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Output tracking control of a flexible robot arm

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Abstract— **In this paper, we address the problem of output feedback tracking control of a flexible robot arm. The robot arm is modeled as an Euler-Bernoulli beam. The beam is clamped to a motor at one end and attached to a force actuator at the other. Based on measurements at the boundaries, a uniformly exponentially stable observer is proposed. Using the information from the observer, a tracking controller which allows the robot arm to follow time-varying references and damp out the elastic vibrations is designed. The existence, uniqueness and stability of solutions of the closed loop system and the observer are based on semigroup theory. Numerical simulation results are included to illustrate the performance of the proposed control laws and the proposed observer. The simulation results are in agreement with the theoretical results.**

I. Introduction

The stabilization problem for mechanical systems described by infinite-dimensional model has been extensively studied by several authors. The idea was first applied by Chen [4] to the systems described by wave equation (e.g. strings), and later extended to the Euler-Bernoulli beam equation and the Timoshenko beam equation by numerous authors, among others [1],[5],[6],[13],[14],[19],[20]. In particular, in [5], Chen et al. showed that a single actuator applied at the free end of the cantilever beam is sufficient to obtain uniformly stabilization of the deflection of the beam. In [14],[20] the orientation and stabilization of a beam attached to a rigid body were studied. Recently, Lynch and Wang [13] applied *flatness* in controller design for a hub-beam system with a tip payload. In [1], Aoustin et al. considered the motion planning and synthesis of a tracking controller of a flexible robot arm using *Mikusinski*'s operational calculus.

Observer design based on Lyapunov theory is well known and widely used for both linear systems and nonlinear systems. In [10],[18],[19] observer design for flexible-link robot described by ODEs is studied. Balas [2] considered observer design for linear flexible structures described by FEM. Demetriou [7], presented a method for construction of observer for linear second order lumped and distributed parameter systems using parameter-dependent Lyapunov functions. Kristiansen [9] applied *contraction* theory [11] in observer design for a class of linear distributed parameter systems. The damping forces were included in the last two cases. Thus, exponentially stable observers can easily be designed. Here, as opposed to the work of [2],[10],[18],[19], observer design for a flexible-link robot is based on an *infinite-dimensional* model. Recently, the present authors [15] designed an exponentially stable observer for a motorized Euler-Bernoulli beam described by a combination of ODE, PDE and a set of static boundary conditions. The stability of the proposed observer was proven using *semigroup theory*.

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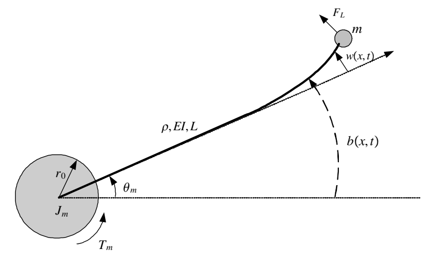

Fig. 1. Flexible robot arm.

In this paper, we extend previous results on regulation to tracking problem and observer design for a one-dimensional beam equation. The beam is clamped to a motor at one end, and attached to a force actuator at the other. We assume that the mass of the force actuator is much smaller than the mass of the motor and the mass of the beam. The dynamics of force actuator at the tip are thus neglected, which is common in literature (e.g. [14]). For this simplified system, we design a uniformly exponentially stable observer, and feedback control laws at the boundaries which allow the beam to follow time-varying references and damp out the elastic displacement of the beam. The existence, uniqueness and stability of solutions of the closed loop system and the proposed observer are based on semigroup theory.

The paper is organized as follows. First, a model for the flexible robot arm is presented. Then, the observer design problem is considered. After that, the planar tracking problem is studied. Finally, simulation results and some concluding remarks are given.

II. System Model

We consider a flexible beam clamped to a motor at one end and attached to a force actuator at the other (Figure 1). The equations for the elastic motion of the system are given as ([3],[15]),

$$\begin{aligned} \rho b_{tt}(x,t) &= -EIw_{xxxx}(x,t), \quad x \in]r_0, L[&& (1)\\ F_L(t) &= -EIw_{xxx}(L,t) && (2)\\ w(r_0,t) &= w_x(r_0,t) = EIw_{xx}(L,t) = 0 && (3) \end{aligned}$$

and the equation of motion for the hub is given by the angular momentum

$$\dot{h}(t) = T_m(t) \tag{4}$$

where

$$\begin{aligned} b(x,t) &= x\theta_m(t) + w(x,t), \quad x \in [r_0, L] && (5)\\ h(t) &= J_m\dot{\theta}_m(t) + \int_{r_0}^{L} \rho x b_t(x,t)\, dx && (6) \end{aligned}$$

$b(x,t)$ denotes the the arc length of the beam at point x and time t, $w(x,t)$ is the elastic displacement of the beam at x and time t, ρ is the mass per unit length of the beam, E is the modulus of elasticity of the beam, I is the area moment of inertia of the beam, r_0 is the clamping location of the beam, L is the length of the beam, θ_m is the angle of the motor, J_m

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is the mass moment of inertia of the motor, $T_m : \mathbb{R}^+ \to \mathbb{R}$ is the boundary control torque generates by the motor and $F_L : \mathbb{R}^+ \to \mathbb{R}$ is the boundary control force generated by the force actuator at the tip of the beam. The subscripts $(\cdot)_t$ and $(\cdot)_x$ denote the partial differential with respect to t and x, respectively. Throughout this paper, the time derivative is also often represented by a *dot*, e.g. $\dot{\theta}_m = d\theta_m/dt$.

