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Passive Identifiers for Boundary Adaptive Control of 3D Reaction-Advection-Diffusion PDEs

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Abstract— **The prevalent identification technique in existing results on adaptive control for PDEs is the "passive," also known as "observer-based" approach. However, it has so far not been used in boundary control problems. In this paper we prove a separation principle/certainty equivalence result for a class of backstepping boundary controllers applied to a class of unstable reaction-advection-diffusion PDEs in 3D.**

I. INTRODUCTION

The prevalent identification technique in existing results [1], [2], [7], [16], [19] on adaptive control for PDEs is the "passive," also known as "observer-based" approach. This approach is appealing due to its simplicity—it employs an observer in the form of a copy of the plant, plus a stabilizing error term—however, it has so far not been used in boundary control problems. In this paper we study the boundary control problem for a class of unstable 3D reaction-advection-diffusion PDEs with unknown coefficients. We employ explicit controllers designed in [18]. We make those controllers adaptive by substituting the parameter estimates from the identifier into the control law. Adaptive controllers designed in this way are referred to as "certainty equivalence." Stability of such controllers is a highly non-trivial question because the parameter estimates make the adaptive controller nonlinear even when the PDE plant is linear. In this paper we prove the separation principle for the adaptive controller consisting of the passive identifier and the backstepping boundary controller.

A 1D version of the same class of systems is considered in [12] using a Lyapunov approach. While the Lyapunov approach necessitates the use of parameter projection and low adaptation gain, such restrictions are not needed with the passive identifiers.

Early works on adaptive control of infinite-dimensional systems were for plants stabilizable by non-identifier based high gain feedback [15], under a relative degree one assumption. State-feedback model reference adaptive control (MRAC) was extended to PDEs in [7], [2], [19], [16], [1] but not for the case of boundary control. Efforts in [5], [20] made use of positive realness assumptions where relative degree one is implicit. Stochastic adaptive LQR with least-squares parameter estimation and state feedback was pursued in [6]. Adaptive control of nonlinear PDEs was studied in [14], [10]. Adaptive controllers for nonlinear systems on lattices were designed in [9]. An experimentally validated adaptive boundary controller for a flexible beam was presented in [4].

Throughout the paper we assume well posedness of the closed loop systems in the interest of space and due to the parabolic character of the system which ensures it. An example on how one would handle it is given in [11].

The paper consists of two parts. First we explore a 1D PDE with only a reaction coefficient unknown to illustrate the methodology of control and identifier design and the proof idea. Then we apply the method to a 3D PDE with all of its parameters unknown.

a) Notation.: The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The temporal norms are denoted by $\mathcal{L}_\infty$ and $\mathcal{L}_2$ for $t \geq 0$. We denote by l_1 a generic function in $\mathcal{L}_1$. The symbols $I_1(\cdot)$, $J_1(\cdot)$ denote the corresponding Bessel functions.

II. ONE DIMENSIONAL HEAT EQUATION

In this section we consider a simple plant to illustrate the main ideas of our approach in a narrative way without extensive notation.

Consider a one-dimensional unstable heat equation

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \tag{1}$$
$$u(0,t) = 0 \tag{2}$$
$$u(1,t) = U(t)\,, \tag{3}$$

with one unknown parameter λ. Our objective is to regulate the state of this system to zero from the boundary with Dirichlet actuation $U(t)$. For $U(t) = 0$ this system can have an arbitrarily large number of unstable eigenvalues.

Denote the estimate of λ by $\hat\lambda$ and consider the following identifier [1]

$$\hat u_t = \hat u_{xx} + \hat\lambda u + \gamma^2 (u - \hat u)\int_0^1 u^2(x)\,dx \tag{4}$$
$$\hat u(0) = 0 \tag{5}$$
$$\hat u(1) = u(1)\,. \tag{6}$$

Such identifiers are often called "observers", but in fact they are not used for state estimation. This identifier employs a copy of the PDE plant and an additional nonlinear term. The term "passive identifier" comes from the fact that an operator from the parameter estimation error $\tilde\lambda = \lambda - \hat\lambda$ to the inner product of u with $u - \hat u$ is strictly passive. The additional term in (4) acts as nonlinear damping whose task is to ensure square integrability of $\dot{\hat\lambda}$ (i.e. in our notation $\dot{\hat\lambda} \in \mathcal{L}_2$). This

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[1] To reduce notational burden we suppress time dependence everywhere and x-dependence where it does not lead to a confusion.

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slows down the adaptation and serves as an alternative to update law normalization.

Consider the error signal $e = u - \hat{u}$ which satisfies the following PDE

$$
\begin{aligned}
e_t &= e_{xx} + \tilde{\lambda} u - \gamma^2 e \|u\|^2 && (7)\\
e(0) &= 0 && (8)\\
e(1) &= 0\,. && (9)
\end{aligned}
$$

With a Lyapunov function

$$V = \frac{1}{2}\int_0^1 e^2(x)\,dx + \frac{\tilde{\lambda}^2}{2\gamma}, \tag{10}$$

we get

$$\dot{V} = -\|e_x\|^2 - \gamma^2\|e\|^2\|u\|^2 + \tilde{\lambda}\int_0^1 e(x)u(x)\,dx - \frac{\tilde{\lambda}\dot{\hat{\lambda}}}{\gamma}. \tag{11}$$

With the update law

$$\dot{\hat{\lambda}} = \gamma\int_0^1 (u(x) - \hat{u}(x))u(x)\,dx\,, \tag{12}$$

we obtain

$$\dot{V} \le -\|e_x\|^2 - \gamma^2\|e\|^2\|u\|^2\,, \tag{13}$$

which implies $V(t) \le V(0)$ so that $\tilde{\lambda}$ and $\|e\|$ are bounded. Integrating (13) with respect to time from zero to infinity we get the properties $\|e_x\|$, $\|e\|\|u\| \in \mathcal{L}_2$. From the update law (12) we get $|\dot{\hat{\lambda}}| \le \gamma\|e\|\|u\|$ and so $\dot{\hat{\lambda}} \in \mathcal{L}_2$.

