Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On the Driven Inverted Pendulum

John Hauser, Alessandro Saccon, and Ruggero Frezza

***Abstract*— We explore the solutions of the driven inverted pendulum system $l\ddot{\varphi} = g \sin\varphi - a_l(t)\cos\varphi$ where $a_l(\cdot)$ is a bounded lateral acceleration. We show that, for lateral accelerations that are constant before some initial time, an inverted trajectory always exists and remains within a diamond shaped region in the state space.**

Functional analytic techniques are also developed to provide further insight into the nature of the inverted pendulum trajectories. Associated to the driven inverted pendulum is a time varying linear system. We show that this system *always* possesses an exponential dichotomy, allowing for the development of a successive approximation algorithm for finding the desired inverted pendulum trajectory. We show that the curve obtained from one iteration of this algorithm is a very good estimate of the required inverted trajectory. As that curve is obtained by filtering the quasi-static angle trajectory by a noncausal time varying low pass filter with weighting function with a shape similar to $h(t) = \exp -\alpha_0|t|$, we find that the current pendulum angle is influenced by the values of the lateral acceleration within only a few seconds of the current time.

These results are important as the driven inverted pendulum is a common susbsystem in systems ranging from motorcycles and bicycles to rockets and aircaft.

I. Introduction

The inverted pendulum is the classic nonlinear control experiment. In addition to being the simplest unstable nonlinear system imaginable, it also provides a simple model for rocket control. Surprisingly, this dynamics continues to appear in numerous other systems of interest. The inverted pendulum driven by a lateral acceleration is clearly present in the dynamic balance of a skier racing down the slope. In much the same way, simple models for exploring bicycle and motorcycle dynamics include the inverted pendulum as a key subsystem, imposing strong constraints on the system performance—see, e.g., [1], [2], [3], and [4]. These dynamics also show up as internal dynamics in aircraft flight dynamics [5].

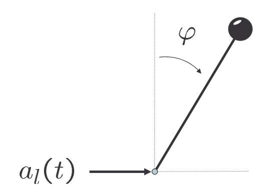
In this paper, we are interested in the following question. Given an arbitrary bounded lateral acceleration trajectory, does there exist a pendulum that remains upright (or inverted) for all time, forward and reverse. It turns out that, for lateral accelerations that are constant before a given initial time, such an upright trajectory always exists. We explore numerous aspects of such upright pendulum trajecotries using both state space and functional analysis techniques.

J. Hauser is with the Department of Electrical and Computer Engineering, University of Colorado, Boulder, CO 80309-0425, USA, hauser@colorado.edu

A. Saccon and R. Frezza are with the Dipartimento di Ingegneria dell'Informazione, Università di Padova, 35131 Padova, Italy, {asaccon,frezza}@dei.unipd.it

One important result is that the current pendulum angle depends only on the lateral accelerations within a few seconds of the current time. This has important implications in, for example, the planning of aggressive motorcycle trajectories.

II. Problem Statement



In this paper, we are interested in the trajectories of an inverted pendulum

$$l\ddot{\varphi} = g \sin\varphi - a_l(t)\cos\varphi \quad (1)$$

driven by a lateral acceleration $a_l(\cdot)$. In particular, given an arbitrary bounded lateral acceleration $a_l(\cdot)$ (with $|a_l(t)| \leq a_L$, $t \in \mathbb{R}$), does there exist a pendulum trajectory that is upright on the infinite interval, i.e., a trajectory $\varphi(\cdot)$ of (1) with $|\varphi(t)| < \phi/2$, $t \in \mathbb{R}$? Is there at most one upright pendulum trajectory (of infinite extent) consistent with a given lateral acceleration trajectory?

III. Geometric Considerations

It is useful to consider the behavior of the driven inverted pendulum in the phase plane. We begin with a necessary condition concerning the region in which an upright pendulum trajectory must lie.

For constant maximum lateral acceleration $a_l(t) \equiv a_L$, the driven inverted pendulum dynamics becomes that of a simple inverted pendulum with conserved (scaled) energy

$$V(\varphi,\dot{\varphi}) = g\cos\varphi + a_L \sin\varphi + l\dot{\varphi}^2/2 = \bar{a}\cos(\varphi-\varphi_L) + l\dot{\varphi}^2/2$$

where $\bar{a} = \sqrt{a_L^2 + g^2}$ and $\tan\varphi_L = a_L/g$. The unstable equilibrium occurs at the static pendulum angle φ_L with energy $V(\varphi_L, 0) = \bar{a}$. Moreover, the stable and unstable manifolds to that equilibrium are given by the level set $V(\varphi,\dot{\varphi}) = V(\varphi_L, 0)$. The picture is symmetric (with appropriate changes) at constant minimum acceleration $a_l(t) \equiv -a_L$. These features combine to determine the diamond depicted in figure 1.

