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Kalman Filtering by Minimax Criterion with Uncertain Noise Intensity Functions

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Abstract— **The problem of minimax filtering is examined for linear continuous-time observation models with uncertain intensities of non-stationary white noises. For designing algorithms of minimax filtering, the method of dual optimization is used together with the techniques of the maximum principle. It is shown that the Kalman filter is a minimax one if its coefficients are defined by the least favorable noise intensity. The explicit form of the minimax filter is derived in the case of scalar state and observation processes with arbitrarily correlated disturbances. The results of numerical modeling are also presented.**

I. INTRODUCTION

One of the major methods for designing robust estimates and filters is provided by the approach of minimax optimization [1]–[6]. In the game-theoretic framework the problem of robust estimation for linear stochastic systems with uncertain second-order moments has received considerable attention. For a lot of regression and discrete-time observation models the effective algorithms of minimax estimation and filtering have been designed using the method of dual optimization [3], [4], [7], [8]. According to the last, a minimax solution should be sought as an optimal one with the least favorable characteristics which form a solution of the respective dual optimization problem. This algorithm leads to the solution of the minimax problem if certain conditions of regularity are fulfilled. The recent investigations [9]–[12] have shown that similar sufficient conditions can be also derived in the infinite-dimensional case.

In this paper, we develop the approach described above to the problem of minimax filtering for linear continuous-time systems of Kalman's type. It is shown that the method of dual optimization can be applied to determining the minimax filter with respect to both local and integral mean-square-error criteria. If the *a priori* information on the observation model is expressed via the intensity functions of the white-noise processes, then the dual problem can be represented in the form of an optimal control problem. Using the techniques of the maximum principle, we provide a general solution to the dual variational problem. For a model of particular type, the obtained solution takes an explicit form. For the sake of completeness, we also present the results of computer modeling.

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II. MODEL DESCRIPTION

Consider the following continuous-time linear dynamic system:

$$\begin{cases} \xi(t) = \xi(0) + \int_0^t a(s)\xi(s)\,ds + \int_0^t b(s)\,d\mu(s), \\ \eta(t) = \int_0^t A(s)\xi(s)\,ds + \int_0^t B(s)\,d\nu(s), \end{cases} \tag{1}$$

where $\{\xi(t),\, t \in [0,T]\}$ is an $m \times 1$ random process to be estimated given an $n \times 1$ observable random process $\{\eta(t),\, t \in [0,T]\}$. The initial state $\xi(0)$ is assumed to have zero mean and covariance $D = \mathsf{cov}\{\xi(0)\}$ such that $D \in \mathcal{D}$, where $\mathcal{D}$ is a given set of symmetric positive-semidefinite $m \times m$ matrices. Concerning the noise processes $\mu(t) \in \mathbb{R}^p$ and $\nu(t) \in \mathbb{R}^q$, we suppose that they are of zero mean and uncorrelated with the initial state $\xi(0)$. Furthermore, the compound process $w(t) = \mathrm{col}[\mu(t), \nu(t)]$ is assumed to be with orthogonal increments. In other words, its covariance function can be represented as follows:

$$\mathsf{cov}\{w(t), w(s)\} = \int_0^{\min(t,s)} R(\tau)\,d\tau, \quad t, s \in [0,T].$$

The intensity function $R(t)$ is known to belong to the following class:

$$\mathfrak{R} = \{R\colon [0,T] \to \mathbb{R}^{u \times u}\colon R(\cdot) \text{ is measurable and } R(t) \in \mathcal{R} \text{ for a.e. } t \in [0,T]\}, \tag{2}$$

where $\mathcal{R}$ is some fixed set of symmetric positive-semidefinite $u \times u$ matrices, $u = p + q$.

Notation 1: Elements of the set $\mathcal{R}$ will be represented in the form $R = \begin{pmatrix} R_\mu & R_{\mu\nu} \\ R_{\nu\mu} & R_\nu \end{pmatrix}$.

The matrix functions $a(s) \in \mathbb{R}^{m \times m}$, $A(s) \in \mathbb{R}^{n \times m}$, $b(s) \in \mathbb{R}^{m \times p}$, $B(s) \in \mathbb{R}^{n \times q}$ are supposed to be known and piecewise continuous.

Since system (1) is considered under *a priori* statistical uncertainty, it will be referred to as a *statistically indeterminate observation model*.

The main assumptions on the uncertainty sets $\mathcal{D}$ and $\mathcal{R}$ are presented below.

Assumption A: The sets $\mathcal{D}$ and $\mathcal{R}$ are convex and compact.