For the case of known λ, the following control method has been proposed in [18]: use a transformation

$$
\begin{aligned}
w(x) &= u(x) - \int_0^x k(x,\xi)u(\xi)\,d\xi && (14)\\
k(x,\xi) &= -\lambda\xi\frac{I_1\left(\sqrt{\lambda(x^2-\xi^2)}\right)}{\sqrt{\lambda(x^2-\xi^2)}} && (15)
\end{aligned}
$$

to map (1)–(2) into an exponentially stable system

$$
\begin{aligned}
w_t &= w_{xx} && (16)\\
w(0) &= w(1) = 0\,. && (17)
\end{aligned}
$$

The stabilizing control law is then given by

$$u(1) = -\int_0^1 \lambda\xi\frac{I_1\left(\sqrt{\lambda(1-\xi^2)}\right)}{\sqrt{\lambda(1-\xi^2)}}u(\xi)\,d\xi\,. \tag{18}$$

For the unknown λ we modify the transformation (14) as follows:

$$
\begin{aligned}
\hat{w}(x) &= \hat{u}(x) - \int_0^x k(x,\xi,\hat{\lambda})\hat{u}(\xi)\,d\xi && (19)\\
k(x,\xi,\hat{\lambda}) &= -\hat{\lambda}\xi\frac{I_1\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{\sqrt{\hat{\lambda}(x^2-\xi^2)}} && (20)
\end{aligned}
$$

It maps (4)–(6) into the following target system (see Lemma 4 from Section IV)

$$
\begin{aligned}
\hat{w}_t &= \hat{w}_{xx} + \dot{\hat{\lambda}}\int_0^x \frac{\xi}{2}\hat{w}(\xi)\,d\xi + (\hat{\lambda} + \gamma^2\|u\|^2)e_1 && (21)\\
\hat{w}(0) &= \hat{w}(1) = 0\,, && (22)
\end{aligned}
$$

where

$$e_1 = e - \int_0^x k(x,\xi,\hat{\lambda})e(\xi)\,d\xi. \tag{23}$$

We observe that in comparison to non-adaptive target system (16)–(17) two additional terms appeared, both small in some sense, since the identifier guarantees $\|e\|, \dot{\hat{\lambda}} \in \mathcal{L}_2$. For the analysis we will also need the inverse transformation

$$
\begin{aligned}
\hat{u}(x) &= \hat{w}(x) + \int_0^x l(x,\xi,\hat{\lambda})\hat{w}(\xi)\,d\xi && (24)\\
l(x,\xi,\hat{\lambda}) &= -\hat{\lambda}\xi\frac{J_1\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{\sqrt{\hat{\lambda}(x^2-\xi^2)}}. && (25)
\end{aligned}
$$

Let us denote a bound on $\hat{\lambda}$ by λ_0. The functions $k(x,\xi,\hat{\lambda})$ and $l(x,\xi,\hat{\lambda})$ are bounded, therefore we get the estimates

$$\|e_1\| \le M_1\|e\|, \quad \|u\| \le \|\hat{u}\| + \|e\| \le M_2\|\hat{w}\| + \|e\|\,, \tag{26}$$

where M_1, M_2 are some constants.

To prove boundedness of signals, we estimate

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\hat{w}\|^2 &= -\int_0^1 \hat{w}_x^2\,dx + \dot{\hat{\lambda}}\int_0^1 \hat{w}(x)\int_0^x \frac{\xi}{2}\hat{w}(\xi)\,d\xi\,dx\\
&\quad + (\hat{\lambda} + \gamma^2\|u\|^2)\int_0^1 e_1\hat{w}\,dx\\
&\le -\|\hat{w}_x\|^2 + \frac{|\dot{\hat{\lambda}}|}{2}\|\hat{w}\|^2 + M_1\lambda_0\|\hat{w}\|\|e\|\\
&\quad + \gamma^2 M_1\|u\|(M_2\|\hat{w}\| + \|e\|)\|\hat{w}\|\|e\|\\
&\le -\frac{1}{4}\|\hat{w}\|^2 + \frac{1}{16}\|\hat{w}\|^2 + |\dot{\hat{\lambda}}|^2\|\hat{w}\|^2 + \frac{1}{16}\|\hat{w}\|^2\\
&\quad + 4M_1^2\lambda_0^2\|e\|^2 + \frac{1}{16}\|\hat{w}\|^2\\
&\quad + 8\gamma^4 M_1^2 M_2^2\|u\|^2\|e\|^2\|\hat{w}\|^2 + \frac{\|e\|^2}{16M_2^2}\\
&\le -\frac{1}{16}\|\hat{w}\|^2 + l_1\|\hat{w}\|^2 + l_1, && (27)
\end{aligned}
$$

where l_1 denotes a generic function in $\mathcal{L}_1$. The last inequality follows from the properties $\dot{\hat{\lambda}}, \|u\|\|e\|, \|e\| \in \mathcal{L}_2$. Using Lemma A.2 we get $\|\hat{w}\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. From (26) we get $\|u\|, \|\hat{u}\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$, and (12) implies that $\dot{\hat{\lambda}}$ is bounded.