Proposition 1: Suppose that $\varphi(\cdot)$ is an upright pendulum trajectory ($|\varphi(t)| < \pi/2$, $t \in \mathbb{R}$) satisfying (1) with $|a_l(t)| \leq a_L$, $t \in \mathbb{R}$. Then the phase trajectory $(\varphi(\cdot), \dot{\varphi}(\cdot))$ is contained in the diamond $\mathcal{D}$ given by

$$\mathcal{D} = \{(\varphi,\dot{\varphi}) : |\varphi| \leq \varphi_L, \dot{\varphi}^2 \leq 2\bar{a}(1 - \cos(|\varphi| - \varphi_L))/l\}\ .$$

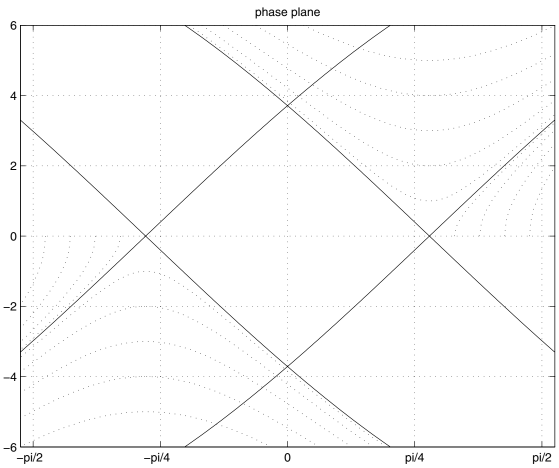


Fig. 1. The diamond $\mathcal{D}$

Proof: The key is to show that even the maximum lateral acceleration (of appropriate sign) is not able to keep the pendulum from falling over (in forward or reverse time) when starting (at some t_0) outside the diamond. Referring to figure 1, we consider the upper right region of initial conditions that must fall over to the right ($+\varphi$) in forward time ($+t$). The other three cases, ($-\varphi, +t$), ($+\varphi, -t$), ($-\varphi, -t$), are similar.

Suppose that, at time t_0, $(\varphi(t_0), \dot{\varphi}(t_0))$ with $|\varphi(t_0)| < \pi/2$ and $\dot{\varphi}(t_0) > 0$ is such that $\varphi(t_0) > \varphi_L$ or $\dot{\varphi}^2(t_0) > 2\bar{a}(1 - \cos(\varphi(t_0) - \varphi_L))/l$ (this is the upper right region). If $a_l(t) = a_L$ for $t \geq t_0$, the energy V is conserved so that the pendulum falls right following the contour $V(\varphi, \dot{\varphi}) = V(\varphi(t_0), \dot{\varphi}(t_0))$, first reaching $\varphi(t) = \pi/2$ at a specific time $t = t_L$. For $a_l(t) \leq a_L$, $t \geq t_0$, the energy V is nondecreasing since $\dot{V}(t, \varphi, \dot{\varphi}) = (a_L - a_l(t))\dot{\varphi}\cos\varphi \geq 0$ in the upper right region so that the pendulum motion follows a curve on or above the contour $V(\varphi, \dot{\varphi}) = V(\varphi(t_0), \dot{\varphi}(t_0))$. It follows that the pendulum falls right to $\varphi(t) = \pi/2$ for the first time at some $t = t_1 \leq t_L$ since the velocity $\dot{\varphi}$ is always at or above that of the pendulum under maximum acceleration. ■

In the case that the lateral acceleration is constant up to some time t_0, we can conclude that there is an upright pendulum trajectory.

Theorem 2: Suppose that $|a_l(t)| \leq a_L$, $t \in \mathbb{R}$, and that there is a $t_0 \in \mathbb{R}$ such that $a_l(t) = a_{l0}$ for $t \leq t_0$. Then there is a pendulum trajectory $\varphi(\cdot)$ of (1) such that $|\varphi(t)| < \pi/2$ for all $t \in \mathbb{R}$.

Proof: The set of initial conditions $(\varphi(t_0), \dot{\varphi}(t_0))$ that remain upright in reverse time belong to the upright component of the unstable manifold to the constant acceleration equilibrium $(\varphi_{l0}, 0) = (\tan^{-1} a_{l0}/g, 0)$. Explicitly, $W^u_{a_{l0}}$ is given by

$$W^u_{a_{l0}} = \{(\varphi, \dot{\varphi}) : |\varphi| < \pi/2, \dot{\varphi} = w_{a_{l0}}(\varphi)\}$$

where

$$w_{a_{l0}}(\varphi) = \mathrm{sign}(\varphi - \varphi_{l0})\sqrt{2\sqrt{a_{l0}^2 + g^2}(1 - \cos(\varphi - \varphi_{l0}))/l}\ .$$

As demonstrated in Proposition 1, $W^u_{a_{l0}}$ contains initial conditions W_R that fall right (e.g., in the upper right region) and initial conditions W_L that fall left. As the connected set $W^u_{a_{l0}}$ cannot be the union of the two disjoint nonempty open sets W_R and W_L, the set of initial conditions leading to upright trajectories

$$W_U = W^u_{a_{l0}} \setminus (W_R \cup W_L)$$

must be nonempty.

