Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005



# On hybrid controllers that induce input-to-state stability with respect to measurement noise*

Ricardo G. Sanfelice and Andrew R. Teel†

*Abstract*— **For a class of nonlinear systems affine in controls and with unknown high frequency gain, we develop a hybrid control strategy that guarantees (practical) global input-to-state stability (ISS) with respect to measurement noise. We provide a design procedure for the hybrid controller and apply it to Freeman's counterexample and minimum-phase relative degree one systems.**

## I. INTRODUCTION

The problem of designing feedback controls for nonlinear systems to guarantee bounded states when errors in measurements are present is relevant in most industrial applications since measurements are taken from sensors that are always corrupted by noise. It is desirable that the feedback law designed for the nominal nonlinear system confers some degree of robustness to the closed-loop system in the presence of measurement noise. More precisely, for the nonlinear system

$$\dot{x} \quad = \quad f(x,u) \tag{1}$$

the problem of robust stabilization under measurement noise consists of finding a feedback law $\kappa$ so that the trajectories of the closed-loop system $\dot{x} = f(x, \kappa(x+e))$ remain bounded for bounded measurement noise $e$.

An alternative weaker property is when for each compact set there exists a positive number such that the trajectories starting from that compact set remain bounded as long as the measurement noise is bounded by that number. This property comes for free when $f$ and $\kappa$ are continuous as can be shown using Kurzweil's converse Lyapunov theorem [10] and techniques in [18]. It can also be induced for any asymptotically controllable system by means of sampling and hold (possibly discontinuous) feedback; see for example [19]. Such feedback laws are nonstandard sample-and-hold strategies and are a special case of hybrid control laws with continuous and discrete dynamics. The control of nonlinear systems by hybrid controllers (controllers with both continuous and discrete dynamics) with the objective of enhancing the robustness properties has been recently addressed in the literature. In [13], the authors propose a hybrid control strategy for the so-called Artstein's circles example that renders the closed-loop system stable and robust to measurement noise when an explicit bound for the measurement noise is available. Prieur and Astolfi in [14] show that global robustness to small enough measurement noise, actuator noises, and exogenous disturbances for non-holonomic chained systems can be obtained by means of a hybrid controller. Related to this is the work by Liberzon [11] where a hybrid controller is proposed for systems with quantized measurements of the state.

When the measurement noise $e$ is not sufficiently small (bounded with unknown bound), it is usually required for the closed-loop trajectories to satisfy a bound that depends on the initial conditions and on the size of the measurement noise $e$, an *input-to-state stability* (ISS) property [16] with respect to $e$. Unfortunately, the existence of a continuous globally stabilizing feedback for the nominal system (1) does not imply global input-to-state stabilizability with respect to measurement noise using the same kind of feedback. Freeman in [5] constructed a nonlinear system that admits a continuous, time-invariant, memoryless globally stabilizing feedback law, but for which no feedback law of that type can prevent finite escape time for arbitrarily small $e$. Further work by Freeman [6] and Fah [4] show that time-varying feedback laws give the ISS property for feedback passive systems and one-dimensional affine systems, respectively, with unknown high frequency gain. Alternative to the *Nussbaum gain* approach in [12], switching strategies for linear systems with unknown high frequency gain under linearly bounded perturbations has been studied in the literature, see Ilchman and Owens [9] and the references therein.

In this paper, we take a hybrid control approach to the problem of robust stabilization of nonlinear systems under measurement noise. With the framework for hybrid systems discussed in [7], [3], [8], we propose a hybrid controller for single-input nonlinear systems affine in controls that have a relative degree one output function with ISS inverse dynamics and satisfy an additional growth condition with respect to measurement noise. For the design of the controller, no information about the bound on the measurement noise is required. The main result we establish is that the closed loop is practically input-to-state stable with respect to measurement noise. This result covers, for example, systems like the counterexample given by Freeman [5] and minimum-phase relative degree one nonlinear systems.

## II. PROBLEM DESCRIPTION

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= x + e \end{aligned} \tag{2}$$

where $x$ is the state, $u$ is the control input, and $e$ is the measurement noise. In general, it is desired that there exist class-$\mathcal{K}_\infty$ functions $\sigma_1, \sigma_2$, and $\sigma_3$ such that for every initial

* Research partially supported by the ARO under Grant DAAD19-03-1-0144, the NSF under Grant CCR-0311084 and Grant no. ECS-0324679, and by the AFOSR under Grant F49620-03-1-0203.

† {rsanfelice,teel}@ece.ucsb.edu; Center for Control, Dynamical Systems, and Computation; Electrical and Computer Engineering Department; University of California; Santa Barbara; CA 93106-9560.

0-7803-9568-9/05/$20.00 ©2005 IEEE

condition $x^0 = x(0)$ and every bounded signal $e(\cdot)$, the trajectories of $\dot{x} = f(x, \kappa(x+e))$ satisfy

$$\begin{aligned} \sup_{t\geq 0}|x(t)| & \leq \max\left\{\sigma_1(|x^0|), \sigma_2\left(\sup_{t\geq 0}|e(t)|\right)\right\} \\ \limsup_{t\to\infty}|x(t)| & \leq \sigma_3\left(\limsup_{t\to\infty}|e(t)|\right) . \end{aligned} \tag{3}$$

In words, it is desired that the trajectories do not grow more than a number that is a function of the initial condition and of the maximum value of the norm of the measurement noise, and that they converge to a value that depends on the asymptotic value of the measurement noise. For continuous-time systems, it was shown in [17] that such a property is equivalent to ISS with respect to measurement noise first introduced by Sontag [16]. In [5], Freeman gave a two-dimensional input-affine smoothly stabilizable example for which no continuous state feedback law exists that induces (3). This motivates considering hybrid feedback controllers.