Assumption B: The regularity condition is fulfilled:

$$\exists\, \varepsilon > 0\colon \quad B(s) R_\nu B^\top(s) \geq \varepsilon I \quad \textit{a.e.} \quad \forall\, R \in \mathcal{R}.$$

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III. Minimax Filter

Given the observation process $\{\eta(t),\ t \in [0,T]\}$, a linear estimate $\widetilde{\xi}(t)$ of the state vector $\xi(t)$ is said to be an *admissible filter* iff there exists a measurable function $g\colon [0,T]\times[0,T] \to \mathbb{R}^{m\times n}$ such that $\int_0^T\int_0^T \|g(t,s)\|^2\,dtds < \infty$ and

$$\widetilde{\xi}(t) = \int_0^t g(t,s)\,d\eta(s) \quad \forall\, t \in [0,T]. \tag{3}$$

In other words, we consider any linear non-anticipative estimates.

Notation 2: By Ξ denote the class of filters (3).

Notation 3: In what follows, $\mathsf{E}\{\cdot \mid D,R\}$ ($\mathsf{cov}\{\cdot \mid D,R\}$ etc) denotes the expectation (covariance, etc) calculated under the assumption that the processes ξ and η satisfy equations (1) with the initial state covariance D and the noise intensity function $R(\cdot)$.

The accuracy of $\widetilde{\xi} \in \Xi$ will be measured by the mean-square-error criterion (m.s.e.c.)

$$J_t(\widetilde{\xi},D,R) = \mathsf{E}\Big\{\|\xi(t)-\widetilde{\xi}(t)\|^2 \mid D,R\Big\},\ t\in[0,T]. \tag{4}$$

Note that functional (4) is a local criterion.

In addition, introduce the integral functional

$$J(\widetilde{\xi},D,R) = \int_0^T J_t(\widetilde{\xi},D,R)\,dt. \tag{5}$$

Definition 1: Let $I(\cdot)$ denote one of criteria (4) *or* (5)*. An admissible filter $\hat{\xi}$ is called minimax w.r.t. the functional $I(\cdot)$ iff*

$$\hat{\xi} \in \arg\min_{\widetilde{\xi}\in\Xi}\ \sup_{D\in\mathcal{D},R\in\mathfrak{R}} I(\widetilde{\xi},D,R). \tag{6}$$

The optimal guaranteed value of $I(\cdot)$ is equal to

$$\hat{I} = \inf_{\widetilde{\xi}\in\Xi}\ \sup_{D\in\mathcal{D},R\in\mathfrak{R}} I(\widetilde{\xi},D,R).$$

In this paper, for designing the algorithms of minimax filtering we are going to use the *method of dual optimization* [5], [13]. To this end, consider the maximization problem

$$(\hat{D},\hat{R}) \in \arg\max_{D\in\mathcal{D},R\in\mathfrak{R}} \underline{I}(D,R), \tag{7}$$

where

$$\underline{I}(D,R) = \inf_{\widetilde{\xi}\in\Xi} I(\widetilde{\xi},D,R).$$

We say that (7) is the *dual* problem w.r.t. the original minimax one (6) if the following duality relation is fulfilled:

$$\inf_{\widetilde{\xi}\in\Xi}\ \sup_{D\in\mathcal{D},R\in\mathfrak{R}} I(\widetilde{\xi},D,R) = \sup_{D\in\mathcal{D},R\in\mathfrak{R}} \underline{I}(D,R).$$

According to the method of dual optimization, the minimax filter $\hat{\xi}$ is sought as an optimal one

$$\hat{\xi} \in \arg\min_{\widetilde{\xi}\in\Xi} I(\widetilde{\xi},\hat{D},\hat{R})$$

given the least favorable combination $(\hat{D},\hat{R})$ of uncertain characteristics.

The exact formulation of the algorithm described above is presented in the following theorem.

Theorem 1: Let $I(\cdot)$ denote one of criteria (4) *or* (5)*. Under Assumptions A and B, the assertions below are valid.*

1) *There exists a solution to the dual problem* (7)*, where*

$$\underline{I}(D,R) = \begin{cases} \mathrm{tr}[\gamma(t)] & \text{if} \quad I(\cdot) = J_t(\cdot), \\ \int_0^T \mathrm{tr}[\gamma(t)]\ dt & \text{if} \quad I(\cdot) = J(\cdot), \end{cases} \tag{8}$$

and $\gamma(\cdot)$ satisfies the Riccati equation a.e. on $[0,T]$:

$$\begin{aligned}\dot{\gamma}(s) &= a(s)\gamma(s) + \gamma(s)a^\top(s) + b(s)R_\mu(s)b^\top(s) - \\ &[b(s)R_{\mu\nu}(s)B^\top(s) + \gamma(s)A^\top(s)](B(s)R_\nu(s)B^\top(s))^{-1}\times \\ &[B(s)R_{\nu\mu}(s)b^\top(s) + A(s)\gamma(s)], \qquad \gamma(0) = D. \end{aligned}\tag{9}$$