For the sake of completeness, we demonstrate that W_R (and hence W_L) is open. Let $(\varphi(t), \dot{\varphi}(t)) = \phi(t; t_0, \varphi_0, \dot{\varphi}_0)$ denote the time varying flow of the driven inverted pendulum starting from an initial condition of $(\varphi_0, \dot{\varphi}_0)$ at time t_0. Suppose that $(\varphi_0, \dot{\varphi}_0) \in W_R$ and let t_1 be the first time that the pendulum reaches $\pi/2$ at $(\varphi_1, \dot{\varphi}_1) = \phi(t_1; t_0, \varphi_0, \dot{\varphi}_0)$, noting that $\dot{\varphi}_1 > 0$. By continuity of the map $(\varphi, \dot{\varphi}) \mapsto \phi(t_1; t_0, \varphi, \dot{\varphi})$, initial conditions near $(\varphi_0, \dot{\varphi}_0)$ on $W^u_{a_{l0}}$ will be carried to a neighborhood of $(\varphi_1, \dot{\varphi}_1)$ at time t_1. Thus, there is an $\epsilon > 0$ such that for all $|\varphi - \varphi_0| < \epsilon$, $\dot{\varphi} = w_{a_{l0}}(\varphi)$, the state $\phi(t_1; t_0, \varphi, \dot{\varphi})$ at time t_1 is either in the upper right region (and soon to fall through $\pi/2$) or such that $\phi_1(t_1; t_0, \varphi, \dot{\varphi}) \geq \pi/2$. ■

IV. Functional Analytic Considerations

Having concluded that, under rather mild conditions, there the driven inverted pendulum (1) possesses an upright trajectory, we now seek to understand the characteristics of such a trajectory. The following results do not necessarily require that the lateral acceleration be constant before a given time. We remark that such a restriction is very minor from a practical point of view.

At this point it is convenient to write the driven inverted pendulum dynamics (1)

$$l\ddot{\varphi} = g\sin\varphi - a_l(t)\cos\varphi$$

in the form

$$\ddot{\varphi} = a(t)/l\ \sin(\varphi - \varphi_{qs}(t)) \tag{2}$$

where $a(t)$ is the total acceleration $a(t) = \sqrt{g^2 + a_l^2(t)}$ and the *quasi-static* pendulum angle $\varphi_{qs}(t)$ satisfies

$$\tan\varphi_{qs}(t) = a_l(t)/g\ .$$

The dynamics (2) may be viewed as that of an inverted pendulum for which the acceleration analogous to gravity varies in strength and direction.

Defining the variable natural frequency $\alpha(\cdot)$ by $\alpha^2(t) = a(t)/l$, we can write the dynamics as

$$\ddot{\varphi} = \alpha^2(t)\ \sin(\varphi - \varphi_{qs}(t))$$

Motivated by the work of Devasia and Paden [6] on nonlinear system inversion, we rewrite the dynamics as

$$\ddot{\varphi} = \alpha^2(t)\varphi - \alpha^2(t)(\varphi - \sin(\varphi - \varphi_{qs}(t)))$$

and consider this from a functional analytic point of view.

A. Dichotomy

In this section, we study the unstable linear system

$$\ddot{\gamma} = \alpha^2(t)\,\gamma - \alpha^2(t)\,\mu(t) \tag{3}$$

where the variable natural frequency $\alpha(\cdot)$ is bounded above and below with $\alpha_0 \le \alpha(t) \le \alpha_1$, $t \in \mathbb{R}$, for some constants $\alpha_1 \ge \alpha_0 > 0$. We will show that, despite its unstable and time varying nature, the system (3) defines a unique linear operator $\mathcal{A}$ mapping bounded $\mu(\cdot)$ into bounded $\gamma(\cdot)$ (on the infinite interval $(-\infty, \infty)$). Surprisingly, this turns out to be the case regardless of how rapidly $\alpha(\cdot)$ varies—indeed, $\alpha(\cdot)$ need only be measurable.

Consider first the undriven system

$$\ddot{\gamma} = \alpha^2(t)\,\gamma \tag{4}$$

For constant α, this system can be decoupled into exponentially stable and unstable subsystems corresponding to the eigenvalues $\pm\alpha$. The time varying linear system has much the same structure.