For a family of nonlinear systems, we propose a novel hybrid controller that practically achieves (3). Our hybrid controller can be written in the following general form. The continuous dynamics of the controller are

$$\left.\begin{aligned} \dot{x}_c & = \Gamma(y, x_c, q) \\ \dot{q} & = 0 \end{aligned}\right\} \text{when } (\omega_c(x_c), q) \in C_c$$

and the discrete dynamics are

$$\left.\begin{aligned} x_c^+ & = G_c(y, x_c, q) \\ q^+ & = Q(y, x_c, q) \end{aligned}\right\} \text{when } (\omega_c(x_c), q) \in D_c,$$

where $x_c$ is the continuous state, $q$ is the discrete state, $y$ is the measurement of the state, $C_c$ and $D_c$ are sets that define when the controller evolves continuously (flows) and when evolves discretely (jumps), respectively. The output of the controller is the control input for the nonlinear system

$$u = \kappa(x_c, q).$$

More specifically, the controller we propose consists of two discrete modes $q \in \{-1, 1\}$ and a control law $\kappa$ with sign determined by the sign of $q$. The function $\kappa$ is designed so that its magnitude grows as the norm of the measurement of the state $x$ grows. The basic idea behind our strategy is that of switching $q$ with the mapping $q^+ = -q$ when the measurements are larger than an auxiliary state that keeps track of the measurements at switching times. If the measurement of the state is larger than twice the value of the auxiliary state, then $q$ is toggled. Since the measurement noise is bounded, for sufficiently large value of the state $x$ and for some $q$, the control law is such that it causes the norm of the trajectories of the closed-loop system to decrease. We will show that this strategy is successful even for the case when the high frequency gain is unknown.

We now introduce notation, the basic concepts, and the framework for hybrid systems used throughout the paper.

## III. PRELIMINARIES

In this paper we consider hybrid systems discussed by the authors et al. in [7], [8] and later explored in [2], [15] where converse Lyapunov results and invariance principles were established. Below, for completeness, we summarize this framework.

Solutions to hybrid systems, when they exist, can evolve continuously (flow) and/or discretely (jump) depending on the continuous and discrete dynamics and the sets where those dynamics apply. We treat the number of jumps as an independent variable $j$ and we parameterize the state by $(t, j)$. A solution is a function defined on subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0} := [0, +\infty) \times \{0, 1, 2, ...\}$. A subset $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0}$ is a *compact hybrid time domain* if

$$\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \ldots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid domain. A *hybrid arc* (or *hybrid trajectory*) is a pair $(x, \operatorname{dom} x)$ consisting of a hybrid time domain $\operatorname{dom} x$ and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(t, j)$ is absolutely continuous in $t$ for a fixed $j$ and $(t, j) \in \operatorname{dom} x$. We will not mention $\operatorname{dom} x$ explicitly, and understand that with each hybrid arc comes a hybrid time domain. A hybrid arc is said to be *complete* if $\operatorname{dom} x$ is unbounded, *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded, and *maximal* if there does not exist another hybrid arc $x'$ such that $x$ is a truncation of $x'$ to some proper subset of $\operatorname{dom} x'$.

The state of a hybrid system is often given by a "continuous" variable and a "discrete" one. We will not explicitly distinguish between the two. The set of potential values of the discrete variable, often consisting of descriptive elements like "off" or "on", can be identified with a subset of integers. This leads to more compact notation.

A hybrid system $\mathcal{H}$ will be given on a state space $O$ by set-valued mappings $F$ and $G$ describing, respectively, the continuous and the discrete dynamics, and sets $C$ and $D$ where these dynamics may occur. For definition of solutions and conditions that $F, G, C, D$ need to satisfy for solutions to exists consult [8]. Similarly, a hybrid arc $x$ and a measurement noise signal $e$ are a *solution pair* $(x, e)$ *to the hybrid system* $\mathcal{H}$ if $\operatorname{dom} x = \operatorname{dom} e$ and

(S1) For all $j \in \mathbb{N}$ and almost all $t$ such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in F(x(t, j), e(t, j)) \tag{4}$$

(S2) For all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j+1) \in G(x(t, j), e(t, j)). \tag{5}$$

For solutions $x(t, j)$ on the hybrid time domain $\operatorname{dom} x$ and a compact hybrid time domain $S := \operatorname{dom} x \cap [0, T] \times \{0, 1, \ldots, J\})$, $(T, J) \in \operatorname{dom} x$, we define by $|\omega(x)|_\infty^S := \sup_{(t,j)\in S} \omega(x(t, j))$. We also define the norms

$$|x|_\infty = \sup_{(t,j)\in \operatorname{dom} x} |x(t, j)|, \quad |x|_a = \limsup_{t+j\to\infty} |x(t, j)|,$$

where $|\cdot|$ denotes the Euclidean norm.

By input-to-state stability with respect to $e$ for hybrid systems we mean the following.

*Definition 3.1:* (ISS with respect to measurement noise) A hybrid system $\mathcal{H}$ is input-to-state stable with respect to $e$

if there exist functions $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{K}_\infty$ such that for each $x^0 \in O$, each solution pair $(x, e)$ to $\mathcal{H}$ satisfies

$$|x|_\infty \le \max\left\{\sigma_1(|x^0|), \sigma_2(|e|_\infty)\right\}, \quad |x|_a \le \sigma_3\left(|e|_a\right). \quad \blacksquare$$

Note that this property is the same as the one given in (3). Equivalences between this ISS characterization and other ISS characterization for hybrid systems are discussed in [1]. In this paper, we are interested in practical input-to-state stability (pISS) for the family of hybrid systems $\mathcal{H}^k$ where $k$ is a parameter.

