets, however, does not fit within the above-mentioned framework. A relevant notion for contingency and viability for collections of sets was developed in [22] and we employ it below.

A. Viability domains and kernels

Let a collection of compact sets $\mathcal{E}$ in $\mathbf{R}^n$ be fixed and let $Z \in comp(\mathbf{R}^n)$.

Definition. A mapping $L(\cdot) \in C(Z, comp(\mathbf{R}^n))$ is called a (continuous) contingent field to $\mathcal{E}$ at Z if

$$\liminf_{h \to 0+} \inf_{\tilde{E} \in \mathcal{E}} \sup_{x \in Z} dist\left(L(x), \frac{\tilde{E} - x}{h}\right) = 0.$$

The set of all contingent fields to $\mathcal{E}$ at Z will be denoted by $\mathcal{T}_\mathcal{E}(Z)$. The following lemma is a straightforward consequence of the definition.

Lemma 3.1: The (set-valued) mapping $L(\cdot) \in C(Z; comp(\mathbf{R}^n))$ belongs to $\mathcal{T}_\mathcal{E}(Z)$ if and only if there are sequences $h_k \to 0+$, $\gamma_k \to 0$, and $E_k \in \mathcal{E}$ such that

$$(I + h_k L(\cdot))(Z) \subset E_k + h_k\gamma_k\mathcal{B} \quad \forall k$$

(here and below, I is the identity mapping).

In the particular case of the collection $\mathcal{E}_K = \{\{x\};\ x \in K\}$ of all single points from a given set $K \subset \mathbf{R}^n$, the set $\mathcal{T}_\mathcal{E}(Z)$ is nonempty if and only if $Z = \{x\}$ is a singleton and every contingent field is single-valued. In fact, the contingent field at $\{x\}$ is defined at the single point x only, and therefore can be identified with a vector $l \in \mathbf{R}^n$. It is straightforward that $l \in \mathcal{T}_{\mathcal{E}_K}(\{x\})$ if and only if $l \in T_K(x)$, where $T_K(x)$ is the usual (Bouligand) contingent cone to K at x (see, e.g., [3]). In the case of collections that do not consist only of singletons, we use the term *contingent field* (in contrast to "contingent vector") to stress that *mappings* defined on Z are considered "tangent" objects, rather than constant vectors only. The tangent fields possess many of the properties of the tangent vectors (see [22, Proposition 1]).

Below $\mathcal{M}$ will be another (possibly empty) collection of compact sets in $\mathbf{R}^n$ that will be interpreted later as a "target." Let $\mathcal{L}$ be a family of mappings from $C(\mathbf{R}^n; comp(\mathbf{R}^n))$.

Then $\mathcal{L}_{|Z}$ will denote the set of restrictions of the mappings from $\mathcal{L}$ to the set $Z \in comp(\mathbf{R}^n)$.

Definition. The collection $\mathcal{E}$ is called a *viability domain with target* $\mathcal{M}$ for $\mathcal{L}$ if

$$\mathcal{T}_{\mathcal{E}}(E) \cap \mathcal{L}_{|E} \neq \emptyset \quad \forall E \in \mathrm{cl}\,(\mathcal{E}) \setminus \mathcal{M}.$$

Obviously $\mathcal{E}$ is a viability domain if and only if $\mathrm{cl}\,(\mathcal{E})$ is a viability domain.

If a collection $\mathcal{E} \subset comp(\mathbf{R}^n)$ is not a viability domain for $\mathcal{L}$, then it may happen that it contains a collection $\mathcal{E}' \subset \mathcal{E}$ which is a viability domain.

Theorem 3.2: Let the closed collections $\mathcal{E}$ and $\mathcal{M}$ of compact sets and the closed convex family $\mathcal{L} \subset C(\mathbf{R}^n; comp(\mathbf{R}^n))$ be given. Suppose that $\mathcal{E}$ satisfies Conditions B.1–B.3 and that for every $Z \in comp(\mathbf{R}^n)$, the family $\mathcal{L}_{|Z}$ is equi-Lipschitz and uniformly bounded. Then there exists a (possibly empty) closed collection $\mathcal{E}_0 \subset \mathcal{E}$ which is a viability domain for $\mathcal{L}$ with target $\mathcal{M}$ and which contains every other viability domain in $\mathcal{E}$ with target $\mathcal{M}$.