Choosing as state $(x_1, x_2) = (\gamma, \dot{\gamma})$, the dynamics (4) are represented by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha^2(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A(t)\,x\,. \tag{5}$$

Suppose, for the moment, that $c(\cdot)$ and $d(\cdot)$ are positive bounded C^1 curves, bounded away from zero, and consider the time varying coordinate transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -c(t) & d(t) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = S(t)z\,.$$

In z coordinates, we have

$$\dot{z} = S(t)^{-1}[A(t)S(t) - \dot{S}(t)]z = \tilde{A}(t)z$$

where the transformed system matrix is given by

$$\tilde{A}(t) = \frac{1}{c+d} \begin{bmatrix} -(cd + \alpha^2 + \dot{c}) & \dot{d} + d^2 - \alpha^2 \\ \dot{c} - c^2 + \alpha^2 & cd + \alpha^2 - \dot{d} \end{bmatrix}.$$

Choosing $c(\cdot)$ and $d(\cdot)$ to satisfy the Riccati equations

$$\begin{aligned} \dot{c} - c^2 + \alpha^2(t) &= 0 && (6) \\ \dot{d} + d^2 - \alpha^2(t) &= 0 && (7) \end{aligned}$$

we see that the transformed system matrix reduces to

$$\tilde{A}(t) = \begin{bmatrix} -c(t) & 0 \\ 0 & d(t) \end{bmatrix}.$$

Lemma 3: The Riccati equations (6) and (7) each possess a positive bounded solution satisfying

$$\alpha_0 \le c(t), d(t) \le \alpha_1$$

for all $t \in \mathbb{R}$.

Proof: Note that $c(t)$ is the *value* of the scalar optimal control problem

$$\begin{aligned} c(t) \;=\; \min \; & \int_t^\infty \alpha^2(\tau)x^2(\tau) + u^2(\tau)\,d\tau \\ & \dot{x} = u, \qquad x(t) = 1 \end{aligned}$$

and that this value is bounded above and below by the values of the stationary optimal control problems obtained by replacing $\alpha(\cdot)$ in the incremental cost by α_1 and α_0, respectively. The positive values of the stationary optimal control problems are easily seen to be precisely α_1 and α_0. The result for $d(\cdot)$ follows easily by reversing time. ■

The required Riccati equation solution, $c(\cdot)$ for example, is the limit, as T goes to ∞, of trajectories obtained by solving (6) backwards from $c(T) = 0$, see [7, chapter 3] for more details.

Note that the absolutely continuous $c(\cdot)$ and $d(\cdot)$ are nonlinearly filtered versions of $\alpha(\cdot)$—one that anticipates and one that lags. We see that

Proposition 4: The linear system (5) with $\alpha(\cdot)$ as above is reducible to the decoupled linear system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c(t) & 0 \\ 0 & d(t) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{8}$$

Moreover, the z_1 and z_2 subsystems are exponentially stable and unstable, respectively, satisfying

$$\begin{aligned} |z_1(t)| &\le |z_1(t_0)|e^{-\alpha_0(t-t_0)}, && t \ge t_0 \\ |z_2(t)| &\le |z_2(t_0)|e^{+\alpha_0(t-t_0)}, && t \le t_0 \end{aligned}$$

so that (5) admits an exponential dichotomy.

Proof: To conclude reducibility [8], we need to show that $S(\cdot)$ defined above is bounded with bounded inverse. Reducibility is clear as the elements of $S(\cdot)$ are bounded by α_1 while the elements of $S^{-1}(\cdot)$ are bounded by $\alpha_1/2\alpha_0$.

The state transition function for the z_1 subsystem satisfies

$$|\Phi_{11}(t,\tau)| = |\exp\{-\int_\tau^t c(s)ds\}| \le e^{-\alpha_0(t-\tau)}$$

since $c(t) \ge \alpha_0$ for all t. The z_2 property is similarly derived. ■

A sufficient condition [8], [9] for the existence of an exponential dichotomy is that the eigenvalues of the frozen systems $A(t)$ are uniformly bounded away from the $j\omega$ axis and that $A(t)$ be *sufficiently slowly varying*. The above result is somewhat surprising (and much stronger) in that there is *no* requirement that $\alpha(t)$ be slowly varying.