*Definition 3.2:* (pISS with respect to measurement noise) The family of hybrid systems $\mathcal{H}^k$ is practically input-to-state stable with respect to $e$ if there exist functions $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{K}_\infty$ such that for each $\epsilon > 0$, each $x^0 \in O$, there exists $k > 0$ so that every solution pair $(x, e)$ to $\mathcal{H}^k$ satisfies

$$\begin{aligned} |x|_\infty &\le \max\left\{\sigma_1(|x^0|), \sigma_2(|e|_\infty), \epsilon\right\} \\ |x|_a &\le \max\left\{\sigma_3\left(|e|_a\right), \epsilon\right\} \end{aligned} \quad \blacksquare$$

## IV. Robust Hybrid Controller

We consider nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= x + e \end{aligned} \qquad (6)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous and $f(0) = 0$. The output $y$ is the measurement of the state $x$ in the presence of the measurement noise $e$.

*Assumption 4.1:* For the system (6), suppose that there exist locally Lipschitz functions $h : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ and $h_1 : \mathbb{R}^n \to \mathbb{R}$; class-$\mathcal{K}_\infty$ functions $\alpha, \rho_1, \rho_2, \rho_3, \rho_4$; and $\omega : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ that satisfy

1) $\alpha(|x|) \le \max\{\omega(x), |h_1(x)|\}$ for all $x \in \mathbb{R}^n$;
2) $|\langle \nabla h_1(x), g(x)\rangle| \ge \mu > 0$ for almost all $x \in \mathbb{R}^n$;
3) $|h(x+e) - |h_1(x)|| \le \max\{\rho_1(|e|), \rho_2(\omega(x))\}$ for all $x, e \in \mathbb{R}^n$;
4) Every classical trajectory $x$ of the system (6) starting at $x^0 = x(0) \in \mathbb{R}^n$ satisfies for all $T \ge 0$
$$\begin{aligned} |\omega(x)|_\infty^T &\le \max\left\{\rho_3(|x^0|), \rho_4(|h_1(x)|_\infty^T)\right\} \\ |\omega(x)|_a &\le \rho_4\left(|h_1(x)|_a\right); \end{aligned}$$
5) For some $\epsilon > 0$, $14(1+\epsilon)\rho_2 \circ \rho_4(r) < r$ for all $r > 0$.

*Remark 4.2:* The second condition above guarantees that the relative degree of the "output" function $h_1$ is equal to one and the fourth condition combined with the first condition is a type of minimum phase condition with respect to the "output" $h_1$. The fifth condition is a small gain-type of condition where the number "14" is somewhat arbitrary, as the function $\rho_2$ in the third condition can typically be scaled arbitrarily. $\blacksquare$

In this section, we first describe a hybrid controller that renders the closed-loop system practically ISS with respect to measurement noise. We provide a step-by-step procedure for the construction of such controller and discuss the main properties of the closed-loop system.

### A. Hybrid Controller

The hybrid controller, denoted as $\mathcal{H}_c^k$, consists of continuous states $z \in \mathbb{R}$, $\xi \in \mathbb{R}_{\ge 0}$, and $\tau \in \mathbb{R}_{\ge 0}$, and a discrete state $q \in \{-1, 1\}$. The state $z$ asymptotically tracks a filtered version of the measurement of the state $h(x+e)$ while the state $\xi$ defines the threshold for $|z|$ where transitions between both discrete modes $q = 1$, $q = -1$ occur. The state $\tau$ is a timer that is reset to zero after every jump and that enables the jumps after it reaches a given nonzero threshold. The output of the controller is given by

$$u = \kappa(|z|, q) := q \frac{\gamma_x^{-1}(|z|)}{\mu}$$

where $\gamma_x \in \mathcal{K}_\infty$ is to be determined and the constant $\mu > 0$ is defined in Assumption 4.1.2.

The continuous dynamics of the controller are given by

$$\begin{aligned} \dot{z} &= -\operatorname{sign}(z - h(x+e))\,\gamma_z^{-1}(|z - h(x+e)|) \\ \dot{\xi} &= 0, \quad \dot{\tau} = -\tau + \tau^*, \quad \dot{q} = 0, \end{aligned}$$

where $\gamma_z \in \mathcal{K}_\infty$ and $\tau^* \in \mathbb{R}_{>0}$ are design parameters. The continuous evolution of the system will be effective when $(\xi, |z|)$ is in the set

$$C := \left\{(\alpha, \beta) \in \mathbb{R}^2_{\ge 0} \;\middle|\; \frac{\alpha}{2} \le \beta \le 2\alpha\right\}$$

or when $(\tau, |z|)$ is in the set

$$\mathcal{T}_C := \left\{(\alpha, \beta) \in \mathbb{R}^2_{\ge 0} \mid \alpha \le T_k(\beta)\right\}$$

where $T_k : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ is a continuous, nonincreasing function to be chosen. This function forces the closed-loop solutions to flow for at least $T_k(|z|)$ seconds after every jump. The jump map for the controller is defined as follows. When $(\xi, |z|)$ satisfies $|z| \ge 2\xi$, i.e. $(\xi, |z|)$ belongs to the set

$$D_u := \left\{(\alpha, \beta) \in \mathbb{R}^2_{\ge 0} \mid \beta \ge 2\alpha\right\},$$