Definition. The (possibly empty) collection $\mathcal{E}_0 \subset \mathcal{E}$ obtained in the above theorem is called the *viability kernel of* $\mathcal{E}$ *with target* $\mathcal{M}$ for $\mathcal{L}$ and will be denoted by $Viab_{\mathcal{L}}(\mathcal{E};\mathcal{M})$.

B. Existence of a viable solution tube

In this subsection we address the key issue of the viability theory specified below in Theorem 3.3 in the present framework of collections of sets and solution tubes: the existence of a solution tube in a viability domain.

We consider a specific family of mappings $\mathcal{L} \subset C(\mathbf{R}^n; comp(\mathbf{R}^n))$ having the affine form

$$\mathcal{L} = \{F_0(\cdot) + B(\cdot)u;\ u \in U\}, \tag{7}$$

where $F_0 \in C(\mathbf{R}^n; comp(\mathbf{R}^n))$, $U \subset \mathbf{R}^r$, and $B(\cdot)$ is an $(n \times r)$-matrix function. Below in this section we suppose that

(i) $\mathcal{E} \subset comp(\mathbf{R}^n)$ satisfies Conditions B.1–B.3;

(ii) U is convex and compact, $B(\cdot)$ is locally Lipschitz;

(iii) $F_u(x) \stackrel{def}{=} F_0(x) + B(x)u$ satisfies Condition A uniformly in $u \in U$.

With the family $\mathcal{L}$ we associate the controlled differential inclusion

$$\dot{x} \in F_0(x) + B(x)u(t), \quad u(t) \in U. \tag{8}$$

Let us denote by $\mathcal{U}_{[s,\tau]}$ the set of all measurable $u(\cdot) : [s,\tau] \mapsto U$. As a consequence of Theorem 2.1, for every $s \geq 0$, every $E \in \mathcal{E}$, and every $u(\cdot) \in \mathcal{U}_{[s,+\infty)}$, inclusion (8) has a unique solution tube $E_{u(\cdot)}[s,E](\cdot)$ in the collection $\mathcal{E}$ on $[0,+\infty)$, starting from E at time s. Similarly, as before, the solution tube in $comp(\mathbf{R}^n)$ will be denoted by $X_{u(\cdot)}[s,E](\cdot)$.

Definition. The subset $\mathcal{E}'$ of $\mathcal{E}$ is *inclusion-complete* (in $\mathcal{E}$) if the inclusions $E \in \mathcal{E}'$, $E' \subset E$, and $E' \in \mathcal{E}$ together imply $E' \in \mathcal{E}'$.

The following definition adapts the terminology from [1].

Definition. Let $\mathcal{E}'$ and $\mathcal{M}$ be given closed subsets of $\mathcal{E}$. The collection $\mathcal{E}'$ enjoys the *viability property* with target $\mathcal{M}$ with respect to (8) if and only if for every $E \in \mathcal{E}' \setminus \mathcal{M}$ there exists $u(\cdot) \in \mathcal{U}_{[0,+\infty)}$ such that the solution tube $E_{u(\cdot)}(\cdot) = E_{u(\cdot)}[0,E](\cdot)$ either satisfies $E_{u(\cdot)}(t) \in \mathcal{E}'$ for all $t \geq 0$, or there is $T > 0$ such that $E_{u(\cdot)}(t) \in \mathcal{E}'$ on $[0,T]$ and $E_{u(\cdot)}(T) \in \mathcal{M}$.

Theorem 3.3: Let Conditions (i)–(iii) from the beginning of this subsection hold. The closed inclusion-complete collection $\mathcal{E}' \subset \mathcal{E}$ enjoys the viability property with target $\mathcal{M}$ with respect to (8) if and only if it is a viability domain with target $\mathcal{M}$.