Applying (1)-(3), (5)-(6) and integration by parts to (4), we get the equations of motion

$$\rho b_{tt} = -EIw_{xxxx}, \quad x \in]r_0, L[\tag{7}$$

$$J_m\ddot{\theta}_m = -r_0 EI\, w_{xxx}|_{r_0} + EIw_{xx}|_{r_0} - LF_L + T_m \tag{8}$$

$$w|_{r_0} = w_x|_{r_0} = w_{xx}|_L = 0, \quad F_L = -\,EIw_{xxx}|_L \tag{9}$$

In this paper, we consider the following problems:

Problem 1: *Given the system (7)-(9) and measurements: $\theta_m(t)$ and $w(L,t)$, $t \geq 0$. Design an observer for the system.*

Problem 2: *Consider the system (7)-(9). Given the time-varying reference trajectories $\theta_d(t)$, $\dot{\theta}_d(t)$ and $\ddot{\theta}_d(t)$, $t \geq 0$. Assume that $\ddot{\theta}_d(t)$ is exponentially decaying or zero. Find the control laws $F_L(t)$ and $T_m(t)$ such that*

$$\lim_{t\to\infty}\{\theta_m(t), \dot{\theta}_m(t)\} = \{\theta_d(t), \dot{\theta}_d(t)\}$$

$$\lim_{t\to\infty}\{w(x,t), \dot{w}(x,t)\} = 0, \quad x \in [0, L]$$

Remark 1: Exponentially decaying $\ddot{\theta}_d(t)$ can for instance be obtained by an exponentially stable reference model with piecewise constant set-points.

III. Observer Design

Let the measurements be denoted as follows: $y_1(t) = \theta_m(t)$ and $y_2(t) = w(L,t)$, $t \geq 0$. Utilizing the *coordinate error feedback* [11], we propose the observer

$$\rho\dot{\hat{b}} = \rho\dot{\hat{b}} - h_d\left[Ly_1 + y_2\right] \cdot \delta_d(x - L), \quad x \in]r_0, L[\tag{10}$$

$$J_m\dot{\hat{\theta}} = J_m\dot{\hat{\theta}} - H_d y_1 \tag{11}$$

$$\rho\ddot{\hat{b}} = -EI\hat{w}_{xxxx} - h_d\dot{\hat{b}} \cdot \delta_d(x - L), \quad x \in]r_0, L[\tag{12}$$

$$J_m\ddot{\hat{\theta}} = -H_d\dot{\hat{\theta}} - H_p\left(\hat{\theta} - y_1\right) - LF_L + EI\hat{w}_{xx}|_{r_0} - r_0\, EI\hat{w}_{xxx}|_{r_0} + T_m \tag{13}$$

with the boundary conditions

$$\hat{w}|_{r_0} = \hat{w}_x|_{r_0} = \hat{w}_{xx}|_L = 0, \quad F_L = -\,EI\hat{w}_{xxx}|_L \tag{14}$$

where h_d, H_p and H_d are positive observer gains; $\hat{b}$, $\hat{w}$ and $\hat{\theta}$ are the estimates of b, w and θ_m, respectively, and $\delta_d(\cdot)$ denotes the *discrete* Dirac delta function, i.e. $\delta_d(0) = 1$, and $\delta_d(x) = 0$ for $\forall x \neq 0$. Note that the coordinate error feedback has similarities with *Luenberger's linear reduced-order observer* [12].

Applying (10)-(11) to (12)-(13) gives the observer dynamics

$$\rho\hat{b}_{tt} = -EI\hat{w}_{xxxx} - h_d\,\tilde{b}_t\Big|_L\,\delta_d(x - L), x \in]r_0, L[\tag{15}$$

$$J_m\ddot{\hat{\theta}} = -H_d\dot{\tilde{\theta}} - H_p\tilde{\theta} - LF_L + EI\,\hat{w}_{xx}|_{r_0} - r_0 EI\,\hat{w}_{xxx}|_{r_0} + T_m \tag{16}$$

and the boundary conditions

$$\hat{w}|_{r_0} = \hat{w}_x|_{r_0} = \hat{w}_{xx}|_L = 0, \quad F_L = -\,EI\hat{w}_{xxx}|_L \tag{17}$$

where $\tilde{b} = \hat{b} - b$, $\tilde{w} = \hat{w} - w$ and $\tilde{\theta} = \hat{\theta} - \theta_m$ denote the observer errors.

Subtracting (15)-(17) by (7)-(9) gives the observer error dynamics

$$\rho\tilde{b}_{tt} = -EI\tilde{w}_{xxxx} - h_d\,\tilde{b}_t\Big|_L\,\delta_d(x - L), \quad x \in]r_0, L[\tag{18}$$

$$J_m\ddot{\tilde{\theta}} = -H_d\dot{\tilde{\theta}} - H_p\tilde{\theta} - r_0 EI\,\tilde{w}_{xxx}|_{r_0} + EI\,\tilde{w}_{xx}|_{r_0} \tag{19}$$

$$\tilde{w}|_{r_0} = \tilde{w}_x|_{r_0} = \tilde{w}_{xx}|_L = \tilde{w}_{xxx}|_L = 0 \tag{20}$$

Let $\mathbf{q} = (\tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t) = (q_1, q_2, q_3, q_4)$. The observer error dynamics (18)-(20) can be compactly written as

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}, \quad t > 0; \ \mathbf{q}(0) \in H \tag{21}$$

where

$$\mathbf{A}\mathbf{q} = \begin{bmatrix} q_2 \\ -\frac{1}{J_m}(*) \\ q_4 \\ (**) \end{bmatrix}, \quad \forall\mathbf{q} \in D(\mathbf{A})$$

and

$$(*) = H_p q_1 + H_d q_2 + r_0 EI\, q_{3,xxx}|_{r_0} - EI\, q_{3,xx}|_{r_0}$$

$$(**) = -\frac{EI}{\rho}q_{3,xxxx} + \frac{x}{J_m}(*) - \frac{h_d}{\rho}\left(Lq_2 + q_4|_L\right)\delta_d(x - L)$$

Let $\Omega =]r_0, L[$. Define the spaces

$$H = \mathbb{R}^2 \times H_0^2(\Omega) \times L_2(\Omega)$$

$$D(\mathbf{A}) = \left\{\mathbf{q} \in \mathbb{R}^2 \times H_0^4(\Omega) \times H_0^2(\Omega)\Big| \ q_{3,xx}|_L = q_{3,xxx}|_L = 0\right\}$$

where

$$L_2(\Omega) = \{f \mid \int_\Omega |f|^2\, dx < \infty\} \tag{22}$$

$$H_0^k(\Omega) = \left\{f|\ f, f', ..., f^{(k)} \in L_2(\Omega),\ f|_{r_0} = f'|_{r_0} = 0\right\} \tag{23}$$