Applying the dichotomy transformation to the driven system (3), we see that the relationship between $\mu(\cdot)$ and $\gamma(\cdot)$ can be expressed as

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} -c(t) & 0 \\ 0 & d(t) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \frac{\alpha^2(t)}{c(t)+d(t)} \\ \frac{-\alpha^2(t)}{c(t)+d(t)} \end{bmatrix} \mu(t) \\ \gamma &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned} \tag{9}$$

Working in a noncausal fashion over the infinite interval, we find that

$$\begin{aligned}\gamma(t) = & \int_{-\infty}^{t} \exp\left\{-\int_{\tau}^{t} c(s)ds\right\} \frac{\alpha^2(\tau)}{c(\tau)+d(\tau)}\mu(\tau)d\tau \\ & + \int_{t}^{\infty} \exp\left\{-\int_{t}^{\tau} d(s)ds\right\} \frac{\alpha^2(\tau)}{c(\tau)+d(\tau)}\mu(\tau)d\tau \\ =: & \int_{-\infty}^{\infty} h(t,\tau)\ \mu(\tau)d\tau \end{aligned} \tag{10}$$

is a bounded solution of (3) for each bounded $\mu(\cdot)$. Denoting the map $\mu(\cdot) \mapsto \gamma(\cdot)$ in (10) by $\mathcal{A}$, we see that

Theorem 5: $\mathcal{A}$ is a bounded linear operator with

$$\|\mathcal{A}\| = 1 .$$

For each bounded input $\mu(\cdot)$, the output $\gamma(\cdot) = \mathcal{A}[\mu(\cdot)]$ is the *unique* bounded solution of (3).

Proof: Uniqueness follows from the fact that (9) has no bounded nontrivial homogeneous solution [9]. Now, by (3), we see that $\gamma(t) \equiv 1$ is the bounded response to the bounded input $\mu(t) \equiv 1$. Since the time-varying impulse response $h(t,\tau)$ is positive for all t,τ, the norm of $\mathcal{A}$ is given by $\|\mathcal{A}\| = \sup_t \mathcal{A}[1](t) = 1$. ∎

Note that when α is constant, $\mathcal{A}$ is the linear time-invariant system with impulse response $h(t) = \alpha/2\ e^{-\alpha|t|}$ and frequency response $\hat{h}(\omega) = \alpha^2/(\alpha^2+\omega^2)$: it is a noncausal low pass filter with no phase shift. It is interesting that the time varying system shares the property that the maximum amplification (of 1!) is achieved at DC. Furthermore, as we shall see, $\mathcal{A}$ for time varying $\alpha(\cdot)$ also low pass filters its input in a manner that we can use to our advantage.

B. Contraction and Approximation

We are interested in showing that, given a smooth bounded lateral acceleration trajectory $a_l(\cdot)$ as described above, there is a bounded pendulum trajectory $\varphi(\cdot)$ satisfying (1). Equivalently, we seek a bounded differential pendulum trajectory $\theta(\cdot)$ satisfying

$$\ddot{\theta} = a(t)/h\ \sin\theta - \ddot{\varphi}_{qs}(t) . \tag{11}$$

To this end, we consider the use of a contraction as in the work of Devasia and Paden [6] on nonlinear system inversion.

To this end, choosing $\alpha^2(t) = a(t)/h$, the differential pendulum dynamics (11) is equal to

$$\begin{aligned}\ddot{\theta} &= \alpha^2(t)\sin\theta - \ddot{\varphi}_{qs}(t) \\ &= \alpha^2(t)\theta - \alpha^2(t)\left\{\theta - \sin\theta + \ddot{\varphi}_{qs}(t)/\alpha^2(t)\right\} .\end{aligned}$$

Defining the nonlinear operator $\mathcal{N}$

$$\theta(\cdot) \mapsto \mathcal{A}[\theta(\cdot) - \sin\theta(\cdot) + \ddot{\varphi}_{qs}(\cdot)/\alpha^2(\cdot)] =: \mathcal{N}[\theta(\cdot)] ,$$

we see that

Proposition 6: A bounded curve $\theta(\cdot)$ is a solution of (11) if and only if it is a fixed point of $\mathcal{N}$, i.e.,

$$\theta(\cdot) = \mathcal{N}[\theta(\cdot)] .$$

Thus, we seek a fixed point of the nonlinear operator $\mathcal{N}$. The key is to find an invariant neighborhood of the form $\bar{B}_\delta = \{\theta(\cdot) : \|\theta(\cdot)\| \le \delta\}$ on which $\mathcal{N}$ is a contraction. The following identity is interesting and useful.

Lemma 7: $\mathcal{A}[\ddot{\varphi}(\cdot)/\alpha^2(\cdot)] = \mathcal{A}[\varphi(\cdot)] - \varphi(\cdot)$ for all bounded $\varphi(\cdot)$.