C. Dini derivative and viability

In this subsection, $\mathcal{E}$ will always be a closed collection of nonempty compact sets in $\mathbf{R}^n$ and $J : \mathcal{E} \to \mathbf{R}$ will be a lower semicontinuous function (abbreviated as l.s.c.). Let $E \in \mathcal{E}$ be fixed and let $F : E \mapsto comp(\mathbf{R}^n)$ be a set-valued field on E.

Definition. We define the lower Dini derivative of J at E in the direction of the field F as

$$D^-_{\mathcal{E}} J(E;F) \stackrel{def}{=} \liminf_{h,\delta \to 0+} \inf \left\{ \frac{J(E') - J(E)}{h};\ E' \in \mathcal{E},\ (I + hF)(E) \subset E' + h\delta\mathcal{B} \right\}.$$

For $\mathcal{E}$ and J as above, we define as usual

$$epiJ \stackrel{def}{=} \{(E,\nu);\ E \in \mathcal{E},\ \nu \in \mathbf{R} :\ J(E) \leq \nu\} \subset \mathcal{E} \times \mathbf{R},$$

which is also a closed collection of compact sets.

The following claims are essential for the next section.

Lemma 3.4: For every $E \in \mathcal{E}$ and $F : E \mapsto comp(\mathbf{R}^n)$, the following two conditions are equivalent:
(i) $D^-_{\mathcal{E}} J(E;F) \leq 0$; (ii) $(F,0) \in \mathcal{T}_{epiJ}(E, J(E))$.

Lemma 3.5: For every fixed $E \in \mathcal{E}$, the mapping

$$L \to D^-_{\mathcal{E}} J(E;L) \tag{9}$$

defined in the space $C(E; comp(\mathbf{R}^n))$ of continuous set-valued fields on E is l.s.c.

Let $\mathcal{M} \subset \mathcal{E}$ be closed and let $\mathcal{L}$ be a closed family of mappings $L : \mathbf{R}^n \Rightarrow \mathbf{R}^n$ such that, for every $Z \in comp(\mathbf{R}^n)$, the family $\mathcal{L}_{|Z}$ is equi-Lipschitz and uniformly bounded.

Corollary 3.6: $epiJ$ is a viability domain with a target $\mathcal{M} \times \mathbf{R}$ for $\mathcal{L} \times \{0\}$ if and only if

$$\min_{L \in \mathcal{L}} D^-_{\mathcal{E}} J(E; L_{|E}) \leq 0 \quad \forall E \in \mathcal{E} \setminus \mathcal{M}.$$

The use of "min" in the above inequality, means that the infimum is attained.

IV. Optimal control in the presence of observable and unobservable uncertainties

We consider the minimization problem for (1), (2) formulated in the introduction. In addition we introduce a set $M \subset [0,T] \times \mathbf{R}^n$, which will determine the termination time of the control process, as described below. The following suppositions will hold for the rest of the section.

Condition C.5: U, Y, and $\bar{V}$ are compact subsets of finite dimensional vector spaces, U is convex, and the mapping $y \to V(y) \subset \bar{V}$ is compact valued and Lipschitz continuous.

Condition C.6: The function $f : \mathbf{R}^n \times U \times Y \times \bar{V} \mapsto \mathbf{R}^n$ has the form

$$f(x,u,y,v) = f_0(x,y,v) + B(x,y)u,$$

where f_0 and B are continuous, locally Lipschitz in x uniformly with respect to the other variables. The sets $f_0(x,y,V(y))$ are convex. f has linear growth with respect to x, uniformly in u,y,v.