In H, we define the inner product

$$\langle\mathbf{q}, \mathbf{z}\rangle_H = \int_\Omega EIq_{3,xx}z_{3,xx}\, dx + H_p q_1 z_1 + \int_\Omega \rho\left(xq_2 + q_4\right)\left(xz_2 + z_4\right)dx + J_m q_2 z_2 \tag{24}$$

where $\mathbf{q} = (q_1, \ldots, q_4) \in H$ and $\mathbf{z} = (z_1, \ldots, z_4) \in H$. The energy of (21) can be expressed as

$$\mathcal{E}_{obs} = \frac{1}{2}\langle\mathbf{q}, \mathbf{q}\rangle_H = \frac{1}{2}\|\mathbf{q}\|_H^2 = \frac{1}{2}\int_\Omega EI\tilde{w}_{xx}^2\, dx + \frac{1}{2}H_p\tilde{\theta}^2 + \frac{1}{2}\int_\Omega \rho\left(x\dot{\tilde{\theta}} + \tilde{w}_t\right)^2 dx + \frac{1}{2}J_m\dot{\tilde{\theta}}^2 \tag{25}$$

where $\mathbf{q} = (\tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t) \in H$. It can be verified that $(H, \langle\cdot,\cdot\rangle_H)$ is a Hilbert space. We have the result:

Theorem 1: Consider the abstract problem (21). The operator $\mathbf{A}$ generates a C_0-semigroup $\{e^{\mathbf{A}t}\}_{t\geq0}$ of contractions on H. The strong solution of (21) is exponentially stable for $\forall\mathbf{q}(0) \in D(\mathbf{A})$.

Proof: To show the first assertion, we apply the *Lumer-Phillips* theorem (see e.g. [16]). The time derivative of (25) along the solution trajectories of (21) is

$$\dot{\mathcal{E}}_{obs} = -H_d\dot{\tilde{\theta}}^2 - h_d\tilde{b}_t(L)^2 \leq 0 \tag{26}$$

where integration by parts has been successively applied. Hence, $\mathbf{A}$ is dissipative.

To show that the range of the operator $\lambda\mathbf{I}-\mathbf{A}$ is onto H for some $\lambda>0$, we will first argue that $\mathbf{A}$ is compact. Let $\mathbf{g}=(g_1,\ldots,g_4)\in H$ be given. Consider the equation

$$\mathbf{A}\mathbf{q}=\mathbf{g} \tag{27}$$

It can be verified that the solution of (27) is

$$\begin{aligned} q_1 &= -\frac{H_d}{H_p}g_1-\frac{J_m}{H_p}g_2+\frac{EI}{H_p}\left(q_{3,xx}|_{r_0}-r_0\ q_{3,xxx}|_{r_0}\right)\\ q_2 &= g_1\\ q_3(x) &= -\frac{\rho}{EI}\int_{r_0}^{x}\int_{r_0}^{\xi_1}\int_{r_0}^{\xi_2}\int_{r_0}^{\xi_3} g_4(\xi_4)\,d\xi_4 d\xi_3 d\xi_2 d\xi_1\\ &\quad -\frac{x^3}{3!}\frac{h_d}{EI}\left(Lg_1+g_3|_L\right)\\ &\quad -\frac{x^5}{5!}\frac{\rho}{EI}g_2+\sum_{i=0}^{3}c_i x^i\ ,\quad x\in[r_0,L]\\ q_4(x) &= g_3(x),\qquad x\in[r_0,L] \end{aligned}$$

where $c_0,\ldots,c_3$ are uniquely determined by the boundary conditions (20). Hence, (27) has a unique solution $\mathbf{q}\in D(\mathbf{A})$. It follows that $\mathbf{A}^{-1}$ exists and maps H into $\mathbb{R}^2\times H_0^4(\Omega)\times H_0^2(\Omega)$. Moreover, since $\mathbf{A}^{-1}$ maps every bounded set of H into bounded sets of $\mathbb{R}^2\times H_0^4(\Omega)\times H_0^2(\Omega)$, and the embedding of the latter space into H is compact (see e.g. Lemma 1.2.2, p. 14, [17]), it follows that $\mathbf{A}^{-1}$ is compact. Note that this also proves that the spectrum of $\mathbf{A}$ consists only of isolated eigenvalues (see e.g. p. 187, [8]), which implies that $(\lambda\mathbf{I}-\mathbf{A})^{-1}:H\to H$ is compact for any λ in the resolvent set of $\mathbf{A}$. Consider now the equation

$$(\lambda\mathbf{I}-\mathbf{A})\,\mathbf{q}=\mathbf{A}(\lambda\mathbf{A}^{-1}-\mathbf{I})\mathbf{q}=\mathbf{g} \tag{28}$$

for some given $\mathbf{g}\in H$. By *contraction mapping theorem*, it follows that (28) has a unique solution $\mathbf{q}\in D(\mathbf{A})$ for $0<\lambda<||\mathbf{A}^{-1}||^{-1}$. Thus, $\lambda\mathbf{I}-\mathbf{A}:H\to H$ is thus onto for $0<\lambda<||\mathbf{A}^{-1}||^{-1}$. By (Th. 4.5, p. 15, [16]), $\lambda\mathbf{I}-\mathbf{A}:H\to H$ is onto for all $\lambda>0$.

Since $(H,\langle\cdot,\cdot\rangle_H)$ is a Hilbert space, it follows from (Th. 4.6, p. 16, [16]) that $D(\mathbf{A})$ is dense in H, i.e. $\overline{D(\mathbf{A})}=H$. Hence, $\mathbf{A}$ generates a C_0-semigroup of contractions on H.