In order to get pointwise in x boundedness we need to show the boundedness of $\|\hat{w}_x\|$ and $\|e_x\|$:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 \hat{w}_x^2\,dx &= \int_0^1 \hat{w}_x\hat{w}_{xt}\,dx = -\int_0^1 \hat{w}_{xx}\hat{w}_t\,dx\\
&= -\int_0^1 \hat{w}_{xx}^2\,dx - \frac{\dot{\hat{\lambda}}}{2}\int_0^1 \hat{w}_{xx}\int_0^x \xi w(\xi)\,d\xi\,dx\\
&\quad - (\hat{\lambda} + \gamma^2\|u\|^2)\int_0^1 e_1\hat{w}_{xx}\,dx\\
&\le -\frac{1}{8}\|\hat{w}_x\|^2 + \frac{|\dot{\hat{\lambda}}|^2\|\hat{w}\|^2}{4} + (\lambda_0 + \gamma^2\|u\|^2)^2 M_1\|e\|^2. && (28)
\end{aligned}
$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 e_x^2\,dx \le -\|e_{xx}\|^2 + |\tilde\lambda|\|e_{xx}\|\|u\| - \gamma^2\|e_x\|^2\|u\|^2$$
$$\le -\frac{1}{8}\|e_x\|^2 + \frac{1}{2}|\tilde\lambda|^2\|u\|^2 \,. \qquad (29)$$

Since the right hand sides of (28) and (29) are square integrable, using Lemma A.2 we get $\|\hat w_x\|, \|e_x\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. From (24) we get $\|\hat u_x\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and therefore $\|u_x\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. By Agmon's inequality we get the boundedness of u and $\hat u$ for all $x \in [0,1]$.

To show the regulation of u to zero, note that

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 \quad \le \quad -\|e_x\|^2 + |\tilde\lambda|\|e\|\|u\| < \infty \qquad (30)$$

The boundedness of $(d/dt)\|w\|^2$ follows from (27). Using Lemma A.1 (which is an alternative to Barbalat's lemma) we get $\|\hat w\| \to 0$, $\|e\| \to 0$ as $t \to \infty$. It follows from (24) that $\|\hat u\| \to 0$ and therefore $\|u\| \to 0$. Using Agmon's inequality and the fact that $\|u_x\|$ is bounded, we get the regulation of $u(x,t)$ to zero for all $x \in [0,1]$:

$$\lim_{t\to\infty}\max_{x\in[0,1]}|u(x,t)| \le \lim_{t\to\infty}(2\|u\|\|u_x\|)^{1/2} = 0\,. \qquad (31)$$

The result can be summarized in the following theorem.

Theorem 1: Consider the system (1)–(2) with the controller

$$u(1) = -\int_0^1 \hat\lambda\xi \frac{I_1\left(\sqrt{\hat\lambda(1-\xi^2)}\right)}{\sqrt{\hat\lambda(1-\xi^2)}}\hat u(\xi)\,d\xi\,. \qquad (32)$$

If a closed loop system that consists of (1), (2), (32), identifier (4)–(6), and update law (12), has a classical solution $(\hat\lambda,\ u,\ \hat u)$, then for any $\hat\lambda(0)$ and any initial conditions $u_0, \hat u_0 \in L_2(0,1)$, the signals $\hat\lambda$, u, $\hat u$ are bounded and u is regulated to zero for all $x \in [0,1]$:

$$\lim_{t\to\infty}\max_{x\in[0,1]}|u(x,t)| = 0. \qquad (33)$$

III. 3D Plant with Unknown Diffusivity, Advection, and Reactivity

We present now results for a more general plant in a three-dimensional setting:

$$u_t = \varepsilon(u_{xx} + u_{yy} + u_{zz}) + b_1 u_x + b_2 u_y + b_3 u_z + \lambda u \quad (34)$$

for $(x,y,z) \in \Omega$, where the domain Ω is a cylinder with top and bottom of arbitrary shape Γ (Fig. 1). This configuration of the domain Ω is essential because it allows us to view the problem as many 1D problems with $0 \le x \le 1$ and fixed y, z. We assume Dirichlet boundary conditions on the boundary $\partial\Omega$,

$$u \quad = \quad 0\,, \quad (x,y,z) \in \partial\Omega\backslash\{x=1\}, \qquad (35)$$

except at the top of the cylinder $x = 1$ where the actuation is applied,

$$u(1,y,z) \quad = \quad U(t,y,z)\,, \quad (y,z) \in \Gamma. \qquad (36)$$

The parameters $\varepsilon > 0$, b_1, b_2, b_3, λ are assumed to be unknown.