The admissible sets $\mathcal{U}_{[t,\theta]}$, $\mathcal{Y}_{[t,\theta]}$, and $\mathcal{V}_{[t,\theta]}(y(\cdot))$ are defined in the introduction. We recall the notion of *nonanticipative strategy* on $[t,\theta]$. This is any mapping $\alpha : \mathcal{Y}_{[t,\theta]} \mapsto \mathcal{U}_{[t,\theta]}$ that satisfies the nonanticipativity condition

$$\begin{aligned}&\forall\, y_1, y_2 \in \mathcal{Y}_{[t,\theta]},\ \forall\, \tau \in (t,\theta]\\ &\text{if } y_1(s) = y_2(s) \quad \text{for a.e. } s \in [t,\tau],\\ &\text{then } \alpha(y_1)(s) = \alpha(y_2)(s) \quad \text{for a.e. } s \in [t,\tau].\end{aligned}$$

Let $\mathcal{A}_{[t,\theta]}$ denote the set of all such strategies on $[t,\theta]$.

Let us consider first the case of a fixed-end time T discussed in the introduction. With the system (1) and any fixed $u \in \mathcal{U}_{[t,T]}$ and $y \in \mathcal{Y}_{[t,T]}$, we associate the differential inclusion

$$\dot{x} \in f(x, u(s), y(s), V(y(s))). \tag{10}$$

Similarly, as before, we denote by $X_{u,y}[t,E](\cdot)$ the solution tube of (10) in $comp(\mathbf{R}^n)$, starting from the set E at time t. Moreover, for a compact set $Z \subset \mathbf{R}^n$ we define $G : comp(\mathbf{R}^n) \mapsto \mathbf{R}$ as

$$G(T,Z) = \sup_{z\in Z} g(T,z).$$

Then, obviously, definition (4) is equivalent to

$$I(t,E) = \inf_{\alpha\in\mathcal{A}_{[t,T]}} \sup_{y\in\mathcal{Y}_{[t,T]}} G(T, X_{\alpha(y),y}[t,E](T)). \tag{11}$$

In this formulation of the original problem there is no unobservable uncertainty. We passed to a problem with complete information (here y is an observable disturbance) but over the solution tubes to differential inclusion (10).

Because of the complexity of the problem so obtained, we can restrict the consideration to the solution tubes of (10) in a given collection of sets $\mathcal{E}$ instead of the whole $comp(\mathbf{R}^n)$. Thus we come up with the more general problem, formulated below in the case of a target that determines the termination time. We also suppose the following condition.

Condition C.7: The collection $\mathcal{E}$ satisfies Conditions B.2–B.4. The set $M \subset [0,+\infty) \times \mathbf{R}^n$ is closed and contains $t \times \mathbf{R}^n$ for every $t \geq T$. The function $g(\cdot) : M \mapsto \mathbf{R}$ is l.s.c. Moreover, the following property holds: For every $(t,x_0) \in M$ with $t < T$, for every $s \in (t,T]$, $y \in \mathcal{Y}_{[t,T]}$, and $u \in \mathcal{U}_{[t,T]}$, the inclusion (10) has a trajectory $x(\cdot)$ with $x(t) = x_0$ for which $g(s,x(s)) \geq g(t,x_0)$.

Formulation of the general problem. Let Conditions C.5–C.7 be satisfied. For any fixed $u \in \mathcal{U}_{[t,T]}$ and $y \in \mathcal{Y}_{[t,T]}$, we denote by $E_{u,y}[t,E](\cdot)$ the solution tube in the collection $\mathcal{E}$ starting from $E \in \mathcal{E}$ at time t. We define also the target collection $\mathcal{M} \stackrel{def}{=} \{(t,E);\ E \in \mathcal{E},\ (t,E) \subset M\}$, which is nonempty since $(T,\mathcal{E}) \subset \mathcal{M}$, closed, and inclusion complete in $\mathcal{E}$. For a given tube $E(\cdot) \in C([0,T];\mathcal{E})$ the termination time is determined as

$$T(E(\cdot)) = \min\{t \geq 0;\ (t,E(t)) \in \mathcal{M}\}.$$

We consider the following minimization problem for the initial pair $(t,E) \notin \mathcal{M}$:

$$I_{\mathcal{E}}(t,E) \stackrel{def}{=} \inf_{\alpha\in\mathcal{A}_{[t,T]}} \sup_{y\in\mathcal{Y}_{[t,T]}} G(\tau, E(\tau)), \tag{12}$$

$$\text{where } E(\cdot) \stackrel{def}{=} E_{\alpha(y),y}[t,E](\cdot),\ \tau = T(E(\cdot)) \tag{13}$$