2) *The minimax filter is defined by the Kalman equations:*

$$\begin{aligned}\hat{\xi}(t) = \int_0^t \big\{ &a(s)\hat{\xi}(s)\,ds + [b(s)\hat{R}_{\mu\nu}(s)B^\top(s) + \hat{\gamma}(s)A^\top(s)]\times \\ &(B(s)\hat{R}_\nu(s)B^\top(s))^{-1}[d\eta(s) - A(s)\hat{\xi}(s)\,ds]\big\}, \end{aligned}\tag{10}$$

where $\hat{\gamma}(\cdot)$ satisfies (9) *with $D = \hat{D}$ and $R(\cdot) = \hat{R}(\cdot)$.*

3) *The guaranteed value of the criterion is equal to $\underline{I}(\hat{D},\hat{R})$.*

4) *The minimax filter $\hat{\xi}$ and the least favorable characteristics $(\hat{D},\hat{R})$ form a saddle point for the functional $I(\cdot)$ on the product of Ξ and $\mathcal{D}\times\mathfrak{R}$:*

$$I(\hat{\xi},D,R) \le I(\hat{\xi},\hat{D},\hat{R}) \le I(\widetilde{\xi},\hat{D},\hat{R})$$

$$\forall\,\widetilde{\xi}\in\Xi \quad \textit{and} \quad \forall\,(D,R)\in\mathcal{D}\times\mathfrak{R}.$$

Proof: The proof of Theorem 1 is based on the following game-theoretic result (see Lemma 3, [11]): the pair $\hat{\xi}$ and $(\hat{D},\hat{R})$ form a saddle point for $I(\cdot)$ on the product of Ξ and $\mathcal{D}\times\mathfrak{R}$, whenever the following conditions hold:

a) given any $\widetilde{\xi}\in\Xi$, $I(\widetilde{\xi},\cdot)$ is a concave function on the convex set $\mathcal{D}\times\mathfrak{R}$;

b) $(\hat{D},\hat{R}) \in \arg\max_{D\in\mathcal{D},R\in\mathfrak{R}} \underline{I}(D,R)$;

c) $\hat{\xi} = \widetilde{\xi}(\hat{D},\hat{R})$, where $\widetilde{\xi}(D,R) \in \arg\min_{\widetilde{\xi}\in\Xi} I(\widetilde{\xi},D,R)$;

d) $I(\widetilde{\xi}(D^\alpha,R^\alpha),D,R) \to I(\hat{\xi},D,R)$ as $\alpha \downarrow 0$ for every $(D,R)\in\mathcal{D}\times\mathfrak{R}$, where $D^\alpha = (1-\alpha)\hat{D} + \alpha D$ and $R^\alpha = (1-\alpha)\hat{R} + \alpha R$.

Therefore, the proof falls into several steps.

a) In fact, $I(\widetilde{\xi},D,R)$ is linear in (D,R), while $\mathcal{D}\times\mathfrak{R}$ is convex by Assumption A.

b) In order to prove the existence of (7) it suffices to show that both the functional $\underline{I}(\cdot)$ is upper semicontinuous and the set $\mathcal{D}\times\mathfrak{R}$ is compact w.r.t. the weak topology of $\mathbb{R}^{m\times m}\times L_1([0,T],\mathbb{R}^{u\times u})$.

It is easy to check that $\mathfrak{R}$ is uniformly integrable, since each function $R\in\mathfrak{R}$ is with values in the bounded set $\mathcal{R}\subset\mathbb{R}^{u\times u}$. Then, due to the Dunford–Pettis theorem [14], $\mathfrak{R}$ is relatively compact w.r.t. the weak topology of

$L_1([0,T],\mathbb{R}^{u\times u})$. In addition $\mathfrak{R}$ is strongly closed. Indeed, if the sequence $\{R^n\}\subset\mathfrak{R}$ converges to a function R w.r.t. L_1-norm, then there exists a subsequence $\{n_k\}$ such that $R(t)=\lim_{k\to\infty}R^{n_k}(t)$ for a.e. $t\in[0,T]$. Hence, $R(t)\in\mathcal{R}$ a.e. on $[0,T]$, i.e., $R\in\mathfrak{R}$. Furthermore, $\mathfrak{R}$ is weakly closed as a convex set. Now we can notice that $\mathfrak{R}$ is a weakly compact subset of $L_1([0,T],\mathbb{R}^{u\times u})$.