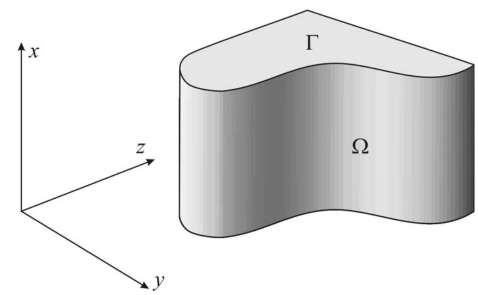

Fig. 1. The domain Ω for the plant (34).

For the notational convenience let us use the following notation later in this section:

$$\Delta u \;=\; u_{xx} + u_{yy} + u_{zz}\,, \quad \nabla u = (u_x, u_y, u_z)^T$$
$$\mathbf{b} \;=\; (b_1, b_2, b_3)^T$$
$$\|u\|^2 \;\triangleq\; \iint_\Gamma \int_0^1 u^2(x,y,z)\,dxdydz \triangleq \int_\Omega u^2\,d\Omega$$
$$\|\nabla u\|^2 \triangleq \int_\Omega \nabla u \cdot \nabla u\,d\Omega\,. \qquad (37)$$

We will employ the following "observer"

$$\hat u_t \;=\; \hat\varepsilon\Delta\hat u + \hat{\mathbf{b}}\cdot\nabla\hat u + \hat\lambda u + \gamma^2(u-\hat u)\|\nabla u\|^2,$$
$$(x,y,z) \in \Omega \qquad (38)$$
$$\hat u \;=\; 0\,, \quad (x,y,z) \in \partial\Omega\backslash\{x=1\} \qquad (39)$$
$$\hat u \;=\; u\,, \quad x = 1,\ (y,z) \in \Gamma\,. \qquad (40)$$

There are two main differences compared to 1D case with one parameter in Section II. First, since the diffusion coefficient ε is unknown we must use projection to ensure $\hat\varepsilon > \underline{\varepsilon} > 0$. We define the projection operator as

$$\mathrm{Proj}_{\underline{\varepsilon}}\{\tau\} = \begin{cases} 0 & ,\ \hat\varepsilon = \underline{\varepsilon} \text{ and } \tau < 0 \\ \tau & ,\ \text{else}\,. \end{cases} \qquad (41)$$

Although this operator is discontinuous it is possible to introduce a small boundary layer instead of a hard switch which will avoid dealing with Filippov solutions and noise due to frequent switching of the update law (see [11] for more details). However, we use (41) here for notational clarity. Note that $\hat\varepsilon$ does not require the projection from above and all other parameters do not require projection at all.

Second, we can see in (38) that while the diffusion and advection coefficients multiply the operators of $\hat u$, the reaction coefficient multiplies u in the observer. This is necessary in order to eliminate any λ-dependence in the error system so that it is stable.

The error signal $e = u - \hat u$ satisfies the following PDE:

$$e_t \;=\; \hat\varepsilon\Delta e + \hat{\mathbf{b}}\cdot\nabla e + \tilde\varepsilon\Delta u + \tilde{\mathbf{b}}\cdot\nabla u + \tilde\lambda u$$
$$-\gamma^2 e\|\nabla u\|^2, \quad (x,y,z) \in \Omega \qquad (42)$$
$$e \;=\; 0\,, \quad (x,y,z) \in \partial\Omega\,. \qquad (43)$$

Using a Lyapunov function

$$V = \frac{1}{2}\int_\Omega e^2\,d\Omega + \frac{\tilde\varepsilon^2}{2\gamma_1} + \frac{|\tilde{\mathbf{b}}|^2}{2\gamma_2} + \frac{\tilde\lambda^2}{2\gamma_3} \qquad (44)$$

we get

$$\dot{V} = -\hat{\varepsilon}\|\nabla e\|^2 - \gamma^2\|e\|^2\|\nabla u\|^2 + \tilde{\varepsilon}\int_\Omega e\Delta u\, d\Omega + \int_\Omega e(\tilde{\mathbf{b}}\cdot\nabla u)\, d\Omega + \tilde{\lambda}\int_\Omega eu\, d\Omega - \frac{1}{\gamma_0}\tilde{\varepsilon}\dot{\hat{\varepsilon}} - \frac{1}{\gamma_1}\tilde{\mathbf{b}}\cdot\dot{\hat{\mathbf{b}}} - \frac{1}{\gamma_2}\tilde{\lambda}\dot{\hat{\lambda}}. \quad (45)$$

With update laws

$$\dot{\hat{\varepsilon}} = -\gamma_0 \mathrm{Proj}_{\underline{\varepsilon}}\left\{\int_\Omega \nabla u\cdot\nabla(u-\hat{u})\, d\Omega\right\} \quad (46)$$

$$\dot{\hat{\mathbf{b}}} = \gamma_1\int_\Omega (u-\hat{u})\nabla u\, d\Omega \quad (47)$$

$$\dot{\hat{\lambda}} = \gamma_2\int_\Omega (u-\hat{u})u\, d\Omega, \quad (48)$$

where $\gamma_0, \gamma_1, \gamma_2 > 0$ we get

$$\dot{V} \le -\underline{\varepsilon}\|\nabla e\|^2 - \gamma^2\|e\|^2\|\nabla u\|^2, \quad (49)$$

which implies $V(t) \le V(0)$ so that $\tilde{\varepsilon}$, $|\tilde{\mathbf{b}}|$, $\tilde{\lambda}$, $\|e\|$ are bounded. Integrating (49) with respect to time from zero to infinity we get square integrability of $\|\nabla e\|$, $\|e\|\|\nabla u\|$, which, together with the update laws (46)–(48), gives square integrability of $|\dot{\hat{\mathbf{b}}}|$ and $\dot{\hat{\lambda}}$.