If $(t,E) \in \mathcal{M}$, then by definition $I_{\mathcal{E}}(t,E) = G(t,E)$. The role of the last part of Condition C.3 is to ensure that once the target is achieved, the continuation of the process would not lead to a better guaranteed result. This supposition is obviously fulfilled in the case $\mathcal{M} = \{T\} \times \mathcal{E}$ (fixed-end time) or $G(t,E) = t$ (minimal time problem). Obviously we have $I_{\mathcal{E}}(t,E) \geq I_{comp(\mathbf{R}^n)}(t,E)$. The difference between these two values is the price for the simplification we make by passing to the collection $\mathcal{E}$. In some cases, however, equality may take place even for simple subcollections of $comp(\mathbf{R}^n)$, as shown by a discrete-time example in [19].

Some basic properties. The following propositions hold under Conditions C.1–C.3.

Proposition 4.1: (Dynamic Programming.) For every $E \in \mathcal{E}$, $t \in [0,T)$, and $s \in (t,T]$,

$$I_{\mathcal{E}}(t,E) = \inf\nolimits_{\alpha\in\mathcal{A}_{[t,s]}}\ \sup\nolimits_{y\in\mathcal{Y}_{[t,s]}} I_{\mathcal{E}}(\tau, E(\tau)),$$

where $E(\cdot) \stackrel{def}{=} E_{\alpha(y),y}[t,E](\cdot),\ \tau \stackrel{def}{=} \min\{s, T(E(\cdot))\}$.

Proposition 4.2: The value function $I_{\mathcal{E}} : [0,T] \times \mathcal{E}$ is l.s.c.

Proposition 4.3: For every $(t,E) \in [0,T) \times \mathcal{E}$ there is $\alpha \in \mathcal{A}_{[t,T]}$ for which the infimum in (12) is attained (that is an optimal strategy).

Characterizations of the value function. The first theorem below gives a characterization of the value function as the unique minimal Dini supersolution of the associated HJI equation. Define the extended closed collection of compact sets $\hat{\mathcal{E}}$ in $\mathbf{R} \times \mathbf{R}^n$ as

$$\hat{\mathcal{E}} \stackrel{def}{=} \{(t,E);\ t \geq 0,\ E \in \mathcal{E}\}.$$

Theorem 4.4: Under Conditions C.5–C.7, the value function $I_{\mathcal{E}}$ is the unique minimal l.s.c. solution of the differential inequality

$$\sup_{y\in Y} \min_{u\in U} D^-_{\hat{\mathcal{E}}} J(t,E;(1, f(\cdot,u,y,V(y)))) \leq 0 \tag{14}$$

$\forall\, (t,E) \in \hat{\mathcal{E}} \setminus \mathcal{M}$, with the side condition

$$J(t,E) \geq G(t,E) \quad \forall\, (t,E) \in \mathcal{M}. \tag{15}$$

That is,

$$I_{\mathcal{E}}(t,E) = \min\{J(t,E);\ J - \text{l.s.c. solution of } (14),\ (15)\}.$$

The next theorem extends the result in [7], the latter concerning the case of complete information (the disturbance v is not present). It shows that one can apply for calculation of $I_{\mathcal{E}}$ an extension, similar to that in [21], of the viability kernel algorithm [26], [8].