Given a fixed estimate $\widetilde{\xi}\in\Xi$, the continuity of $I(\widetilde{\xi},\cdot)\colon\mathbb{R}^{m\times m}\times L_1([0,T],\mathbb{R}^{u\times u})\to\mathbb{R}$ can be checked directly. Hence, $I(\widetilde{\xi},\cdot)$ is weakly continuous as a linear functional. Then, $\underline{I}(\cdot)=\inf_{\widetilde{\xi}\in\Xi}I(\widetilde{\xi},\cdot)$ is upper semicontinuous w.r.t. the weak topology.

c) By $\widetilde{\xi}(D,R)=\{\widetilde{\xi}(t\mid D,R),t\in[0,T]\}$ denote the Kalman filter satisfying equations (9)–(10) with the initial state covariance D and the noise intensity function R. Then, $\widetilde{\xi}(D,R)$ is an optimal filter w.r.t. both local and integral criteria over the class of all linear non-anticipative estimates (e.g., see [15]). Hence, condition c) is fulfilled for any variant of functionals (4) or (5).

d) Under Assumption B Condition d) is proved in [12].

This completes the proof of Theorem 1. ■

IV. Dual Problem

In this section, we are going to examine the dual problem (7). Provided by Theorem 1, (7) can be treated as an optimal control problem for system (9) with the terminal or integral functional (8) maximized w.r.t. the state $\gamma(\cdot)$ and control $R(\cdot)$ subject to constraints (2).

The statement below contains the result of applying the maximum principle to the variational problem (7). For the sake of briefness, we will use the following notation.

Notation 4: Denote the right side of the Riccati equation (9) by $\Gamma(s,\gamma,R)$ with replacing $\gamma(s)$ by γ and $R(s)$ by R. In addition, let

$$\Psi(s,\gamma,R,\psi)=\nabla_\gamma(\mathrm{tr}[\psi\Gamma(s,\gamma,R)])=\psi[a(s)-K(s,\gamma,R)A(s)]+[a(s)-K(s,\gamma,R)A(s)]^\top\psi,$$

where $\psi\in\mathbb{R}^{m\times m}$ and

$$K(s,\gamma,R)=[b(s)R_{\mu\nu}B^\top(s)+\gamma A^\top(s)](B(s)R_\nu B^\top(s))^{-1}.$$

Theorem 2: Under Assumptions A and B, suppose that the set $\mathcal{D}$ contains a maximal element $\bar{D}$, i.e., $D\le\bar{D}\ \forall D\in\mathcal{D}$.

1) *If $I(\cdot)=J_t(\cdot)$, then $(\bar{D},\hat{R})$ is a solution to the dual problem* (7) *iff the following relations are fulfilled:*

$$\hat{R}(s)\in\arg\max_{R\in\mathcal{R}}\mathrm{tr}[\psi(s)\Gamma(s,\hat{\gamma}(s),R)]\,, \tag{11}$$

$$\dot{\hat{\gamma}}(s)=\Gamma(s,\hat{\gamma}(s),\hat{R}(s)),\quad \hat{\gamma}(0)=\bar{D}, \tag{12}$$

$$\dot{\psi}(s)=-\Psi(s,\hat{\gamma}(s),\hat{R}(s),\psi(s)),\quad \psi(t)=I, \tag{13}$$

for a.e. $s\in[0,t]$.

2) *If $I(\cdot)=J(\cdot)$, then $(\bar{D},\hat{R})$ is a solution to the dual problem* (7) *iff relations* (11), (12)*, and*

$$\dot{\psi}(s)=-\Psi(s,\hat{\gamma}(s),\hat{R}(s),\psi(s))-I,\quad \psi(T)=O, \tag{14}$$

are fulfilled a.e. on $[0,T]$.

Proof: First note that the functional $I(\cdot)$ is monotonous in D, i.e., $I(\widetilde{\xi},D)\le I(\widetilde{\xi},\bar{D})$ whenever $D\le\bar{D}$ and $\widetilde{\xi}\in\Xi$. Hence, $\hat{D}=\bar{D}$, since $\bar{D}$ is the maximal element on $\mathcal{D}$.

1) Now we suppose that problem (7) is considered with the local criterion (4).

Introduce the Pontryagin function

$$H(s,\gamma,R,\psi)=\mathrm{tr}[\psi\Gamma(s,\gamma,R)]\,, \tag{15}$$

where $s\in[0,t]$, $\gamma\in\mathbb{R}^{m\times m}$, $\psi\in\mathbb{R}^{m\times m}$, and $R\in\mathbb{R}^{u\times u}$.

Let us assume that $\hat{R}(\cdot)$ is a solution to the following problem of optimal control:

$$f(\gamma(t))=-\mathrm{tr}[\gamma(t)]\to\min\quad\text{subject to} \tag{16}$$

$$\dot{\gamma}(s)=\Gamma(s,\gamma(s),R(s)),\quad\gamma(0)=\bar{D}, \tag{17}$$

$$R(s)\in\mathcal{R}\quad\text{for a.e. } s\in[0,t]. \tag{18}$$

Since $\nabla_\gamma H(s,\gamma,R,\psi)=\Psi(s,\gamma,R,\psi)$ and $\nabla_\gamma f(\gamma)=-I$, from the maximum principle [16] it follows that there exists a function $\psi\colon[0,t]\to\mathbb{R}^{m\times m}$ such that relations (11)–(13) are fulfilled.