$(\tau, |z|)$ satisfies $\tau \ge T_k(|z|)$, i.e. $(\tau, |z|)$ belongs to the set

$$\mathcal{T}_{D_u} := \left\{(\alpha, \beta) \in \mathbb{R}^2_{\ge 0} \mid \alpha \ge T_k(\beta)\right\},$$

and $(\xi, \tau, |z|) \notin D_\circ$ where

$$D_\circ := \left\{(\alpha, \beta, \lambda) \in \mathbb{R}^3_{\ge 0} \mid \alpha = \lambda = 0, \beta \ge T_k(0)\right\},$$

jumps will be enabled with mapping

$$z^+ = h(x+e), \quad \xi^+ = |z|, \quad \tau^+ = 0, \quad q^+ = -q.$$

This jump mapping resets the tracking state $z$ to the current measurement of the state $x$, updates the threshold to the current value of $|z|$ at the jump, resets the timer to zero, and switches the control law. In the event that $(\xi, |z|)$ satisfies $\frac{\xi}{2} \ge |z|$, i.e. $(\xi, |z|)$ belongs to the set

$$D_l := \left\{(\alpha, \beta) \in \mathbb{R}^2_{\ge 0} \;\middle|\; \frac{\alpha}{2} \ge \beta\right\},$$

$(\tau, |z|)$ satisfies $\tau \ge \frac{T_k(|z|)}{2}$, i.e. $(\tau, |z|)$ belongs to the set

$$\mathcal{T}_{D_l} := \left\{(\alpha, \beta) \in \mathbb{R}^2_{\ge 0} \;\middle|\; \alpha \ge \frac{T_k(\beta)}{2}\right\},$$

and $(\xi, \tau, |z|) \notin D_\circ$, jumps will be enabled with mapping equal to the one above except that the control law is not switched, i.e. $q^+ = q$. In the situation that $(\xi, \tau, |z|) \in D_\circ$ a jump will occur with jump mapping

$$z^+ = h(x+e), \quad \xi^+ = |z|, \quad \tau^+ = 0, \quad q^+ \in \{-1, 1\},$$

where the update law for $q$ is set-valued, meaning that the mode can choose to switch or not.

Summarizing, the controller $\mathcal{H}_c^k$ is given by

$$\begin{aligned} u &= q\frac{\gamma_x^{-1}(|z|)}{\mu} \\ \dot{z} &= -\operatorname{sign}(z - h(x+e))\, \gamma_z^{-1}(|z - h(x+e)|) \\ \dot{\xi} &= 0, \quad \dot{\tau} = -\tau + \tau^*, \quad \dot{q} = 0 \end{aligned} \tag{7}$$

when $(\xi, |z|) \in C$ or $(\tau, |z|) \in \mathcal{T}_C$,

$$\left.\begin{aligned} z^+ &= h(x+e) \\ \xi^+ &= |z| \\ \tau^+ &= 0 \\ q^+ &= -q \end{aligned}\right\} \begin{aligned} &\text{when } (\xi, |z|) \in D_u \\ &\text{and } (\tau, |z|) \in \mathcal{T}_{D_u} \\ &\text{and } (\xi, \tau, |z|) \notin D_\circ \end{aligned} \tag{8}$$

$$\left.\begin{aligned} z^+ &= h(x+e) \\ \xi^+ &= |z| \\ \tau^+ &= 0 \\ q^+ &= q \end{aligned}\right\} \begin{aligned} &\text{when } (\xi, |z|) \in D_l \\ &\text{and } (\tau, |z|) \in \mathcal{T}_{D_l} \\ &\text{and } (\xi, \tau, |z|) \notin D_\circ \end{aligned} \tag{9}$$

$$\left.\begin{aligned} z^+ &= h(x+e) \\ \xi^+ &= |z| \\ \tau^+ &= 0 \\ q^+ &\in \{-1, 1\} \end{aligned}\right\} \text{ when } (\xi, \tau, |z|) \in D_\circ\ . \tag{10}$$

### B. Controller design

We present a procedure of finding functions $\gamma_x, \gamma_z \in \mathcal{K}_\infty$, $T_k$, and $\tau^*$ that define the controller $\mathcal{H}_c^k$.

**Step 1** Find a function $\alpha_x \in \mathcal{K}_\infty$ satisfying

$$|\langle \nabla h_1(x), f(x) \rangle| \leq \alpha_x(|x|)$$

for almost all $x \in \mathbb{R}^n$. This function exists since $h_1$ is locally Lipschitz, $f$ is continuous, and $f(0) = 0$.

**Step 2** Find a function $\gamma_x \in \mathcal{K}_\infty$ satisfying

$$\gamma_x^{-1}(r) > \max\{\rho_8(4r), \rho_6(4r)\} \qquad \text{for all } r > 0,$$

where $\rho_6 = \alpha_x \circ \alpha^{-1}, \rho_8 = \rho_6 \circ \rho_4$.

**Step 3** Find a function $\alpha_z \in \mathcal{K}_\infty$ satisfying

$$\left| \langle \nabla |h_1(x)|, f(x) + g(x)\frac{\gamma_x^{-1}(|h_1(x)| + \zeta)}{\mu} \rangle \right| \leq \alpha_z(\max\{|x|, |\zeta|\})$$

for almost all $x \in \mathbb{R}^n, \zeta \in \mathbb{R}$. Since $\gamma_x^{-1} \in \mathcal{K}_\infty$; $x \mapsto |h_1(x)|$ is locally Lipschitz; $f, g$ are continuous; and $f(0) = 0$; the existence of $\alpha_z$ is guaranteed.