Proof: Let $\gamma_0(\cdot) = \mathcal{A}[\varphi(\cdot)]$ and $\gamma_2(\cdot) = \mathcal{A}[\ddot{\varphi}(\cdot)/\alpha^2(\cdot)]$ and note that

$$\ddot{\gamma}_2(t) - \ddot{\gamma}_0(t) = \alpha^2(t)(\gamma_2(t) - \gamma_0(t)) - \ddot{\varphi}(t) + \alpha^2(t)\varphi(t) .$$

Rearranging, we have

$$\ddot{\gamma}_2(t) - \ddot{\gamma}_0(t) + \ddot{\varphi}(t) = \alpha^2(t)(\gamma_2(t) - \gamma_0(t) + \varphi(t))$$

which, by uniqueness, implies that $\gamma_2(t) = \gamma_0(t) - \varphi(t)$. ∎

Define the curve

$$\eta(\cdot) := \mathcal{A}[\varphi_{qs}(\cdot)] - \varphi_{qs}(\cdot) = \mathcal{A}[\ddot{\varphi}_{qs}(\cdot)/\alpha^2(\cdot)]$$

with size $\bar{\eta} = \|\eta(\cdot)\|$.

Theorem 8: If

$$\bar{\eta} = \|\mathcal{A}[\ddot{\varphi}_{qs}(\cdot)/\alpha^2(\cdot)]\| < 1 \tag{12}$$

then there is a $\delta < \pi/2$ such that $\mathcal{N}$ is a contraction on the invariant set $\bar{B}_\delta$. The unique fixed point $\theta(\cdot)$ of $\mathcal{N}$ in $\bar{B}_\delta$ is a bounded trajectory of (11) so that $\varphi(\cdot) = \varphi_{qs}(\cdot) + \theta(\cdot)$ is a bounded trajectory of (1). Furthermore, if

$$\sin^{-1}\bar{\eta} - \bar{\eta} < \pi/2 - \|\mathcal{A}[\varphi_{qs}(\cdot)]\| \tag{13}$$

then the trajectory is upright, $\|\varphi(\cdot)\| < \pi/2$. In particular, if $\|\varphi_{qs}(\cdot)\| \le 1$ (or $\|\mathcal{A}[\varphi_{qs}(\cdot)]\| \le 1$) with $\bar{\eta} < 1$, the trajectory $\varphi(\cdot)$ is upright.

Proof: Recall that $f(\delta) := \delta - \sin\delta$ and $f'(\cdot)$ are monotonically increasing on $[0,\pi]$ so that $f(\cdot)$ is Lipschitz continuous on $[0,\delta_0] \subset [0,\pi]$ with Lipschitz constant $1 - \cos\delta_0$, i.e., $|f(\delta_1) - f(\delta_2)| \le (1-\cos\delta_0)|\delta_1 - \delta_2|$ for all $\delta_1, \delta_2 \in [0,\delta_0] \subset [0,\pi]$.

The set $\bar{B}_\delta$ is invariant under $\mathcal{N}$ if

$$\|\mathcal{N}[\theta(\cdot)]\| \le \|\mathcal{A}\|\ \|\theta(\cdot) - \sin\theta(\cdot)\| + \bar{\eta} \le \delta$$

for all $\theta(\cdot) \in \bar{B}_\delta$. Using $\|\mathcal{A}\| = 1$ and the monotonicity of $f(\cdot)$, we see that $\bar{B}_\delta$, $\delta \in [0,\pi]$, is invariant under $\mathcal{N}$ if

$$\bar{\eta} \le \sin\delta \le 1 . \tag{14}$$

Choosing $\delta < \pi/2$ with $\bar{\eta} \le \sin\delta$, we see that

$$\|\mathcal{N}[\theta_1(\cdot)] - \mathcal{N}[\theta_2(\cdot)]\| \le \rho\|\theta_1(\cdot) - \theta_2(\cdot)\|$$

with $\rho := 1 - \cos\delta < 1$ so that $\mathcal{N}$ is a contraction on the invariant set $\bar{B}_\delta$. In particular, the minimal set is obtained using $\delta = \sin^{-1}\bar{\eta}$ so that $\rho = 1 - \sqrt{1-\bar{\eta}^2} < 1$.