The sufficient conditions for (11)–(13) to define a solution of (16)–(18) can be found in [17, Cor. 5.4.2]. For the problem under consideration, it remains to check that the function $(\gamma,R)\mapsto H(s,\gamma,R,\psi)$ is concave on $\mathbb{R}^{m\times m}\times\mathcal{R}$. According to Lemma 1, we may assume that in (15) ψ is a positive-semidefinite matrix. Then from Lemma 2, it follows that the Pontryagin function can be represented in the form

$$H(s,\gamma,R,\psi)=\min_{F\in\mathbb{R}^{m\times n}}\mathrm{tr}\big[\psi[-I\,F]W(s,\gamma,R)[-I\,F]^\top\big]\,,$$

where the matrix function

$$W(s,\gamma,R)=\begin{pmatrix}W_X(s,\gamma,R) & W_{XY}(s,\gamma,R)\\ W_{YX}(s,\gamma,R) & W_Y(s,\gamma,R)\end{pmatrix}$$

has the following blocks:

$$W_X(s,\gamma,R)=a(s)\gamma+\gamma a^\top(s)+b(s)R_\mu b^\top(s),$$
$$W_{XY}(s,\gamma,R)=b(s)R_{\mu\nu}B^\top(s)+\gamma A^\top(s),$$
$$W_{YX}(\cdot)=W_{XY}^\top(\cdot),\quad W_Y(s,\gamma,R)=B(s)R_\nu B^\top(s).$$

Since $(\gamma,R)\mapsto W(s,\gamma,R)$ is a linear mapping, we obtain that $(\gamma,R)\mapsto H(s,\gamma,R,\psi)$ is concave.

2) The dual problem with the integral criterion (5) is equivalent to the following minimization problem:

$$\int_0^T(-\mathrm{tr}[\gamma(s)])\,ds\to\min \tag{19}$$

subject to (17), (18), and $t=T$. Then, the corresponding Pontryagin function takes the form

$$H(s,\gamma,R,\psi,\lambda)=\mathrm{tr}[\psi\Gamma(s,\gamma,R)+\lambda\gamma]\,,$$

where $\lambda \in \mathbb{R}$.

Assume that $\hat{R}(\cdot)$ is a solution to the optimal control problem (19), (17), (18), and $t = T$. Then, according to the maximum principle there exist a number λ and a function $\psi(\cdot)$ such that relations (11), (12), and

$$\dot{\psi}(s) = -\Psi(s, \hat{\gamma}(s), \hat{R}(s), \psi(s)) - \lambda I, \quad \psi(T) = O, \quad (20)$$

are fulfilled a.e. on $[0, T]$ under the condition

$$|\lambda| + |\psi(s)| \not\equiv 0. \quad (21)$$

Without loss of generality, we may put $\lambda = 1$. Actually, if $\lambda = 0$, the unique solution to (20) is trivial: $\psi \equiv 0$. But this contradicts to (21), whence $\lambda \neq 0$.

Thus, conditions (11), (12), and (14) are necessary for $(\bar{D}, \hat{R})$ to be a solution of the dual problem. The sufficient conditions can be checked similarly to that in the first part of the proof. ■

V. A Model with Arbitrarily Correlated Noises

In this section we study a particular case of the general statistically indeterminate observation model.

Suppose that the processes ξ and η satisfy equations (1) under the following assumptions:

$$\begin{gathered} m = n = p = q = 1, \quad \mathcal{D} = [0, \bar{D}], \\ \mathcal{R} = \{R\colon\ R = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix},\ |r| \le \bar{r}\}, \end{gathered} \quad (22)$$

where $\bar{D} \ge 0$ and $0 \le \bar{r} \le 1$ are given parameters.

Thus, we consider the observation model with scalar state, observation, and disturbing processes. The initial state variance is majorized above: $\mathsf{D}\{\xi(0)\} \le \bar{D}$. The noise auto- and cross- intensities are also bounded: $R_\mu(s) \le 1$, $R_\nu(s) \le 1$, and $R^2_{\mu\nu}(s) \le \min\{R_\mu(s) R_\nu(s), \bar{r}^2\}$ a.e. on $[0, T]$.

In particular, the state and observation noises have arbitrary correlation if $\bar{r} = 1$.

Concerning Assumptions A, B introduced in Section II, we can state that the sets $\mathcal{D}$ and $\mathcal{R}$ are convex and compact by definition, while the regularity condition holds whenever

$$\exists\, \varepsilon > 0\colon \quad B^2(s) \ge \varepsilon \quad \text{for a.e. } s \in [0, T]. \quad (23)$$

Let us introduce some additional notation.