**Step 4** For $\epsilon > 0$ given in Assumption 4.1.5, find a function $\gamma_z \in \mathcal{K}_\infty$ satisfying

$$\gamma_z^{-1}\left(\frac{\epsilon}{1+\epsilon} r\right) > \rho_7(r) \qquad \text{for all } r > 0,$$

where

$$\rho_7(r) = \max\left\{\alpha_z \circ \alpha^{-1} \circ \rho_4(14r), \alpha_z \circ \alpha^{-1}(14r), \alpha_z(r)\right\}\ .$$

**Step 5** Find a function $\alpha_y \in \mathcal{K}_\infty$ and constant $k_1 \geq 0$ satisfying

$$|\langle \nabla h_1(x), g(x) \rangle| \leq \alpha_y(|x|) + k_1$$

for almost all $x \in \mathbb{R}^n$. This function exists since $h_1$ is locally Lipschitz and $g$ is continuous.

**Step 6** Find a function $\gamma_y \in \mathcal{K}_\infty$ satisfying

$$\gamma_y^{-1}(r) > \max\{\rho_{12}(r), \rho_{13}(r)\} \qquad \text{for all } r > 0,$$

where $\rho_{12} = \alpha_y \circ \alpha^{-1}, \rho_{13} = \rho_{12} \circ \rho_4$.

**Step 7** Pick $k > 0$ and find a continuous nonincreasing function $T_k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ that is upper bounded by

$$\tilde{T}(|r|) := \begin{cases} \max\{T'(k), T'(|r|)\} & \text{if } |r| \leq k \\ T'(|r|) & \text{otherwise} \end{cases}$$

where

$$T'(|r|) = \frac{|r|}{\gamma_x^{-1}(k_2|r|) + \frac{1}{\mu}\left(k_1 + \gamma_y^{-1}(k_3|r|)\right)\gamma_x^{-1}(k_2|r|) + k}$$

where $k_2 = 5$ and $k_3 = 2$. Note that in most cases $\gamma_x^{-1}$ and $\gamma_y^{-1}$ grow faster than linear; in this case $\lim_{|r| \to \infty} T_k(|r|) = 0$. Note that for small $r$, $T_k(|r|)$ is bounded away from zero.

**Step 8** Find $\tau^*$ so that $\tau^* \geq T_k(|r|)$ for all $r \in \mathbb{R}_{\geq 0}$.

### C. Analysis of the closed-loop system $\mathcal{H}_{cl}^k$

The closed-loop (hybrid) system, denoted by $\mathcal{H}_{cl}^k$, with state $\chi := [x, z, \xi, \tau, q]^T$ in the state space $O := \mathbb{R}^n \times \mathbb{R}^3 \times Q_O$ where $Q_O := (-2, 0) \cup (0, 2)$, has continuous dynamics given by equations (6) and (7) when $(\xi, |z|) \in C$ or $(\tau, |z|) \in \mathcal{T}_C$, and discrete dynamics as defined in (8)-(10) with the addition of the jump mapping $x^+ = x$. For clarity in the exposition, the sets where the flows are enabled $(C, \mathcal{T}_C)$ and the sets where the jumps are enabled $(D_u, \mathcal{T}_{D_u}, D_l, \mathcal{T}_{D_l})$ are subsets of $\mathbb{R}^2_{\geq 0}$ rather than of $O$, but they can be easily rewritten as such. Moreover, since the temporal regularization introduced by the function $T_k$ in the sets $\mathcal{T}_C, \mathcal{T}_{D_u}$, and $\mathcal{T}_{D_l}$ forces the flows to occur after every jump for $T_k(|z|) > 0$ seconds, bounded solutions to $\mathcal{H}_{cl}^k$ are non-Zeno. We now state the main result.

*Theorem 4.3: (pISS of $\mathcal{H}_{cl}^k$) The family of hybrid systems $\mathcal{H}_{cl}^k$ is practically input-to-state stable with respect to measurement noise $e$, i.e. there exist functions $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{K}_\infty$ such that for each $\epsilon > 0$, each $x^0 \in O$, there exists $k > 0$ so that every solution pair $(\chi, e)$ to $\mathcal{H}_{cl}^k$ satisfies*

$$\begin{aligned} |\chi|_\infty &\leq \max\{\sigma_1(|\chi^0|), \sigma_2(|e|_\infty), \epsilon\} \\ |\chi|_a &\leq \max\{\sigma_3(|e|_a), \epsilon\}\ . \end{aligned}$$

■

This theorem not only states that our hybrid control strategy guarantees practical input-to-state stability with respect to measurement noise without knowing explicit bounds on its norm, but also establishes robustness with respect to uncertainty in the high frequency gain of the system. Moreover, note that the parameter $k$ is a design parameter that has to be chosen in Step 7 and it can be picked arbitrarily small.

In the next section we illustrate by examples the applicability of this hybrid controller followed by a sketch of the proof of Theorem 4.3.

## V. Examples

*Example 5.1:* (Freeman's counterexample) We now consider the example given by R. Freeman in [5] where the author presents a single-input second order nonlinear system affine in controls that admits a continuous, time-invariant, memoryless globally stabilizing feedback law, but it is the case that no feedback law of the same type can prevent the state of the system from remaining far away from its equilibrium point for arbitrarily small measurement noise. Freeman's system is given by

$$\begin{aligned}\dot{x} &= f(x) + g(x)u := \left(I + 2\Theta\left(\tfrac{\pi}{2}\right) xx^T\right)\Theta(x^T x) \\ &\left(\begin{bmatrix} -1 & 0 \\ 0 & x^T x\end{bmatrix}\Theta(-x^T x)x + \begin{bmatrix}0\\1\end{bmatrix}u\right), \quad y = x + e \end{aligned} \tag{11}$$

where $x := [x_1, x_2]^T \in \mathbb{R}^2$ is the state, $\Theta(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$, and $e$ is the measurement noise.