To show the last assertion, we use a combination of the energy multipliers method and (Th. 4.1, p. 116, [16]). Define

$$\mathcal{V}_{obs}(t)=2\,(1-\varepsilon)\,t\mathcal{E}_{obs}(t)+\mathcal{U}_{obs}(t) \tag{29}$$

where $\varepsilon\in\,]0,1[$ is an arbitrary constant, $\mathcal{E}_{obs}$ is given by (25), and

$$\mathcal{U}_{obs}=2\int_\Omega \rho x\tilde{b}_t\tilde{b}_x\ dx+2J_m\tilde{\theta}\dot{\tilde{\theta}} \tag{30}$$

In the sequel, the following inequalities are frequently used

$$\begin{aligned} ab &\le (\gamma a)^2+\left(\frac{b}{\gamma}\right)^2,\quad \gamma\in\mathbb{R}\backslash\{0\} && (31)\\ (a+b)^2 &\le 2\left(a^2+b^2\right) && (32) \end{aligned}$$

for $\forall a,b\in\mathbb{R}$, and

$$f_x(x,t)\le\left[L\int_{r_0}^{L}|f_{xx}|^2\,dx\right]^{\frac12},\quad \forall x\in[r_0,L] \tag{33}$$

for $\forall f_x\in H_0^1(\Omega)$. Applying (31)-(33) to (30), there exists a constant $C>0$ such that

$$|\mathcal{U}_{obs}(t)|\le C\mathcal{E}_{obs}(t),\quad \forall t\ge 0$$

Hence, the following holds

$$[2\,(1-\varepsilon)\,t-C]\,\mathcal{E}_{obs}(t)\le\mathcal{V}_{obs}(t)\le[2\,(1-\varepsilon)\,t+C]\,\mathcal{E}_{obs}(t) \tag{34}$$

for $\forall t\ge 0$.

Next, differentiation of (29) with respect to time along the solution trajectories of (21) gives

$$\dot{\mathcal{V}}_{obs}(t)=2\,(1-\varepsilon)\,\mathcal{E}_{obs}(t)+2\,(1-\varepsilon)\,t\dot{\mathcal{E}}_{obs}(t)+\dot{\mathcal{U}}_{obs}(t) \tag{35}$$

where $\mathcal{E}_{obs}$ and $\dot{\mathcal{E}}_{obs}$ are given by (25) and (26), respectively, and

$$\dot{\mathcal{U}}_{obs}=\dot{\mathcal{U}}_{1,obs}+\dot{\mathcal{U}}_{2,obs}+\dot{\mathcal{U}}_{3,obs} \tag{36}$$

where

$$\begin{aligned} \dot{\mathcal{U}}_{1,obs} &= 2\int_\Omega \rho x\tilde{b}_{tt}\tilde{b}_x\ dx\\ \dot{\mathcal{U}}_{2,obs} &= 2\int_\Omega \rho x\tilde{b}_t\tilde{b}_{xt}\ dx\\ \dot{\mathcal{U}}_{3,obs} &= 2J_m\dot{\tilde{\theta}}^2+2J_m\tilde{\theta}\ddot{\tilde{\theta}} \end{aligned}$$

Consider now these terms separately.

$\underline{\dot{\mathcal{U}}_{1,obs}}$:

$$\begin{aligned} \dot{\mathcal{U}}_{1,obs} &\le 2EI\ \tilde{\theta}\ \left[r_0\tilde{w}_{xxx}(r_0)-\tilde{w}_{xx}(r_0)\right]-r_0EI\tilde{w}_{xx}(r_0)^2\\ &\quad +2h_dL\gamma_1^2\ \tilde{\theta}^2+2h_dL\left[\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}\right]\ \tilde{b}_t(L)^2\\ &\quad +2h_dL^2\gamma_2^2\int_{r_0}^{L}\tilde{w}_{xx}^2\ dx-3\int_{r_0}^{L}EI\tilde{w}_{xx}^2\ dx \end{aligned}$$

for $\forall\gamma_1,\gamma_2\in\mathbb{R}\backslash\{0\}$, where (31), (33) and integration by parts have been successively applied.

$\underline{\dot{\mathcal{U}}_{2,obs}}$:

$$\dot{\mathcal{U}}_{2,obs}=\rho L\tilde{b}_t(L)^2-\rho r_0\tilde{b}_t(r_0)^2-\int_{r_0}^{L}\rho\tilde{b}_t^2\ dx$$

$\underline{\dot{\mathcal{U}}_{3,obs}}$:

$$\begin{aligned} \dot{\mathcal{U}}_{3,obs} &\le 2J_m\dot{\tilde{\theta}}^2-2EI\ \tilde{\theta}\ \left[r_0\tilde{w}_{xxx}(r_0)-\tilde{w}_{xx}(r_0)\right]\\ &\quad -2H_p\tilde{\theta}^2+2H_d\left[\left(\gamma_3\tilde{\theta}\right)^2+\left(\frac{\dot{\tilde{\theta}}}{\gamma_3}\right)^2\right] \end{aligned}$$

for $\forall\gamma_3\in\mathbb{R}\backslash\{0\}$, where (31) has been applied. Hence,

$$\begin{aligned} \dot{\mathcal{V}}_{obs} &\le -\left[2+\varepsilon-\frac{2h_dL^2}{EI}\gamma_2^2\right]\ \int_{r_0}^{L}EI\tilde{w}_{xx}^2\ dx\\ &\quad -\left[(1+\varepsilon)\,H_p-2h_dL\gamma_1^2-2H_d\gamma_3^2\right]\ \tilde{\theta}^2\\ &\quad -\left[2\,(1-\varepsilon)\,H_dt-(3-\varepsilon)\,J_m-\frac{2H_d}{\gamma_3^2}\right]\dot{\tilde{\theta}}^2\\ &\quad -\left[2\,(1-\varepsilon)\,h_dt-\rho L-\frac{2h_dL}{\gamma_1^2}-\frac{2h_dL}{\gamma_2^2}\right]\tilde{b}_t(L)^2\\ &\quad -r_0EI\tilde{w}_{xx}(r_0)^2-\rho r_0\tilde{b}_t(r_0)^2-\varepsilon\int_{r_0}^{L}\rho\tilde{b}_t^2dx \end{aligned} \tag{37}$$