Lemma 2: The identifier (38)–(40) with update laws (47)–(48) guarantees the following properties:

$$\|\nabla e\|,\ \|e\|\|\nabla u\| \in \mathcal{L}_2, \quad \|e\| \in \mathcal{L}_\infty\cap\mathcal{L}_2, \quad (50)$$

$$\tilde{\varepsilon},\ \tilde{b}_1,\ \tilde{b}_2,\ \tilde{b}_3,\ \tilde{\lambda} \in \mathcal{L}_\infty, \quad \dot{\hat{b}}_1,\ \dot{\hat{b}}_2,\ \dot{\hat{b}}_3,\ \dot{\hat{\lambda}} \in \mathcal{L}_2. \quad (51)$$

We employ the following controller

$$u(1,y,z) = -\int_0^1 \frac{\hat{\lambda}+c}{\hat{\varepsilon}}\xi e^{-\frac{\hat{b}_1(1-\xi)}{2\hat{\varepsilon}}} \times \frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}}\hat{u}(\xi,y,z)\, d\xi \quad (52)$$

with $c \ge 0$, which is a straightforward generalization of the one proposed in [18] for the case of known parameters.

Starting with the result on stability of the identifier, we now turn to proving closed-loop stability. Unfortunately, it is very hard to prove the result in the case of unknown ε. This is because, while the identifier guarantees the properties (50) for $\|e\|$ and $\|\nabla e\|$, it does not provide any estimates for $\|\Delta e\|$ which are required in the case of unknown ε. Therefore for the closed-loop result we assume that ε is known and set $\hat{\varepsilon} = \varepsilon$ everywhere. The update law (46) nevertheless achieves closed-loop stability for unknown ε in simulations, as shown in Section V.

Theorem 3: Consider the plant (34), (35) with the controller (52). If the closed loop system that consists of (34), (35), (52), identifier (38)–(40), and update laws (47), (48) has a classical solution ($\hat{\mathbf{b}}$, $\hat{\lambda}$, u, $\hat{u}$), then for any $\hat{\mathbf{b}}(0)$, $\hat{\lambda}(0)$ and any initial conditions $u_0, \hat{u}_0 \in L_2(\Omega)$, the signals $\hat{\mathbf{b}}$, $\hat{\lambda}$, u, $\hat{u}$ are bounded and u is regulated to zero for all $(x,y,z)\in\Omega$:

$$\lim_{t\to\infty}\max_{(x,y,z)\in\Omega}|u(x,y,z,t)| = 0. \quad (53)$$

IV. Proof of Theorem 3

We will use Poincare and Agmon inequalities

$$\|u\| \le d_1(\Gamma)\|\nabla u\| \quad (54)$$

$$\max_{(x,y,z)\in\Omega}|u| \le d_2(\Gamma)\|u\|_{H_1}^{1/2}\|u\|_{H_2}^{1/2}. \quad (55)$$

Here d_1 and d_2 are constants that depend only on Γ. The main difficulty in proving the result in 3D case compared to 1D case is that we need to show H_2 (instead of H_1) boundedness and H_1 (instead of L_2) regulation in order to have pointwise boundedness and regulation.

A. Target system

We use the following transformation

$$\hat{w}(x,y,z) = \hat{u}(x,y,z) - \int_0^x \hat{k}(x,\xi)\hat{u}(\xi,y,z)\, d\xi \quad (56)$$

$$\hat{k}(x,\xi) = -\frac{\hat{\lambda}+c}{\varepsilon}\xi e^{-\frac{\hat{b}_1(x-\xi)}{2\varepsilon}}\frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}}, \quad (57)$$

which is a generalized version of the transformation presented in [18] for the case of known parameters.

Lemma 4: The transformation (56)–(57) maps (38)–(40) into the target system

$$\hat{w}_t = \varepsilon\Delta\hat{w} + \hat{\mathbf{b}}\cdot\nabla\hat{w} - c\hat{w} + \dot{\hat{b}}_1\Phi_1[\hat{w}] + \dot{\hat{\lambda}}\Phi_2[\hat{w}] + (\hat{\lambda}+\gamma^2\|\nabla u\|^2)e_1, \quad (58)$$

$$\hat{w} = 0, \quad (x,y,z)\in\partial\Omega. \quad (59)$$

where

$$\Phi_i[\hat{w}] = \int_0^x \varphi_i(x,\xi)\hat{w}(\xi,y,z)\, d\xi \quad (60)$$

$$e_1 = e - \int_0^x \hat{k}(x,\xi)e(\xi,y,z)\, d\xi. \quad (61)$$

and

$$\varphi_1(x,\xi) = \frac{x-\xi}{2\varepsilon}\hat{k}(x,\xi) + \frac{1}{2\varepsilon}\int_\xi^x (x-\sigma)\hat{k}(x,\sigma)\hat{l}(\sigma,\xi)\, d\sigma$$

$$\varphi_2(x,\xi) = \frac{\xi}{2\varepsilon}e^{-\frac{\hat{b}_1}{2\varepsilon}(x-\xi)}. \quad (62)$$