Theorem 4.5: Under Conditions C.5–C.7, denote

$$\mathcal{E}^* \stackrel{def}{=} \hat{\mathcal{E}} \times \mathbf{R},$$

$$\mathcal{M}^* \stackrel{def}{=} \{(t, E, \nu);\ (t, E) \in \mathcal{M},\ \nu \geq G(t, E)\}.$$

Define $\mathcal{E}_1^* = \mathcal{E}^*$, and recursively,

$$\mathcal{E}_{k+1}^* = \bigcap_{y \in Y} Viab_{\mathcal{L}_y}(\mathcal{E}_k^*; \mathcal{M}^*), \tag{16}$$

where $\mathcal{L}_y \stackrel{def}{=} \{(1, f(\cdot, u, y, V(y)), 0);\ u \in U\}$. Then

$$epiI_{\mathcal{E}} = \mathcal{E}_\infty \stackrel{def}{=} \bigcap_k \mathcal{E}_k^*.$$

Since the control problem considered above includes minimal time problems, clearly the value function can be discontinuousIf g is locally Lipschitz and the problem with fixed-end time is considered, then $I_{\mathcal{E}}$ is Lipschitz, provided that the solution tubes in $\mathcal{E}$ depend in a certain Lipschitz-like way on the data (see [23, Theorem 2]). However, an example is given in [24], which shows that even in the case of a Lipschitz continuous value function, it may happen that (14) is not satisfied as an equality.

The last theorem shows that finding the value $I_{\mathcal{E}}$ is equivalent to the determination of suitable viability kernels. Notice that the viability kernels in (16) involve the complete information systems $\mathcal{L}_y$ only. The latter systems, however, are set-valued and the numerical algorithms developed in [26], [8], [21] are not directly applicable. At a conceptual level, we may summarize the algorithm for finding the viability kernel $Viab_{\mathcal{L}}(\mathcal{E}; \mathcal{M})$ of a family of set-valued fields $\mathcal{L}$, with target $\mathcal{M}$ in the collection $\mathcal{E}$ as follows. For $h > 0$ and a natural number k, we define recursively

$$\begin{aligned} \mathcal{E}_{i+1}^h \quad := \quad & \{E \in \mathcal{E}_i^h \setminus \mathcal{M};\ \exists L \in \mathcal{L}, \exists \widetilde{E} \in \mathcal{E}_i^h, \\ & (I + hL)(E) \subset \widetilde{E} + ch^2\mathcal{B}\} \cup \mathcal{M}, \quad \mathcal{E}_0^h = \mathcal{E}, \end{aligned}$$

where $c \geq \lambda M/2$, and λ and M are local Lipschitz constant and bound for all $L \in \mathcal{L}$, respectively. Then

$$Viab_{\mathcal{L}}(\mathcal{E}; \mathcal{M}) = \lim_{h \to 0+} \bigcap_i \mathcal{E}_i^h.$$

The above procedure should be applied to the family $\mathcal{L}_y = \{(1, f(\cdot, u, y, V(y)), 0);\ u \in U\}$ in (16) for each $y \in Y$.

Clearly, the algorithm has a high computational complexity, and to make it implementable, one has to choose a simple finitely parameterized collection $\mathcal{E}$.

We mention that Theorem 4.4 can also be used as a base for a numerical solution. In [19] we demonstrate the solvability of a discrete-time Hamilton–Jacobi–Bellman inequality of similar type, where sets of intervals were used for the collection $\mathcal{E}$. Essentially, the approach based on Theorem 4.4 coincides with a particular realization of the viability kernel algorithm.

REFERENCES

[1] J.-P. Aubin, *Viability Theory*, Birkhäuser, Boston, 1991.

[2] J.-P. Aubin, *Mutational and Morphological Analysis. Tools for Shape Evolution and Morphogenesis*, Systems Control Found. Appl., Birkhäuser, Boston, 1999.

[3] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Systems Control Found. Appl., Birkhäuser, Boston, 1990.

[4] J. S. Baras and N. S. Patel, Robust control of set-valued discrete-time dynamical systems, *IEEE Trans. Automat. Control*, vol. 43, 1998, pp. 61–75.

[5] T. Basar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems*, Birkhäuser, Boston, 1995.

[6] J. S. Baras and M. R. James, Partially observed differential games, infinite-dimensional Hamilton–Jacobi–Isaacs equations, and nonlinear H_∞ control, *SIAM J. Control Optim.*, vol. 34, 1996, pp. 1342–1364.