Let $\varepsilon\in\,]0,1[$ be fixed. By choosing $\gamma_1,\gamma_2,\gamma_3$ sufficiently small, the first two terms of (37) become negative for $\forall t\ge 0$. Hence, the following holds

$$\dot{\mathcal{V}}_{obs}(t)\le 0,\quad t\ge t_1 \tag{38}$$

for sufficiently large time t_1,

$$t_1=\max\left\{\frac{3J_m+\frac{2H_d}{\gamma_3^2}}{2\,(1-\varepsilon)\,H_d},\frac{2h_dL\left(\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}\right)+\rho L}{2\,(1-\varepsilon)\,h_d}\right\}$$

By (26), (34) and (38), we have

$$\mathcal{E}_{obs}(t)\le\frac{C}{2\,(1-\varepsilon)\,t-C}\mathcal{E}_{obs}(0),\quad t\ge t_{\max}$$

where

$$t_{\max} = \max\left\{t_1, \frac{C}{2(1-\varepsilon)}\right\}$$

Since $\mathcal{E}_{obs}(t) = \frac{1}{2}||\mathbf{q}(t)||_H^2$, it follows that $||\mathbf{q}(t)||_H < \infty$ for $\forall t \geq 0$ and decays as $O\left(1/\sqrt{t}\right)$ for sufficiently large time. Thus,

$$\int_0^\infty \|\mathbf{q}(t)\|_H^{2p}\, dt = \int_0^\infty \left\|e^{\mathbf{A}t}\mathbf{q}(0)\right\|_H^{2p} dt < \infty$$

for $\forall p > 1$ and $\forall \mathbf{q}(0) \in D(\mathbf{A})$. By density of $D(\mathbf{A})$ in H, the following also holds

$$\int_0^\infty \|\mathbf{q}(t)\|_H^{2p}\ dt < \infty\ , \quad \forall \mathbf{q}(0) \in H$$

for $\forall p > 1$. According to (Th. 4.1, p. 116, [16]), there exist $M \geq 1$ and $\mu > 0$ such that

$$\|\mathbf{q}(t)\|_H \leq Me^{-\mu t}\|\mathbf{q}(0)\|_H\ , \quad \forall \mathbf{q}(0) \in H \tag{39}$$

for $\forall t \geq 0$. This completes the proof. ■

Remark 2: The normal form implies uniform observability, i.e. the origin $(\tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t) = 0$ of the observer error dynamics (18)-(20) is uniformly exponentially stable. This is verified by simulation results below.

IV. Control Formulation

Consider now the Problem 2. Let the control laws be

$$F_L(t) = -k_d\ \dot{w}\Big|_L \tag{40}$$

$$T_m(t) = J_m\ddot{\theta}_d - K_d\left(\dot{\hat{\theta}} - \dot{\theta}_d\right) - K_p(y_1 - \theta_d) + LF_L \tag{41}$$

where k_d, K_p and K_d are positive controller gains. The control laws (40)-(41) are a slight extension of previous proposed controllers for the orientating and stabilizing problem of the beam attached to a rigid body (e.g. [5],[14],[15]). Insertion of (40)-(41) into (7)-(9) gives the error dynamics

$$\begin{aligned} J_m\ddot{\theta}_e &= -K_d\dot{\theta}_e - K_p\theta_e \\ &\quad -K_d\dot{\tilde{\theta}} - r_0 EI\ w_{xxx}|_{r_0} + EI\ w_{xx}|_{r_0} \end{aligned} \tag{42}$$

$$\rho w_{tt} = -EIw_{xxxx} - \rho x\ddot{\theta}_e - \rho x\ddot{\theta}_d \tag{43}$$

and the boundary conditions

$$EIw_{xxx}|_L = k_d\ w_t|_L + k_d\ \tilde{w}_t|_L \tag{44}$$

$$w|_{r_0} = w_x|_{r_0} = w_{xx}|_L = 0 \tag{45}$$

where $\theta_e = \theta_m - \theta_d$ denotes the control error. To show that the equilibrium $(\theta_e, \dot{\theta}_e, w, w_t, \tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t) = 0$ of the closed loop system (18)-(20) and (42)-(45) is stable, the semigroup theory will again be applied.

Let $\mathbf{w} = (\theta_e, \dot{\theta}_e, w, w_t, \tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t) = (w_1, \ldots, w_8)$. Equations (18)-(20) and (42)-(45) can be written as

$$\dot{\mathbf{w}} = \mathcal{A}\mathbf{w} + \mathbf{f}(t)\ , \quad t > 0;\ \mathbf{w}(0) \in \mathcal{H} \tag{46}$$

where

$$\mathcal{A}\mathbf{w} = \begin{bmatrix} w_2 \\ -\frac{1}{J_m}(*) \\ w_4 \\ -\frac{EI}{\rho}w_{3,xxxx} + \frac{x}{J_m}(*) \\ w_6 \\ -\frac{1}{J_m}(**) \\ w_8 \\ (***) \end{bmatrix},\ \forall \mathbf{w} \in D(\mathcal{A})$$

$$\mathbf{f}(t) = \left[0, 0, 0, -x\ddot{\theta}_d(t), 0, 0, 0, 0\right]^\top$$

and

$$\begin{aligned} (*) &= K_p w_1 + K_d w_2 + K_d w_6 \\ &\quad + r_0\ EIw_{3,xxx}|_{r_0} - EIw_{3,xx}|_{r_0} \\ (**) &= H_p w_5 + H_d w_6 \\ &\quad + r_0\ EIw_{7,xxx}|_{r_0} - EIw_{7,xx}|_{r_0} \\ (***) &= -\frac{EI}{\rho}w_{7,xxxx} + \frac{x}{J_m}(**) \\ &\quad -\frac{h_d}{\rho}\left(Lw_6 + w_8|_L\right)\cdot\delta_d(x - L) \end{aligned}$$