Notation 5: Denote

$$r_+ = \begin{cases} r, & r \ge 0, \\ 0, & r < 0, \end{cases} \qquad \mathrm{sat}_{\bar{r}}(r) = \begin{cases} -\bar{r}, & r < -\bar{r}, \\ r, & |r| \le \bar{r}, \\ \bar{r}, & r > \bar{r}, \end{cases}$$

for all $r \in \mathbb{R}$.

The theorem below provides the explicit solution of the minimax filtering problem in the model under consideration.

Theorem 3: Let system (1) *be considered under assumptions* (22) *and* (23).

Then,

1) *the filter*

$$\hat{\xi}(t) = \int_0^t \Big\{ a(s)\hat{\xi}(s)\, ds + \frac{\mathrm{sign}(A(s))(\hat{\gamma}(t)|A(t)| - \bar{r}|b(t)B(t)|)_+}{B^2(t)} \times [d\eta(s) - A(s)\hat{\xi}(s)\, ds] \Big\}, \quad t \in [0, T], \quad (24)$$

is minimax w.r.t. the both local and integral criteria if

$$\dot{\hat{\gamma}}(t) = 2a(t)\hat{\gamma}(t) + b^2(t) - \left(\frac{(\hat{\gamma}(t)|A(t)| - \bar{r}|b(t)B(t)|)_+}{B(t)} \right)^2 \quad \text{for a.e. } t \in [0, T], \quad \hat{\gamma}(0) = \bar{D}. \quad (25)$$

2) *The maximal initial variance $\bar{D}$ and the noise intensity function $\hat{R}(t) = \begin{pmatrix} 1 & \hat{r}(t) \\ \hat{r}(t) & 1 \end{pmatrix}$ form a solution of the dual problem* (7) *with the both local and integral criteria iff*

$$\begin{cases} \hat{r}(t) = \mathrm{sat}_{\bar{r}}(-\hat{\gamma}(t)A(t)(b(t)B(t))^{-1}), & b(t) \neq 0, \\ \hat{r}(t) \in [-\bar{r}, \bar{r}], & b(t) = 0, \end{cases} \quad (26)$$

a.e. on $[0, T]$.

3) *In the least favorable case, the error variance of the minimax estimate satisfies the Riccati equation* (25)*, i.e.,*

$$\mathsf{D}\Big\{\hat{\xi}(t) - \xi(t) \mid \bar{D}, \hat{R}\Big\} = \hat{\gamma}(t).$$

Proof: In order to prove Theorem 3 it suffices to note that (26) provides a solution of (11). Actually, due to Lemma 1 $\psi(s) > 0$ for all s. Hence, (11) is equivalent to the minimization problem

$$\hat{r}(s) \in \arg\min_{r \in [-\bar{r}, \bar{r}]} (b(s) r B(s) + \hat{\gamma}(s) A(s))^2.$$

To obtain expressions (24) and (25) it remains to substitute 1 for R_μ, R_ν and (26) for $R_{\mu\nu}$, $R_{\nu\mu}$ in (10) and (9), respectively. ■

Remark 1: Note that (24) is a Kalman filter with coefficients defined by the least favorable initial variance $\bar{D}$ and cross-intensity (26). Furthermore, the minimax solution presented in Theorem 3 possesses the important feature: the least favorable characteristics $\bar{D}$ and $\hat{R}(\cdot)$ do not depend on the interval $[0, T]$. Hence, at every moment $t \ge 0$, we have

$$J_t(\hat{\xi}, D, R) \le J_t(\hat{\xi}, \bar{D}, \hat{R}) \le J_t(\tilde{\xi}, \bar{D}, \hat{R})$$

for any admissible estimates $\tilde{\xi}$ and parameters D and $R(\cdot)$. This makes it possible to say that the filter $\hat{\xi}$ is *totally minimax*.

VI. Numerical Results

Here we present the results of computer modeling for the scalar observation model described in the previous section.

Let system (1) be defined on the interval $[0, T]$, $T = 5$, with the following coefficients:

$$a(t) = -0.1,\ b(t) = 3,\ A(t) = \sin(t),\ B(t) = 3.$$

Suppose that the maximal initial variance is $\bar{D} = 1$ and the upper bound of the cross-intensity is $\bar{r} = 1$. Thus, correlation between the state and observation disturbances turns to be arbitrary.

Consider several variants of intensity function $r(\cdot)$ (Fig. 1):

$$r_0(t) = \hat{r}(t), \quad r_1(t) = 1, \quad r_2(t) = -1, \quad r_3(t) = 0, \quad r_4(t) = -\sin(t),$$

where the least favorable function $\hat{r}(\cdot)$ is defined by (26).