Proof: Substituting (56) into (38) we get

$$\hat{w}_t = \varepsilon\Delta\hat{w} + \hat{\mathbf{b}}\cdot\nabla\hat{w} - c\hat{w} + (\hat{\lambda}+\gamma^2\|\nabla u\|^2)e_1 - \int_0^x\left(\dot{\hat{b}}_1\hat{k}_{\hat{b}_1}(x,\xi) + \dot{\hat{\lambda}}\hat{k}_{\hat{\lambda}}(x,\xi)\right)\hat{u}(\xi,y,z)\, d\xi \quad (63)$$

To replace $\hat{u}$ with $\hat{w}$ we use an inverse transformation

$$\hat{u} = \hat{w} + \int_0^x \hat{l}(x,\xi)\hat{w}(\xi,y,z)\, d\xi \quad (64)$$

$$\hat{l}(x,\xi) = -\frac{\hat{\lambda}+c}{\varepsilon}\xi e^{-\frac{\hat{b}_1(x-\xi)}{2\varepsilon}} \frac{J_1\left(\sqrt{\frac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\varepsilon}(x^2-\xi^2)}}. \quad (65)$$

Changing the order of integration and computing the inner integrals with the help of [17] we get (58)–(62). ■

B. Boundedness

Let us denote the bounds on $|\hat{\mathbf{b}}|$, $\hat{\lambda}$ by b_0, λ_0. Since $\hat{k}$ and $\hat{l}$ and their derivatives with respect to parameters are bounded functions, we have the estimates

$$\|e_1\| \le M_1\|e\|, \quad \|\nabla u\| \le M_2\|\nabla\hat{w}\| + \|\nabla e\|, \quad (66)$$

where M_1, M_2 are some constants. The functions φ_1, φ_2 are also bounded, let us denote these bounds by $\bar{\varphi}_1$, $\bar{\varphi}_2$.

First we show the boundedness of the L_2-norm:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\hat{w}\|^2 \;&\le\; -\varepsilon\|\nabla\hat{w}\|^2 + (\hat{\lambda}+\gamma^2\|\nabla u\|^2)\int_\Omega e_1\hat{w}\,d\Omega \\ &\quad + \dot{\hat{b}}_1\int_\Omega \hat{w}\Phi_1\,d\Omega + \dot{\hat{\lambda}}\int_\Omega \hat{w}\Phi_2\,d\Omega. \quad (67)\end{aligned}$$

Using the estimate

$$\begin{aligned}\dot{\hat{b}}_1\int_\Omega \hat{w}\Phi_1\,d\Omega \;&\le\; \frac{\varepsilon}{8d_1^2}\|\hat{w}\|^2 + \frac{2}{\varepsilon}d_1^2|\dot{\hat{b}}_1|^2\bar{\varphi}_1^2\|\hat{w}\|^2 \\ &\le\; \frac{\varepsilon}{8}\|\nabla\hat{w}\|^2 + l_1\|\hat{w}\|^2, \quad (68)\end{aligned}$$

and similarly for the term with $\dot{\hat{\lambda}}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\hat{w}\|^2 \;&\le\; -\frac{3\varepsilon}{4}\|\nabla\hat{w}\|^2 + l_1\|\hat{w}\|^2 + M_1\lambda_0\|\hat{w}\|\|e\| \\ &\quad + \gamma^2 M_1\|\nabla u\|(M_2\|\nabla\hat{w}\| + \|\nabla e\|)\|\hat{w}\|\|e\| \\ &\le\; -\frac{3\varepsilon}{4}\|\nabla\hat{w}\|^2 + l_1\|\hat{w}\|^2 + \frac{d_1^2}{\varepsilon}M_1^2\lambda_0^2\|e\|^2 \\ &\quad + \frac{\varepsilon}{4d_1^2}\|\hat{w}\|^2 + \frac{\varepsilon}{4}\|\nabla\hat{w}\|^2 + \frac{\varepsilon}{4M_2^2}\|\nabla e\|^2 \\ &\quad + \frac{2}{\varepsilon}\gamma^4 M_1^2 M_2^2\|\nabla u\|^2\|e\|^2\|\hat{w}\|^2 \\ &\le\; -\frac{\varepsilon}{4}\|\nabla\hat{w}\|^2 + l_1\|\hat{w}\|^2 + l_1. \quad (69)\end{aligned}$$

Using Lemma A.2 we get $\|\hat{w}\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. Integrating (69) with respect to time from zero to infinity we also get $\|\nabla\hat{w}\| \in \mathcal{L}_2$ and therefore $\|\nabla\hat{u}\|, \|\nabla u\| \in \mathcal{L}_2$.