For this system, we design the hybrid controller $\mathcal{H}_c^k$ presented in Section IV-B. Such a controller renders the closed loop pISS with respect to $e$. To the best of our knowledge, no other controller that bestows such property for Freeman's counterexample has been proposed before.

We proceed as in [5] and perform the change of coordinates $\eta(x) = \Theta(-x^T x)x$ where $x \mapsto \Theta(-x^T x)x$ is a diffeomorphism with the properties $\eta(x)^T\eta(x) = x^T x$ and $x = \Theta(\eta(x)^T\eta(x))\eta(x)$. Thus, the system (11) can be written in $\eta(x) := [\eta_1(x)\ \eta_2(x)]^T$ coordinates as

$$\dot{\eta}(x) = \begin{bmatrix} -\eta_1(x) \\ (\eta_1(x)^2 + \eta_2(x)^2)\eta_2(x) + u\end{bmatrix}.$$

We now check that Assumption 4.1 is satisfied. Let $h(x) = |\eta(x)|$, $h_1(x) = \eta_2(x)$, $\alpha(r) = \frac{1}{\sqrt{2}}r$, $\rho_1(r) = \rho_2(r) = 2r$, $\rho_3(r) = |\eta_1(r)|$, $\rho_4 \equiv 0$, $\omega(x) = |\eta_1(x)|$, and $\mu = 1$. Then

1) $\alpha(|x|) = \frac{1}{\sqrt{2}}|x| \leq \max\{|\eta_1(x)|, |\eta_2(x)|\} = \max\{\omega(x), |h_1(x)|\}$ for all $x \in \mathbb{R}^n$;
2) $|\langle \nabla h_1(x), g(x)\rangle| = 1$ for all $x \in \mathbb{R}^n$;
3) $|h(x+e) - |h_1(x)|| = ||x+e| - |\eta_2(x)|| \leq ||\eta(x)| + |e| - |\eta_2(x)|| \leq |\eta_1(x)| + |e| \leq 2\max\{|e|, |\eta_1(x)|\} = \max\{\rho_1(|e|), \rho_2(\omega(x))\}$ for all $x, e \in \mathbb{R}^n$;
4) For every trajectory $x$ of the system (11) starting at $x^0 = x(t_0) \in \mathbb{R}^n$, $|\omega(x)|_\infty = |\eta_1(x)|_\infty \leq |\eta_1(x^0)| = \max\{\rho_3(|x^0|), \rho_4(|h_1(x)|_\infty)\}$ for all $t \geq t_0$ and $\limsup_{t\to\infty}|\omega(x(t))| = 0$;
5) For every $\epsilon > 0$, $14(1+\epsilon)\rho_2 \circ \rho_4(r) = 0 < r \quad \forall r > 0$.

Since Assumption 4.1 is satisfied, we proceed to design the hybrid controller following the steps given in Section IV-B.

**Step 1:** From the definitions of $h_1$ and $f$

$$|\langle \nabla h_1(x), f(x)\rangle| = |(\eta_1(x)^2 + \eta_2(x)^2)\eta_2(x)| \leq |x|^3$$

for almost all $x$. Then, we choose $\alpha_x(r) = r^3$.

**Step 2:** Since $\alpha^{-1}(r) = \sqrt{2}r$ and

$$\rho_6(r) = \alpha_x \circ \alpha^{-1}(r) = 2\sqrt{2}r^3, \quad \rho_8 = \rho_6 \circ \rho_4 \equiv 0$$

we have that $\max\{\rho_8(r), \rho_6(r)\} = 128\sqrt{2}r^3$. Then we choose $\gamma_x(r) = \left(\frac{1}{k_x}r\right)^{\frac{1}{3}}$ where $k_x > 128\sqrt{2}$.

**Step 3:** The following holds

$$\begin{aligned}&|\langle \nabla|h_1(x)|, f(x) + g(x)\gamma_x^{-1}(|h_1(x)| + \zeta)\rangle| \\ &\leq \ |(\eta_1(x)^2 + \eta_2(x)^2)\eta_2(x)| + k_x|\eta(x)|^3 \\ &\leq \ (1+k_x)|x|^3 + k_x|\zeta|^3 \leq 2(1+k_x)\max\{|x|, |\zeta|\}^3\end{aligned}$$

for almost all $x$. Then, we pick $\alpha_z(r) = 2(1+k_x)r^3$.

**Step 4:** First compute $\rho_7$

$$\begin{aligned}\rho_7(r) &= \max\left\{\alpha_z \circ \alpha^{-1} \circ \rho_4(14r), \alpha_z \circ \alpha^{-1}(14r), \alpha_z(r)\right\} \\ &= 2(1+k_x)(14\sqrt{2})^3 r^3.\end{aligned}$$

Then, $\gamma_z$ must satisfy $\gamma_z^{-1}(r) > 2(1+k_x)(14)^3\frac{(1+\epsilon)^3}{\epsilon^3}r^3$. Then, we design $\gamma_z$ to be $\gamma_z(r) = \left(\frac{1}{k_z}r\right)^{\frac{1}{3}}$ for $k_z > 2(1+k_x)(14)^3\frac{(1+\epsilon)^3}{\epsilon^3}$.

**Step 5:** By Assumption 4.1.2, $|\langle \nabla h_1(x), g(x)\rangle| = 1$ for almost all $x \in \mathbb{R}^n$. Then $\alpha_y \equiv 0$ and $k_1 = 1$.