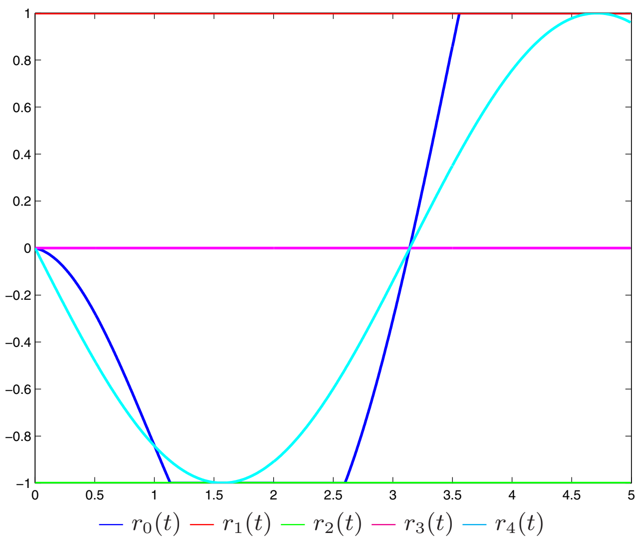

Fig. 1: Several examples of admissible cross-intensity functions.

Let

$$\widetilde{\xi}_i(t) = \widetilde{\xi}(t \mid r_i(\cdot))$$

denote the Kalman filter with coefficients defined by the ith intensity function, $i = 0, \ldots, 4$. Under the assumption $r \equiv r_i$, denote the error variance of the estimate $\widetilde{\xi}_i(t)$ by

$$\gamma_i(t) = \mathsf{D}\left\{\xi(t) - \widetilde{\xi}_i(t) \mid r_i(\cdot)\right\}.$$

Then, $\widetilde{\xi}_0(t) = \hat{\xi}(t)$ is the minimax filter and $\gamma_0(t) = \hat{\gamma}(t)$ is its error variance in the least favorable case.

The next notation is used to present the results of numerical computation for the error variance of the ith Kalman filter if the noise cross-intensity function coincides with $r_j(\cdot)$:

$$D_{ij}(t) = \mathsf{D}\left\{\xi(t) - \widetilde{\xi}_i(t) \mid r_j(\cdot)\right\}, \quad i, j = 0, \ldots, 4.$$

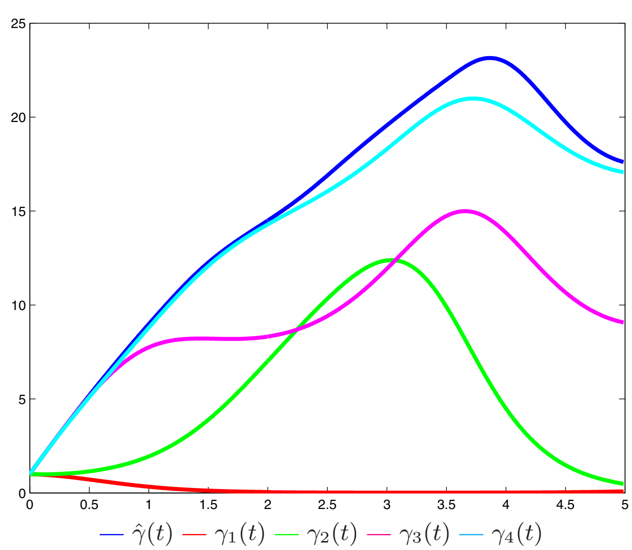

Fig. 2: The error variance of optimal filters for every case of the functions $r_i(\cdot)$.

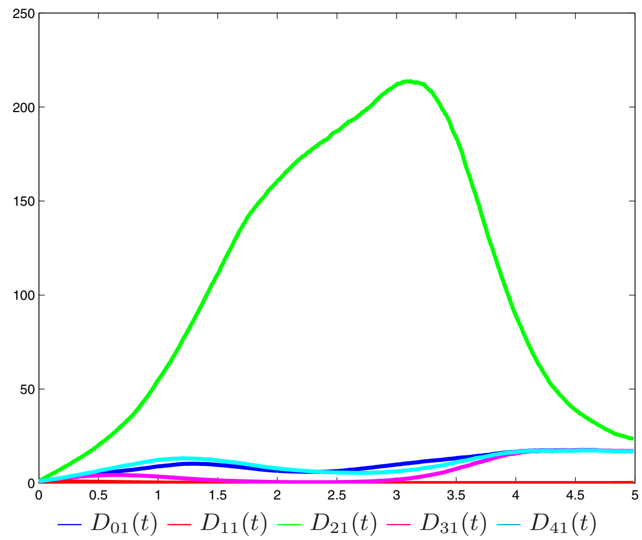

Fig. 3: The error variance for several Kalman filters given the fixed intensity function $r_1(\cdot) \equiv 1$.