Now, the fixed point trajectory $\theta(\cdot)$ satisfies

$$\begin{aligned}\theta(\cdot) &= \mathcal{A}[\theta(\cdot) - \sin\theta(\cdot) + \ddot{\varphi}_{qs}(\cdot)/\alpha^2(\cdot)] \\ &= \mathcal{A}[\theta(\cdot) - \sin\theta(\cdot)] + \mathcal{A}[\varphi_{qs}(\cdot)] - \varphi_{qs}(\cdot)\end{aligned}$$

so that the corresponding *pendulum* trajectory $\varphi(\cdot)$ satisfies

$$\|\varphi(\cdot) - \mathcal{A}[\varphi_{qs}(\cdot)]\| = \|\mathcal{A}[\theta(\cdot) - \sin\theta(\cdot)]\| \le \delta - \sin\delta \tag{15}$$

and

$$\|\varphi(\cdot)\| \le \delta - \sin\delta + \|\mathcal{A}[\varphi_{qs}(\cdot)]\| \ .$$

The result follows with $\delta = \sin^{-1}\bar{\eta}$. ∎

The fixed point differential pendulum trajectory may be computed by the successive approximation method given by

$$\begin{aligned} \theta_0(\cdot) &= 0 \\ \theta_{k+1}(\cdot) &= \mathcal{N}[\theta_k(\cdot)] \\ &= \mathcal{A}[\theta_k(\cdot) - \sin\theta_k(\cdot)] + \mathcal{A}[\varphi_{qs}(\cdot)] - \varphi_{qs}(\cdot) \end{aligned}$$

This algorithm generates a sequence of approximate pendulum trajectories $\{\varphi_k(\cdot)\}$ starting with $\varphi_0(\cdot) = \varphi_{qs}(\cdot)$ and $\varphi_1(\cdot) = \mathcal{A}[\varphi_{qs}(\cdot)]$.

Theorem 8 tells us that if $\bar{\eta} = \|\varphi_1(\cdot) - \varphi_0(\cdot)\| < 1$ and $\|\varphi_1(\cdot)\| < 1$ (or the less conservative condition (13)), then the above algorithm will produce, in the limit, an upright pendulum trajectory.

However the conditions of Theorem 8 are still somewhat conservative. There are many lateral acceleration profiles that violate these conditions and yet the above algorithm seems to converge to a pendulum trajectory solution. Much conservatism arises from neglecting the use of the operator $\mathcal{A}$, which acts largely as a noncausal low pass filter.

Condition (15) and *experience* suggest that the solution trajectory $\varphi(\cdot)$ will be close to $\mathcal{A}[\varphi_{qs}(\cdot)]$ rather than close to $\varphi_{qs}(\cdot)$ as was used in the above Theorem. Working around $\mathcal{A}[\varphi_{qs}(\cdot)]$, write

$$\zeta(\cdot) = \varphi(\cdot) - \mathcal{A}[\varphi_{qs}(\cdot)]$$

so that the nonlinear operator $\mathcal{N}$ becomes

$$\begin{aligned} &\bar{\mathcal{N}}[\zeta(\cdot)] \\ &\quad = \mathcal{A}[\ \ \mathcal{A}[\varphi_{qs}(\cdot)] + \zeta(\cdot) - \varphi_{qs}(\cdot) \\ &\qquad\qquad - \ \sin(\mathcal{A}[\varphi_{qs}(\cdot)] + \zeta(\cdot) - \varphi_{qs}(\cdot))\ \ ] \\ &\quad = \mathcal{A}[\eta(\cdot) + \zeta(\cdot) - \sin(\eta(\cdot) + \zeta(\cdot))] \\ &\quad = \mathcal{A}[\eta(\cdot) - \sin\eta(\cdot)] \\ &\qquad\qquad + \ \mathcal{A}[\zeta(\cdot) - \{\sin(\eta(\cdot) + \zeta(\cdot)) - \sin\eta(\cdot)\}] \end{aligned}$$

where, as above, $\eta(\cdot) = \mathcal{A}[\varphi_{qs}(\cdot)] - \varphi_{qs}(\cdot)$. Defining

$$\epsilon = \|\mathcal{A}[\eta(\cdot) - \sin\eta(\cdot)]\|$$

we see that

$$\|\bar{\mathcal{N}}[\zeta(\cdot)]\| \le \epsilon + \|\zeta(\cdot) - \{\sin(\eta(\cdot) + \zeta(\cdot)) - \sin\eta(\cdot)\}\| \ . \quad (16)$$

Theorem 9: If $\bar{\eta} < \pi/2$ and

$$\epsilon + \sin\bar{\eta} < 1 \quad (17)$$

then there is a $\delta > 0$ such that $\bar{\mathcal{N}}$ is a contraction on the invariant neighborhood $\bar{B}_\delta$ so that the pendulum trajectory $\varphi(\cdot)$ corresponding to the unique fixed point $\zeta(\cdot)$ satisfies

$$\|\varphi(\cdot) - \mathcal{A}[\varphi_{qs}(\cdot)]\| < \delta \ .$$

Proof: The last expression in (16) satisfies

$$\begin{aligned} &\|\zeta(\cdot) - \cos\eta(\cdot)\sin\zeta(\cdot) + \sin\eta(\cdot)(1 - \cos\zeta(\cdot))\| \\ &\quad \le \ \|\zeta(\cdot) - \cos\eta(\cdot)\sin\zeta(\cdot)\| + \|\sin\eta(\cdot)(1 - \cos\zeta(\cdot))\| \\ &\quad \le \ \delta - \cos\bar{\eta}\sin\delta + \sin\bar{\eta}(1 - \cos\delta) \end{aligned}$$