Define the spaces

$$\begin{aligned} \mathcal{H} &= \mathbb{R}^2 \times H_0^2(\Omega) \times L_2(\Omega) \times \mathbb{R}^2 \times H_0^2(\Omega) \times L_2(\Omega) \\ D(\mathcal{A}) &= \big\{\mathbf{w} \in \mathbb{R}^2 \times H_0^4(\Omega) \times H_0^2(\Omega) \\ &\quad \times \mathbb{R}^2 \times H_0^4(\Omega) \times H_0^2(\Omega)\big| \\ &\quad EIw_{3,xxx}|_L = k_d\ w_4|_L + k_d\ w_8|_L \\ &\quad w_{3,xx}|_L = w_{7,xx}|_L = w_{7,xxx}|_L = 0\big\} \end{aligned}$$

where $L_2(\Omega)$ and $H_0^k(\Omega)$ are given by (22) and (23), respectively. In $\mathcal{H}$, we define the inner-product

$$\begin{aligned} \langle \mathbf{w}, \mathbf{z}\rangle_{\mathcal{H}} &= \int_\Omega EIw_{3,xx}z_{3,xx}\ dx + K_p w_1 z_1 \\ &\quad + \int_\Omega \rho(xw_2 + w_4)(xz_2 + z_4)\, dx + J_m w_2 z_2 \\ &\quad + \langle (w_5, \ldots, w_8), (z_5, \ldots, z_8)\rangle_H \end{aligned}$$

where the inner-product $\langle \cdot, \cdot\rangle_H$ is given by (24), $\mathbf{w} = (w_1, \ldots, w_8) \in \mathcal{H}$ and $\mathbf{z} = (z_1, \ldots, z_8) \in \mathcal{H}$. The energy of (46) can be expressed as

$$\mathcal{E} = \frac{1}{2}\langle \mathbf{w}, \mathbf{w}\rangle_{\mathcal{H}} = \mathcal{E}_{CL} + \mathcal{E}_{obs}, \quad \forall \mathbf{w} \in \mathcal{H} \tag{47}$$

where $\mathcal{E}_{obs}$ is given by (25), and

$$\begin{aligned} \mathcal{E}_{CL} &= \frac{1}{2}\int_\Omega EIw_{xx}^2 dx + \frac{1}{2}K_p\theta_e^2 \\ &\quad + \frac{1}{2}\int_\Omega \rho\left(x\dot{\theta}_e + w_t\right)^2 dx + \frac{1}{2}J_m\dot{\theta}_e^2 \end{aligned} \tag{48}$$

It can be verified that $(\mathcal{H}, \langle \cdot, \cdot\rangle_{\mathcal{H}})$ is a Hilbert space.

Let $k_d, H_p, K_p > 0$ be given. Choose the gains according to

$$K_d > k_d L^2 > 0 \tag{49}$$

$$h_d > \max\left\{\frac{K_d^2}{2L^2(K_d - k_d L^2)}, \frac{k_d K_d + (k_d L)^2}{2K_d}\right\} \tag{50}$$

$$H_d > \frac{2K_d(h_d L)^2}{2K_d h_d - K_d k_d - (k_d L)^2} > 0 \tag{51}$$

Then we have the following result:

Theorem 2: Let $h_d, k_d, H_d, K_d > 0$ be given by (49)-(51). Then, $\mathcal{A}$ generates a C_0-semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ of contractions on $\mathcal{H}$, and the semigroup is exponentially stable.

Proof: Using Lumer-Phillips theorem, it is straightforward to show that $\mathcal{A}$ generates a C_0-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ of contractions on $\mathcal{H}$. Note that the time derivative of (47) along the solution trajectories of (46) (with $\mathbf{f} = 0$) is

$$\begin{aligned} \dot{\mathcal{E}} &= \dot{\mathcal{E}}_{obs} - K_d\dot{\theta}_e^2 - K_d\dot{\theta}_e\dot{\tilde{\theta}} \\ &\quad -k_d\left(L\dot{\theta}_e + w_t|_L\right)\left(\tilde{w}_t|_L + w_t|_L\right) \end{aligned}$$

where $\dot{\mathcal{E}}_{obs}$ is given by (26). This can be rewritten as

$$\dot{\mathcal{E}} = -\frac{1}{2}\mathbf{q}^\top \mathbf{Q}\mathbf{q}$$

where

$$\mathbf{Q} = \begin{bmatrix} K_d & k_d L & K_d & 0 \\ k_d L & k_d & 0 & 0 \\ K_d & 0 & 2h_d L^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} K_d & 0 & 0 & k_d L \\ 0 & k_d & 0 & k_d \\ 0 & 0 & 2H_d & 2h_d L \\ k_d L & k_d & 2h_d L & 2h_d \end{bmatrix}$$

and $\mathbf{q} = (\dot{\theta}_e, w_t|_L, \dot{\tilde{\theta}}, \tilde{w}_t|_L)$. It can be verified that $\mathbf{Q} > 0$. Hence, there exists a constant $\lambda > 0$ such that

$$\dot{\mathcal{E}} \leq -\lambda\dot{\theta}_e^2 - \lambda w_t(L)^2 - \lambda\dot{\tilde{\theta}}^2 - \lambda\tilde{w}_t(L)^2 \leq 0 \tag{52}$$

Thus, $\mathcal{A}$ is dissipative.