**Step 6:** Since $\alpha_y \equiv 0$, then $\rho_{12} \equiv \rho_{13} \equiv 0$. Then $\gamma_y \equiv 0$.

**Step 7:** From the previous steps we get

$$T'(|r|) = \frac{|r|}{(k_x k_2^2 + k_1 k_x k_3^3)|r|^3 + k}$$

where $k_x > 128\sqrt{2}$, $k > 0$, $k_1 = 1$, $k_2 = 5$, $k_3 = 2$. For a given $k > 0$, $\tilde{T}(|r|)$ is given by

$$\tilde{T}(|r|) := \begin{cases} \max\{T'(k), T'(|r|)\} & \text{if } |r| \leq k \\ T'(|r|) & \text{otherwise}\end{cases}$$

A continuous nonincreasing function $T_k$ that is upper bounded by $\tilde{T}(|r|)$ is obtained as $T_k(|r|) = \min_{v\in[0,|r|]}\tilde{T}(v)$.

**Step 8:** Pick $\tau^* = \max_{v\in[0,k]}\{T'(k), T'(v)\}$.

Then, the hybrid controller $\mathcal{H}_c^k$ is completely designed. We implement the closed-loop system in Simulink with parameters $k_x = 128\sqrt{2}$, $k_z = 3\cdot 10^6$, $\epsilon = 1000$, $k = 1$, and $\tau^* = 0.02$. We present a simulation of the closed-loop system for initial condition $[x^0, z^0, \xi^0, \tau^0, q^0]^T = [[2, 0.1], 0, 1, 0, -1]^T$. The noise has normal distribution with variance $\sigma = 0.001$. We plot $h(x+e) = |x+e|$, $\xi$, and the set $C$ in the $(\xi, h(x+e))$ plane since it is easy to visualize the evolution of the solutions. Since $\xi$ is constant during flows and changes only on jumps, vertical lines describe the flows while horizontal

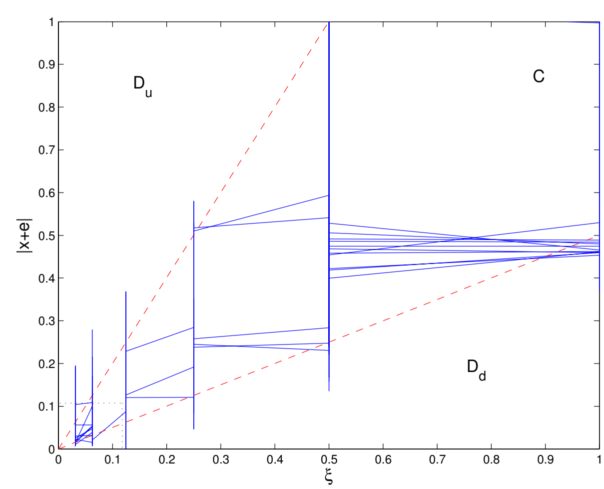

Fig. 1. Closed-loop solution starts at $(1, 2)$ and approaches the dotted box.

lines describe jumps. Once the trajectory has reached the dotted box, it stays jumping and evolving around it due to the measurement noise. Note that every time that the solution leaves the set $C$, flows are still enabled due to the temporal regularization. ■

*Example 5.2:* (minimum-phase, rel. degree one system) Given the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u := \begin{bmatrix} f_0(x) \\ f_1(x) + g_1(x)u \end{bmatrix} \\ y &= x_2 + e \end{aligned}$$

where $x := [x_1, x_2]^T \in \mathbb{R}^n$; $f_0 : \mathbb{R}^n \to \mathbb{R}^{n-1}$; $f_1 : \mathbb{R}^n \to \mathbb{R}$; $f$ is locally Lipschitz continuous and $f(0) = 0$; $g_1 : \mathbb{R}^n \to \mathbb{R}$ is continuous and $|g_1(x)| \geq \mu > 0$ for all $x \in \mathbb{R}^n$; the system $\dot{x}_1 = f_0(x)$ is input-to-state stable with respect to $x_2$, i.e. given $\rho_3, \rho_4 \in \mathcal{K}_\infty$, for every $x_1^0 \in \mathbb{R}^{n-1}$ and every $x_2(\cdot)$

$$\begin{aligned} |x_1|_\infty^S &\leq \max\left\{\rho_3(|x_1^0|), \rho_4(|x_2|_\infty^S)\right\} \\ \limsup_{t\to\infty} |x_1(t)| &\leq \rho_4\left(\limsup_{t\to\infty} |x_2(t)|\right); \end{aligned}$$

$u$ is the control input; and $e$ is the measurement noise.

Let $h(x) = |x_2|$; $h_1(x) = x_2$; $\alpha(r) = \frac{1}{\sqrt{2}} r$; $\rho_1(r) = r$; $\rho_2 \equiv 0$; $\omega(x) = |x_1|$; and $\rho_3, \rho_4$ satisfy the ISS condition above. Assumptions 4.1.1, 4.1.3, and 4.1.5 can be checked as in Example 5.1. Assumption 4.1.2 is satisfied since $|g_1(x)| \geq \mu > 0$, and Assumption 4.1.4 holds since the system $\dot{x}_1 = f_0(x)$ is ISS with respect to $x_2$. The design of the hybrid controller closely follows Step 1 - 8 as in Example 5.1. ■