for $\zeta(\cdot) \in \bar{B}_\delta$, $\delta \in (0, \pi/2)$. Thus, $\bar{B}_\delta$ is invariant under $\bar{\mathcal{N}}$ if

$$\epsilon + \delta - \cos\bar{\eta}\sin\delta + \sin\bar{\eta}(1 - \cos\delta) \le \delta$$

or, equivalently,

$$\epsilon - \cos\bar{\eta}\sin\delta + \sin\bar{\eta}(1 - \cos\delta) \le 0 \ . \quad (18)$$

Minimizing over δ, we see that

$$-1 + \sin\bar{\eta} \le -\cos\bar{\eta}\sin\delta + \sin\bar{\eta}(1 - \cos\delta)$$

with the minimum occuring at $\delta^* = \tan^{-1}(1/\tan\bar{\eta}) = \pi/2 - \bar{\eta}$. When (17) is satisfied, each neighborhood $\bar{B}_\delta$, $\delta \in [\bar{\delta}, \delta^*]$, where the minimal invariant neighborhood is given by the positive $\bar{\delta} < \delta^*$ making (18) an equality, namely,

$$\bar{\delta} = \sin^{-1}(\sin\bar{\eta} + \epsilon) - \bar{\eta} \approx \epsilon/\cos\bar{\eta} \ .$$

Now

$$\begin{aligned} &\|\bar{\mathcal{N}}[\zeta_1(\cdot)] - \bar{\mathcal{N}}[\zeta_2(\cdot)]\| \\ &\quad \le \ \|\zeta_1(\cdot) - \zeta_2(\cdot) - \cos\eta(\cdot)\{\sin\zeta_1(\cdot) - \sin\zeta_2(\cdot)\}\| \\ &\qquad\quad + \|\sin\eta(\cdot)\{\cos\zeta_1(\cdot) - \cos\zeta_2(\cdot)\}\| \\ &\quad \le \ (1 - \cos(\bar{\eta} + \delta))\|\zeta_1(\cdot) - \zeta_2(\cdot)\| \end{aligned}$$

for $\zeta_1(\cdot), \zeta_2(\cdot) \in \bar{B}_\delta$. Thus, $\bar{\mathcal{N}}$ is contractive on the minimal neighborhood $\bar{B}_{\bar{\delta}}$ if

$$\pi/2 > \bar{\eta} + \bar{\delta} = \bar{\eta} + \sin^{-1}(\sin\bar{\eta} + \epsilon) - \bar{\eta}$$

or, since $\sin(\cdot)$ is monotone on $[0, \pi/2]$,

$$\sin\bar{\eta} + \epsilon < 1 \ .$$

∎

V. Conclusion

In this paper, we have shown that the driven inverted pendulum system always possesses an upright pendulum trajectory. Various aspect of the upright trajectory have been explored. An algorithm for calculating the desired trajectory have been presented.

Acknowledgement

We thank Alexandre Megretski for suggesting the result presented in Theorem 2.

References

[1] N. Getz, "Control of balance for a nonlinear nonholonomic nonminimum phase model of a bicycle," *American Control Conference*, 1994.

[2] ——, "Dynamic inversion of nonlinear maps with applications to nonlinear control and robotics," Ph.D. dissertation, University of California at Berkeley, 1995.

[3] R. Frezza and A. Beghi, "Simulating a motorcycle driver," in *New trends in Nonlinear Dynamics and Control and their Applications*, W.Kang, M. Xiao, and C. Borges, Eds. Springer-Verlag, 2003, pp. 175–186.

[4] J. Hauser, A. Saccon, and R. Frezza, "Aggressive motorcycle trajectories," *IFAC Symposium on Nonlinear Control Systems*, 2004.

[5] J. Hauser, S. Sastry, and G. Meyer, "Nonlinear control design for slightly non-minimum phase systems: application to V/STOL aircraft," *Automatica*, vol. 28, pp. 665–679, 1992.

[6] S. Devasia and B. Paden, "Stable inversion for nonlinear nonminimum-phase time-varying systems," *IEEE Transactions on Automatic Control*, vol. 43, pp. 283–288, 1998.

[7] B. D. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Prentice-Hall, 1990.

[8] W. A. Coppel, "Dichotomies and reducibility," *Journal of Differential Equations*, vol. 3, pp. 500–521, 1967.

[9] ——, *Dichotomies in Stability Theory*. Berlin: Springer-Verlag, 1978.