To show that $\{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable, we apply again the energy multipliers method and (Th. 4.1, p. 116, [16]). Define the functional

$$\mathcal{V}(t) = \mathcal{V}_{CL}(t) + \mathcal{V}_{obs}(t) \tag{53}$$

where $\mathcal{V}_{obs}$ is given by (29), and

$$\mathcal{V}_{CL}(t) = 2(1-\varepsilon)\, t\mathcal{E}_{CL}(t) + \mathcal{U}_{CL}(t) \tag{54}$$

$\varepsilon \in]0,1[$ is an arbitrary constant, $\mathcal{E}_{CL}$ is given by (48), and

$$\mathcal{U}_{CL} = 2\int_\Omega \rho x\left(x\dot{\theta}_e + w_t\right)(\theta_e + w_x)\,dx + 2J_m\dot{\theta}_e\theta_e \tag{55}$$

Applying (31)-(33) to (53), there exists a constant $\mathtt{C} > 0$ such that the following holds

$$[2(1-\varepsilon)t - \mathtt{C}]\,\mathcal{E}(t) \leq \mathcal{V}(t) \leq [2(1-\varepsilon)t + \mathtt{C}]\,\mathcal{E}(t) \tag{56}$$

for $\forall t \geq 0$. By successively application of integration by parts and (31)-(33), it can be shown that the time derivative of (53) satisfies the inequality

$$\begin{aligned}
\dot{\mathcal{V}} \;\leq\; & -\left[2+\varepsilon-\frac{2h_d L^2}{EI}\gamma_2^2\right]\int_{r_0}^L EI\tilde{w}_{xx}^2\,dx \\
& -\left[(1+\varepsilon)H_p - 2h_d L\gamma_1^2 - 2H_d\gamma_3^2\right]\tilde{\theta}^2 \\
& -\Bigg[2(1-\varepsilon)\lambda t - (3-\varepsilon)J_m - \frac{2H_d}{\gamma_3^2} \\
& -\frac{2K_d}{\gamma_9^2} - 2L^2\left(\rho L + 2h_d L\left(\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}\right)\right)\Bigg]\dot{\tilde{\theta}}^2 \\
& -\Bigg[2(1-\varepsilon)\lambda t - 2Lk_d\left(\frac{1}{\gamma_6^2}+\frac{1}{\gamma_7^2}\right) \\
& -2\left(\rho L + 2h_d L\left(\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}\right)\right)\Bigg]\tilde{w}_t(L)^2 \\
& -r_0 EI\tilde{w}_{xx}(r_0)^2 - \rho r_0\tilde{b}_t(r_0)^2 - \varepsilon\int_{r_0}^L \rho\tilde{b}_t\,dx \\
& -\left[2+\varepsilon-\frac{2k_d L^2}{EI}\left(\gamma_5^2+\gamma_7^2\right)\right]\int_{r_0}^L EIw_{xx}^2\,dx \\
& -\left[(1+\varepsilon)K_p - 2Lk_d\left(\gamma_4^2+\gamma_6^2\right) - 2K_d\left(\gamma_8^2+\gamma_9^2\right)\right]\theta_e^2 \\
& -\left[2(1-\varepsilon)\lambda t - (3-\varepsilon)J_m - 2\rho L^3 - \frac{2K_d}{\gamma_8^2}\right]\dot{\theta}_e^2 \\
& -\left[2(1-\varepsilon)\lambda t - 2\rho L - 2Lk_d\left(\frac{1}{\gamma_4^2}+\frac{1}{\gamma_5^2}\right)\right]w_t(L)^2 \\
& -r_0 EIw_{xx}(r_0)^2 - \rho r_0^3\dot{\theta}_e^2 - \varepsilon\int_{r_0}^L \rho\left[x\dot{\theta}_e + w_t\right]^2 dx
\end{aligned}$$

for $\forall\gamma_1,\ldots\gamma_9 \in \mathbb{R}\setminus\{0\}$ and $\forall t \geq 0$. Again, let $\varepsilon \in]0,1[$ be fixed, and choose $\gamma_1,\ldots,\gamma_9$ *sufficiently* small such that the following holds

$$\dot{\mathcal{V}}(t) \leq 0, \quad t \geq \mathtt{t}_{\max} \tag{57}$$

for *sufficiently* large time $\mathtt{t}_{\max} > 0$. Using the same argumentation as in the proof of Theorem 1, we get

$$\|\mathbf{w}(t)\|_{\mathcal{H}} \leq \mathtt{K}e^{-\kappa t}\|\mathbf{w}(0)\|_{\mathcal{H}}, \quad \forall\mathbf{w}(0) \in \mathcal{H} \tag{58}$$

for some constants $\mathtt{K} \geq 1$ and $\kappa > 0$. ■

Proposition 3: The abstract problem (46) has a unique strong solution for $\forall\mathbf{w}(0) \in D(\mathcal{A})$, and the unique strong solution tends exponentially to zero.

Proof: Since $\ddot{\theta}_d$ is exponentially decaying or zero, there exist $\mathtt{C} > 0$ and $\upsilon > 0$ such that, $|\ddot{\theta}_d(t)| \leq \mathtt{C}e^{-\upsilon t}, \forall t \geq 0$. Thus, $\mathbf{f} : [0,\infty[\to \mathcal{H}$ is continuous and strong continuous derivative on $[0,\infty[$. Hence, it follows from standard results of semigroup theory (see e.g. Th. 1.2, p. 184, [16]) that (46) has a unique strong solution $\mathbf{w}(t)$ defined on $t \in [0,\infty[$. Since every strong solution is also a weak solution, the strong solution $\mathbf{w}(t)$ of (46) satisfies the integral equation

$$\mathbf{w}(t) = e^{\mathcal{A}t}\mathbf{w}(0) + \int_0^t e^{\mathcal{A}(t-s)}\mathbf{f}(s)\,ds, \quad t \geq 0 \tag{59}$$

where $\{e^{\mathcal{A}t}\}_{t\geq 0}$ is the C_0-semigroup of contractions generated by $\mathcal{A}$. For the case $\upsilon \neq \kappa$, we have

$$\begin{aligned}
\|\mathbf{w}(t)\|_{\mathcal{H}} &\leq \left\|e^{\mathcal{A}t}\mathbf{w}(0)\right\|_{\mathcal{H}} + \int_0^t \left\|e^{\mathcal{A}(t-s)}\mathbf{f}(s)\right\|_{\mathcal{H}} ds \\
&\leq \mathtt{K}e^{-\kappa t}\|\mathbf{w}(0)\|_{\mathcal{H}} + \frac{\mathtt{CK}}{\kappa-\upsilon}\left(e^{-\upsilon t} - e^{-\kappa t}\right)
\end{aligned} \tag{60}$$

for $\forall t \geq 0$. Obviously, $||\mathbf{w}(t)||_{\mathcal{H}}$ tends exponentially to zero as $t \to \infty$ for $\forall\mathbf{w}(0) \in \mathcal{H}$. Similarly, for $\upsilon = \kappa$. ■

Remark 3: If the desired angular acceleration $\ddot{\theta}_d(t)$ is bounded, but not exponentially decaying or zero, then it follows from the analysis above that $||\mathbf{w}(t)||_{\mathcal{H}}$ is bounded; but $||\mathbf{w}(t)||_{\mathcal{H}}$ does not tend to zero. This is verified by the simulation results below.