[7] P. Cardaliaguet, A differential game with two players and one targe, *SIAM J. Control Optim.*, vol. 34, 1996, pp. 1441–1460.

[8] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre, "Set-valued numerical analysis for optimal control and differential games", in *Stochastic and Differential Games*, Ann. Internat. Soc. Dynam. Games 4, Birkhäuser, Boston, 1999, pp. 177–247.

[9] F. L. Chernousko, *Estimation of the Phase State of Dynamical Systems*, Nauka, Moscow, 1988 (in Russian).

[10] F. L. Chernousko and D. Ya. Rokityanskii, Ellipsoidal bounds on reachable sets of dynamical systems with matrices subjected to uncertain perturbations, *J. Optim. Theory Appl.*, vol. 104, 2000, pp. 1–19.

[11] L. Doyen, Filippov and invariance theorems for mutational equations for tubes, *Set-Valued Anal.*, vol. 1, 1993, pp. 289–303.

[12] L. C. Evans and P. E. Souganidis, Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations, *Indiana Univ. Math. J.*, vol. 33, 1984, pp. 773–797.

[13] A. Gorre, Evolutions of tubes under operability constraints, *J. Math. Anal. Appl.*, vol. 216, 1997, pp. 1–22.

[14] G. Häckl, Numerical approximation of reachable sets and control sets, *Random Comput. Dynam.*, vol. 1, 1992–93, pp. 371–394.

[15] L. Jaulin, M. Kieffer, I. Braems, and E. Walter, Guaranteed non-linear estimation using constraint propagation on sets, *Internat. J. Control*, vol. 74, 2001, pp. 1772–1782.

[16] A. B. Kurzhanski and I. Vályi, *Ellipsoidal Calculus for Estimation and Control*, Birkhäuser, Boston, 1997.

[17] A. B. Kurzhanski and P. Varaiya, Ellipsoidal techniques for reachability analysis: Internal approximation, *Systems Control Lett.*, vol. 41, 2000, pp. 201–211.

[18] F. Lempio and V. M. Veliov, Discrete approximations to differential inclusions, *GAMM Mitt. Ges. Angew. Math. Mech.*, vol. 21, 1998, pp. 101–135.

[19] R. Moitié, M. Quincampoix, and V. M. Veliov, Optimal control of discrete-time uncertain systems with imperfect measurement, *IEEE Trans. Automat. Control*, vol. 47, 2002, pp. 1909–1914.

[20] A. I. Panasyuk, Quasidifferential equations in a complete metric space under conditions of the Caratheodory type. I, *Differential Equations*, vol. 31, 1995, pp. 901–910 (translation from Differ. Uravn., vol. 31, 1995, pp. 962–972).

[21] M. Quincampoix and V. M. Veliov, "Viability with a target: Theory and applications", in *Applications of Mathematics in Engineering*, B. I. Cheshankov and M. D. Todorov, eds., Heron Press, Sofia, 1998, pp. 47–54.

[22] M. Quincampoix and V. M. Veliov, Open-loop viable control under uncertain initial state information, *Set-Valued Anal.*, vol. 7, 1999, pp. 55–87.

[23] M. Quincampoix and V. M. Veliov, Solution tubes to differential equations within a collection of sets, *Control Cybernet.*, vol. 31, 2002, pp. 847–862.

[24] M. Quincampoix and V. M. Veliov, Optimal Control of Uncertain Systems with Incomplete Information for the Disturbances, *SIAM J. Control Optim.*, vol. 43, 2005, pp. 1373–1399.

[25] A. Rapaport and P. Bernhard, On a planar pursuit game with imperfect knowledge of a coordinate, *Automatique Productique Informatique Industrielle*, vol. 29, 1995, pp. 575–601 (in French).

[26] P. Saint-Pierre, Approximation of the viability kernel, *Appl. Math. Optim.*, 29, 1994, pp. 187–209.