Now let us show H_1 boundedness. In this case it is enough to consider e and $\hat{w}$ systems separately. First,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla e\|^2 &= \int_\Omega \nabla e_t\nabla e\,d\Omega = -\int_\Omega e_t\Delta e\,d\Omega \\ &\le -\varepsilon\|\Delta e\|^2 + b_0\|\Delta e\|\|\nabla e\| + |\tilde{\mathbf{b}}|\|\Delta e\|\|\nabla u\| \\ &\quad + |\tilde{\lambda}|\|\Delta e\|\|u\| - \gamma^2\|\nabla e\|^2\|\nabla u\|^2 \\ &\le -\varepsilon\|\Delta e\|^2 + \frac{\varepsilon}{4}\|\Delta e\|^2 + \frac{b_0^2}{\varepsilon}\|\nabla e\|^2 + \frac{\varepsilon}{4}\|\Delta e\|^2 \\ &\quad + \frac{|\tilde{\mathbf{b}}|^2}{\varepsilon}\|\nabla u\|^2 + \frac{\varepsilon}{4}\|\Delta e\|^2 + \frac{|\tilde{\lambda}|^2}{\varepsilon}\|u\|^2 \\ &\le -\frac{\varepsilon}{4}\|\Delta e\|^2 + l_1. \quad (70)\end{aligned}$$

Using Lemma A.2 we get $\|\nabla e\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. Second,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\hat{w}\|^2 \;&\le\; -\varepsilon\|\Delta\hat{w}\|^2 - \int_\Omega \Delta\hat{w}\,(\hat{\mathbf{b}}\cdot\nabla\hat{w})\,d\Omega \\ &\quad - \dot{\hat{b}}_1\int_\Omega \Delta\hat{w}\Phi_1\,d\Omega - \dot{\hat{\lambda}}\int_\Omega \Delta\hat{w}\Phi_2\,d\Omega \\ &\quad + (\hat{\lambda}+\gamma^2\|\nabla u\|^2)\int_\Omega e\Delta\hat{w}\,d\Omega. \quad (71)\end{aligned}$$

Using the estimates

$$\begin{aligned}\int_\Omega \Delta\hat{w}\,(\hat{\mathbf{b}}\cdot\nabla\hat{w})\,d\Omega \;&\le\; b_0\|\Delta\hat{w}\|\|\nabla\hat{w}\| \le \frac{\varepsilon}{8}\|\Delta\hat{w}\|^2 + l_1, \\ \dot{\hat{b}}_1\int_\Omega \Delta\hat{w}\Phi_1\,d\Omega \;&\le\; \frac{\varepsilon}{8}\|\Delta\hat{w}\|^2 + l_1\|\hat{w}\|^2, \quad (72)\end{aligned}$$

and similarly for the term with $\dot{\hat{\lambda}}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\hat{w}\|^2 \;&\le\; -\frac{5\varepsilon}{8}\|\Delta\hat{w}\|^2 + l_1\|\hat{w}\|^2 + l_1 \\ &\quad + \gamma^2 M_1 M_2\|\nabla u\|\|\nabla\hat{w}\|\|\Delta\hat{w}\|\|e\| \\ &\quad + \gamma^2 M_1\|\nabla u\|\|\nabla e\|\|\Delta\hat{w}\|\|e\| \\ &\quad + M_1\lambda_0\|\Delta\hat{w}\|\|e\| \\ &\le -\frac{5\varepsilon}{8}\|\Delta\hat{w}\|^2 + l_1 + \frac{\varepsilon\|\Delta\hat{w}\|^2}{4} + \frac{2M_1^2\lambda_0^2}{\varepsilon}\|e\|^2 \\ &\quad + \frac{2}{\varepsilon}\gamma^4 M_1^2 M_2^2\|\nabla u\|^2\|e\|^2\|\nabla\hat{w}\|^2 \\ &\quad + \frac{2}{\varepsilon}\gamma^4 M_1^2\|\nabla u\|^2\|\nabla e\|^2\|e\|^2 + \frac{\varepsilon}{8}\|\Delta\hat{w}\|^2 \\ &\le -\frac{\varepsilon}{4}\|\Delta\hat{w}\|^2 + l_1\|\nabla\hat{w}\|^2 + l_1. \quad (73)\end{aligned}$$

Using Lemma A.2 we get $\|\nabla\hat{w}\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and therefore $\|\nabla\hat{u}\|, \|\nabla u\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. Integrating (70), (73) we also get $\|\Delta e\|, \|\Delta\hat{w}\| \in \mathcal{L}_2$ and therefore $\|\Delta\hat{u}\|, \|\Delta u\| \in \mathcal{L}_2$.

Note that from the above properties and (70)–(73) it follows that $(d/dt)\|\nabla e\|^2$ and $(d/dt)\|\nabla\hat{w}\|^2$ are bounded. By Lemma A.1 we get $\|\nabla e\|, \|\nabla\hat{w}\| \to 0$ and therefore $\|\nabla\hat{u}\|, \|\nabla u\| \to 0$ as $t \to \infty$.