## VI. Sketch of the Proof of Theorem 4.3

Let $\chi$ be a maximal solution to $\mathcal{H}_{cl}^k$ and note that any truncation to a compact hybrid time domain $S := \operatorname{dom}\chi \cap [0,T] \times \{0,\ldots,J\}$, $(T,J) \in \operatorname{dom}\chi$, is bounded. Therefore, any signal of the closed-loop system $\mathcal{H}_{cl}^k$ restricted to $S$ is bounded. Define $c_1 := |\langle\nabla h_1(x), f(x)\rangle|_\infty^S$, $c_2 := |z - |h_1(x)||_\infty^S$, and $c_3 := \frac{1}{\mu}|\langle\nabla h_1(x), g(x)\rangle|_\infty^S - 1$. Observe that with this definition, $|z| \in |h_1(x)| + c_2\mathbb{B}$ on $S$, where $\mathbb{B}$ denotes the unit ball in $\mathbb{R}$. Then, the properties of the function $h_1$ let us rewrite the dynamics of $h_1(x)$ on $S$ as follows

$$\dot{h}_1(x) \in c_1\mathbb{B} + q\Omega\gamma_x^{-1}\left((|h_1(x)| + c_2\mathbb{B}) \cap \mathbb{R}_{\geq 0}\right)$$

where $\Omega$ is a compact connected subset of $\Omega^- \cup \Omega^+$ where $\Omega^- := [-1 - c_3, -1]$ and $\Omega^+ := [1, 1 + c_3]$. Similarly, the continuous and discrete dynamics of the other states can be derived as function of $c_2$. The closed-loop system $\mathcal{H}_{cl}^k$ can be rewritten with these dynamics that are enabled on sets that depend on $c_2$ as well. It turns out that the solutions to the truncated closed loop can be easily studied in the $(\xi, |h_1(x)|)$ plane since during flows $\xi$ remains constant and the trajectories $|h_1(x)|$ are vertical lines, and at jumps $\xi$ is mapped to a point in $|h_1(x)| + c_2\mathbb{B}$ and $|h_1(x)|$ remains constant. It can be shown that $|\chi|_\infty^S$ is upper-bounded by a class-$\mathcal{K}_\infty$ function that depends on the initial state, $c_1$, $c_2$, and $k$. Using inequalities in Step 1 - 8 of the controller design procedure, the bound on $|\chi|_\infty^S$ depends only on $|\chi^0|$, the size of the measurement noise, and $k$. Assuming that the noise is bounded, the bound on $|\chi|_\infty^S$ can be extended to the entire domain of $\chi$. By Proposition 2.1 in [8], $\chi$ being maximal and bounded implies that it is also complete. Then $|\chi|_a$ is well-defined and using again the design procedure, it is bounded by a class-$\mathcal{K}_\infty$ function that depends on $|e|_a$ and $k$. ■

## VII. Conclusions

We presented a novel hybrid control strategy that can be applied to a large class of nonlinear systems affine in controls and with unknown high frequency gain. The main property that our controller achieves is practical input-to-state stability with respect to measurement noise of the closed-loop hybrid system. The design of our controller does not require the knowledge of explicit bounds on the measurement noise. Our controller is of high-gain type which is common in the control of minimum phase system with limited information of the system. By examples, we have shown the methodology for the design of the hybrid controller and provided simulations results. A version of this paper with detailed proofs and the simulation files for Example 5.1 can be found at the first author's website.

## References


[1] C. Cai and A. R. Teel. Results on input-to-state stability for hybrid systems. In *Proc 44th IEEE Conference on Decision and Control*, 2005.

[2] C. Cai, A.R. Teel, and R. Goebel. Converse Lyapunov theorems and robust asymptotic stability for hybrid systems. In *Proc 24th American Control Conference*, 2005.

[3] P. Collins. A trajectory-space approach to hybrid systems. In *Proc. MTNS 2004*.

[4] N.C. Fah. Input-to-state stability with respect to measurement disturbances for one-dimensional systems. *ESAIM-COCV*, 4:99–121, 1999.

[5] R. Freeman. Global internal stabilizability does not imply global external stabilizability for small sensor disturbances. *IEEE Trans. Aut. Cont.*, 40:2119–2122, 1995.

[6] R. Freeman. Time-varying feedback for the global stabilization of nonlinear systems with measurement disturbances. In *Proc. European Control Conf. Brussels, Belgium*, 1997.

[7] R. Goebel, J.P. Hespanha, A.R. Teel, C. Cai, and R.G. Sanfelice. Hybrid systems: Generalized solutions and robust stability. In *Proc. NOLCOS*, pages 1–12, 2004.

[8] R. Goebel and A.R. Teel. Results on solution sets to hybrid systems with applications to stability theory. In *Proc. 24th American Control Conference*, 2005.

[9] A. Ilchmann and D.H. Owens. Threshold switching functions in high-gain adaptive control. *IMA Journal of Mathematical Control & Information*, 8(4):409–429, 1991.

[10] J. Kurzweil. On the inversion of ljapunov's second theorem on stability of motion. *Amer. Math. Soc. Trans. Ser. 2*, 24:19–77, 1956.

[11] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39:1543–1554, 2003.

[12] R. Nussbaum. Some remarks on a conjecture in parameter adaptive control. *Sys. & Cont. Lett.*, 3:243–246, 1983.

[13] C. Prieur. *Lectures Notes in Control and Information Sciences*, chapter A Robust globally asymptotically stabilizing feedback: the example of the Artstein's circles, pages 279–300. Springer Verlag, London, 2000.

[14] C. Prieur and A. Astolfi. Robust stabilization of chained systems via hybrid control. *IEEE Trans. Aut. Cont.*, 48:1768–1772, 2003.

[15] R.G. Sanfelice, R. Goebel, and A.R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *Submitted*, 2005.

[16] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Aut. Cont.*, 34:435–443, 1989.

[17] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Trans. Aut. Cont.*, 41(9):1283–1294, 1996.

[18] E.D. Sontag. Further facts about input to state stabilization. *IEEE Trans. Aut. Cont.*, 35(4):473 – 476, 1990.

[19] E.D. Sontag. Clocks and insensitivity to small measurment errors. *ESAIM-COCV*, 4:537–557, 1999.