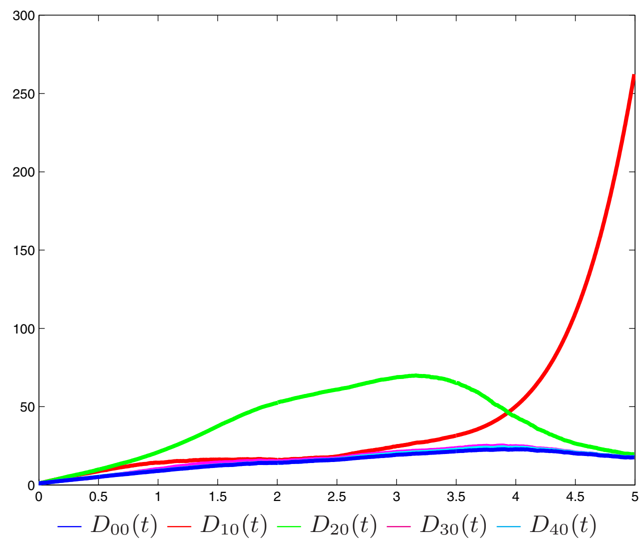

Fig. 4: The error variance for several Kalman filters given the least favorable intensity function $r_0(\cdot) = \hat{r}(\cdot)$.

The table presented below provides an illustration for the following saddle-point inequalities:

$$D_{0j}(T) \leq D_{00}(T) \leq D_{i0}(T) \quad \forall\, i, j.$$

Table 1: The terminal error variance given all possible combinations of the filters and intensity functions under consideration.

$D_{ij}(T)$	0	4	3	2	1
0	17.88	17.44	9.64	1.92	17.12
4	17.99	17.30	9.54	1.72	16.99
3	18.00	17.46	9.33	1.25	17.24
2	19.71	18.74	12.32	0.49	23.55
1	268	280	287	599	0.08

Appendix

Lemma 1: Let $c\colon [0,T] \to \mathbb{R}^{m\times m}$ *be a bounded measurable function.*

1) *If* $\psi\colon [0,T] \to \mathbb{R}^{m\times m}$ *is a solution to one of the following Cauchy problems:*

$$\dot\psi(t) = \psi(t)c(t) + c^\top(t)\psi(t), \quad \psi(0) = I, \tag{27}$$

or

$$\dot\psi(t) = \psi(t)c(t) + c^\top(t)\psi(t) - I, \quad \psi(T) = O, \tag{28}$$

then $\psi(t) \geq O$ *for all* $t \in [0,T]$.

2) *In the case* $m = 1$*, the solutions of* (27) *and* (28) *are positive functions:* $\psi(t) > 0$ *for all* $t \in [0,T)$.

Proof: 1) It is sufficient to note that the functions:

$$\psi(t) = \Phi^\top(t,0)\Phi(t,0) \tag{29}$$

and

$$\psi(t) = \int_t^T \Phi^\top(t,s)\Phi(t,s)\,ds \tag{30}$$

are the solutions of (27) and (28), resp., if

$$\begin{cases} \dfrac{d\Phi(t,s)}{dt} = \Phi(t,s)c(t), & \text{for a.e. } t \geq s, \\ \Phi(s,s) = I \quad \forall\, s. \end{cases}$$

Now it is clear that (29) and (30) are positive-semidefinite matrices at each moment $t \geq 0$.

2) If $m = 1$, then (29) and (30) take the form: $\psi(t) = \exp\bigl\{2\int_0^t c(\tau)\,d\tau\bigr\}$, $\psi(t) = \int_t^T \exp\bigl\{2\int_s^t c(\tau)\,d\tau\bigr\}\,ds$. ∎

Lemma 2: Let $W = \begin{pmatrix} W_X & W_{XY} \\ W_{YX} & W_Y \end{pmatrix}$ *be a symmetric matrix such that* $W_X \in \mathbb{R}^{m\times m}$*,* $W_Y \in \mathbb{R}^{n\times n}$*, and* $W_Y > O$*.*

Then, for any positive-semidefinite $m \times m$ *matrix* ψ

$$\min_{F\in\mathbb{R}^{m\times n}} \operatorname{tr}\bigl[\psi[-I\;F]W[-I\;F]^\top\bigr] = \operatorname{tr}\bigl[\psi(W_X - W_{XY}W_Y^{-1}W_{YX})\bigr]\,.$$

Proof: Let us check that the following inequality

$$[-I\;F]W[-I\;F]^\top \geq W_X - W_{XY}W_Y^{-1}W_{YX} \tag{31}$$

holds for any matrix F and turns to be an equality for a certain one.

The difference between the left and right sides of (31) can be represented in the form

$$\Delta = FW_YF^\top - FW_{YX} - W_{XY}F^\top + W_{XY}W_Y^{-1}W_{YX} = = (FW_Y - W_{XY})W_Y^{-1}(FW_Y - W_{XY})^\top.$$

Now we can see that $\Delta \geq O$ for any F and $\Delta = O$ for $F = W_{XY}W_Y^{-1}$. ∎

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