In order to prove pointwise boundedness in 3D we need to show that the H_2 norms of the signals are bounded. After careful estimates (which we omit here due to a lack of space) one can show that the following inequality holds:

$$\begin{aligned}&\frac{1+8\lambda_0^2 M_1^2 d_1^4\varepsilon^{-2}}{2}\frac{d}{dt}\|e_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\hat{w}_t\|^2 \\ &\le -\frac{\varepsilon}{16}\|\nabla\hat{w}_t\|^2 - \frac{\varepsilon}{4}\|\nabla e_t\|^2 + l_1\|\hat{w}_t\|^2 + \|e_t\|^2 + l_1. \quad (74)\end{aligned}$$

By Lemma A.2 $\|\hat{w}_t\|, \|e_t\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and therefore $\|\hat{u}_t\|, \|u_t\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. From (38) and (34) we get $\|\Delta\hat{u}\|, \|\Delta u\| \in \mathcal{L}_\infty \cap \mathcal{L}_2$. Using now Agmon's inequality (55) we get the regulation result:

$$\lim_{t\to\infty}\max_{(x,y,z)\in\Omega}|u| \le d_2\lim_{t\to\infty}\|u\|_{H_1}^{1/2}\|u\|_{H_2}^{1/2} = 0. \quad (75)$$

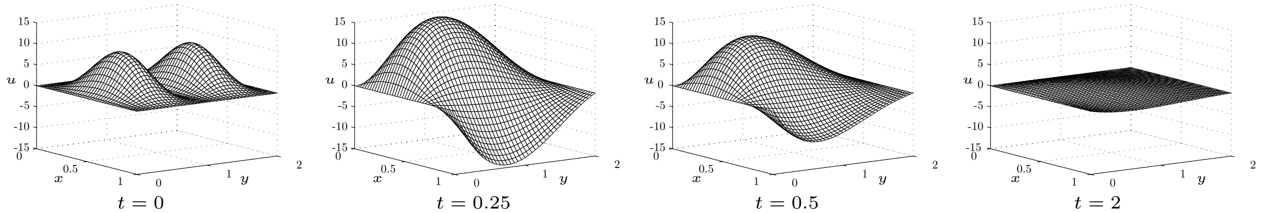

Fig. 2. The closed loop state for the plant (76) at different times.

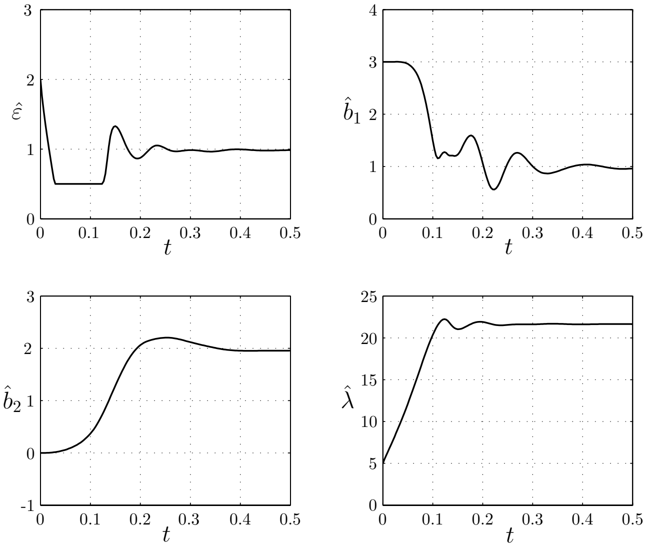

Fig. 3. The parameter estimates for the plant (76).

V. SIMULATIONS

For the demonstration of the design with passive identifier we consider a 2D plant with four unknown parameters ε, b_1, b_2, and λ:

$$u_t = \varepsilon(u_{xx} + u_{yy}) + b_1 u_x + b_2 u_y + \lambda u \tag{76}$$

on the rectangle $0 \leq x \leq 1$, $0 \leq y \leq L$ with actuation applied on the side with $x = 1$ and Dirichlet boundary conditions on the other three sides. The adaptive laws (46)–(48) are modified in a straightforward way from the 3D to the 2D setting. We set the simulation parameters to $\varepsilon = 1$, $b_1 = 1$, $b_2 = 2$, $\lambda = 22$, $L = 2$. With this choice the plant has two unstable eigenvalues at 8.4 and 1. Initial estimates are set to $\hat{\varepsilon}(0) = 2$, $\hat{b}_1(0) = 3$, $\hat{b}_2(0) = 0$, $\hat{\lambda}(0) = 5$ and the bound on $\hat{\varepsilon}$ from below is $\underline{\varepsilon} = 0.5$. The initial conditions for the plant and the observer are $u(x, y, 0) = 10\sin^2(\pi x)\sin^2(\pi y)$ and $\hat{u}(x, y, 0) \equiv 0$. The results of the simulation are presented in Fig. 2 (several snapshots of the state) and Fig. 3 (estimates of the unknown parameters). All estimates come close to the true values at approximately $t = 0.5$ and after that the controller stabilizes the system.

APPENDIX

Lemma A.1 (Lemma 3.1 in [14]): Suppose that the function $f(t)$ defined on $[0, \infty)$ satisfies the following conditions:

(i) $f(t) \geq 0$ for all $t \in [0, \infty)$,

(ii)$f(t)$ is differentiable on $[0, \infty)$ and there exists a constant M such that $f'(t) \leq M$ for all $t \geq 0$,

(iii) $\int_0^\infty f(t)\,dt < \infty$.

Then we have $\lim_{t\to\infty} f(t) = 0$.

Lemma A.2 (Lemma B.6 in [13]): Let v, l_1, and l_2 be real-valued functions defined on R_+, and let c be a positive constant. If l_1 and l_2 are nonnegative and in $\mathcal{L}_1$ and satisfy the differential inequality

$$\dot{v} \leq -cv + l_1(t)v + l_2(t), \quad v(0) \geq 0 \tag{A.1}$$

then $v \in \mathcal{L}_\infty \bigcap \mathcal{L}_1$.

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