V. Simulation

To simulate the system (7)-(9), with the feedback control laws (40)-(41) and the proposed observer (10)-(14), the finite-element method with hermitian basis functions has been applied. The beam was divided into 10 elements. The system parameters used in the simulations are: $L = 1$ [m], $\rho = 2.43$ [kg/m], $E = 70 \times 10^9$ [N/m^2], $I = 6.75 \times 10^{-8}$ [m^4], $r_0 = 0.1$ [m], $J_m = 0.5$ [kgm^2]. The controller gains and the observer gains used in simulations are: $K_p = 80$, $K_d = 50$, $k_d = 10$, $H_p = 100$, $H_d = 50$, $h_d = 40$. The initial conditions for the plant and observer are: $\theta_m(0) = 0$, $\hat{\theta}(0) = -15^0$, $\dot{\theta}_m(0) = \dot{\hat{\theta}}(0) = 0$, and $w(x,0) = w_t(x,0) = \hat{w}(x,0) = \hat{w}_t(x,0) = 0$, $x \in [r_0, L]$. We turned on the observer at time $t = 2$ seconds.

The simulation results for a sine reference signal, with amplitude 30^0 and frequency 1 [rad/sec] are shown in Figure 2-5. The reference trajectory θ_d, the angle of the motor θ_m and the estimate of θ_m are shown in Figure 2. The elastic displacement of the beam and the elastic displacement of the observer at $x = L$ are shown in Figure 3. The observer error of $w(x,t)$ at the nodes 2, 5, 8 and 11 are shown in Figure 4. We observer that the vibrations are damped quickly out, and the observer converges as expected exponentially to the plant. As remarked earlier, since $\ddot{\theta}_d$ is not decaying, the elastic displacement of the beam does not tend to zero, but oscillate with the same frequency as the reference trajectory θ_d after the transient vibrations are damped out (Figure 3).

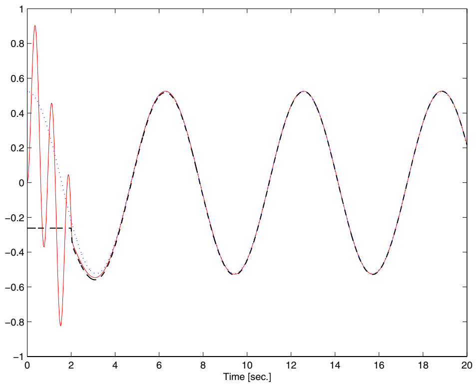

Fig. 2. θ_m [—], θ_d [$\cdots$] and $\hat{\theta}_m$ [- -].

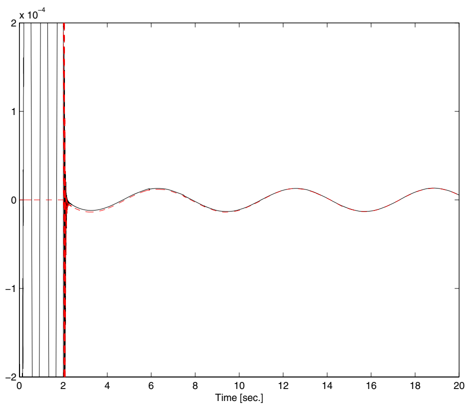

Fig. 3. $w(L,t)$ [—] and $\hat{w}(L,t)$ [- -].

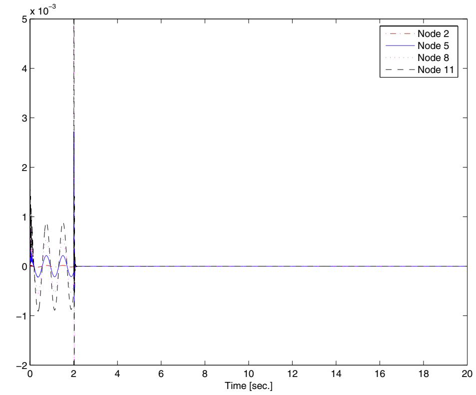

Fig. 4. Observer error at the nodes 2, 5, 8 and 11.

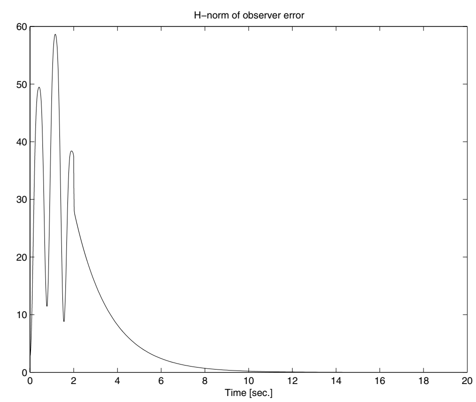

Fig. 5. H-norm of the observer error vector $\mathbf{q} = (\tilde{\theta}, \dot{\tilde{\theta}}, \tilde{w}, \tilde{w}_t)$.

VI. Conclusions

In this paper, we studied the planar tracking problem and the observer design for a flexible robot arm. The robot arm is modeled as an Euler-Bernoulli beam. The beam is clamped to a motor at one end and attached to a force actuator at the other. Based on measurements at the boundaries, a uniformly exponentially stable observer is proposed. Using the information from the observer and the measurements, a tracking controller is designed. The existence, uniqueness and stability of solutions of the observer and the closed loop system are based on semigroup theory. Numerical simulation results are included to illustrate the performance of the proposed control laws and the proposed observer. The simulation results are in agreement with the theoretical results.

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