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A self-optimizing switching control scheme for uncertain ARMAX systems

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Abstract— **In this paper, we propose a switching control scheme for uncertain AutoRegressive Moving Average eXogenous (ARMAX) systems, which guarantees stability of the switched control system and self-optimality. The adopted switching control logic is based on the Extended Least Squares (ELS) parameter estimation procedure for controller selection and switching. An asymptotically vanishing dither noise is added to the control input so as to introduce appropriate excitation of the system dynamics for consistently estimating the system parameters.**

I. INTRODUCTION

Suppose that a system has to be regulated by choosing a controller in some *candidate controllers set*. In a standard optimal control setting, the performance achieved by applying a certain candidate controller to the system is typically measured by a (positive) cost criterion J: the lower the value for J, the more satisfactory the control performance (J can be, for instance, an H_2 or H_∞ cost). If the system is known, then the optimal controller can be computed by minimizing J over the candidate controllers set.

Consider now the case when the system is not known. If the uncertainty on the system description is large, traditional robust control methodologies based on a worst-case approach do not provide, in general, a controller with satisfactory performance. Suppose that a *parametric set of admissible models* is introduced to model the uncertainty on the system description. Then, the problem of selecting the best controller according to J can be addressed by introducing a state variable representing the unknown parameter vector, and determining the optimal controller according to J for the so-obtained augmented state-space representation of the system. The resulting controller incorporates a self-adjusting mechanism, in that it selects a control input that realizes an appropriate compromise between the control and the identification objectives (dual action, see e.g. [1]). However, such an optimal dual control problem is doable only in a few simple cases where computing the solution to the optimization problem is actually feasible.

A computationally feasible approach to the design of self-adjusting controllers is the so-called *switching control* approach to adaptive control originally introduced in [2] and further developed in, e.g., [3]-[8]. A switching control scheme is typically composed of an inner loop where a candidate controller is connected in closed-loop with the system, and an outer loop where a *supervisor*, based on input-output data collected from the controlled system, decides which controller to place in feedback with the system and when to switch to a different one.

The candidate controller to switch to is typically selected through an *estimator-based* procedure ([4], [5]). Specifically, at any switching time, the supervisor selects the candidate controller that is optimal for the best estimated model for the system, according to the certainty equivalence principle. As for the *switching times*, they are chosen so as to avoid that switching is too fast with respect to the system settling time. In particular, in the *dwell time switching* method, the switching rate is slowed down by making a dwell time elapse between consecutive switching times ([4], [5], [8], [9]). This makes it easier guaranteeing closed-loop stability with respect to standard adaptive control methods, where the controller is continuously modified and overshoots of consequent transients may sum up thus causing instability.

The idea underlying the estimator-based approach to switching is that, as the amount of data collected from the system increases, the estimated system better resembles the behavior of the actual system. Hence, by imposing a specific desired behavior on the estimated system, one actually imposes that behavior on the underlying system (*self-tuning property*). If the estimated system is an accurate description of the true system, this ultimately results in applying to the underlying system the candidate controller that is optimal for it (*self-optimality property*).

It is well known (see, e.g., [10]-[14]) that self-optimality does not hold true for general control laws based on the minimization of multi-step performance indexes. As a matter of fact, the interplay between identification and control in a certainty equivalence adaptive control scheme may result in the convergence of the parameter estimate to a parameterization different from the true one in absence of suitable excitation conditions (see, e.g., [12], [15]-[17]). When a cost criterion other than the output variance is considered, this identifiability problem results in a strictly suboptimal performance. In particular, the identifiability problem is significant in infinite-horizon Linear Quadratic Gaussian (LQG) control and, in fact, in [14] it is proven that for a state space system subject to Gaussian noise the set of the parameterizations leading to optimality of LQG control is strictly contained in the set of the potential convergence points.

In the literature, different switching control schemes have been introduced for different classes of systems. However,

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performance results for these schemes are mainly confined to the assessment of adaptive stabilization. For what concerns uncertain stochastic systems in particular, self-optimizing minimum-variance switching control schemes have been proposed for AutoRegressive Moving Average eXogenous (ARMAX) systems with known input-output delay in [18], [19]. However, the minimum-variance control law calls for the restrictive –and often unrealistic– assumption that the system is minimum-phase. Self-optimality results have been obtained for LQG switching control of AutoRegressive eXogenous (ARX) systems by using either the cost-biased method ([9]) or the attenuating dither-noise method ([21]), without requiring the minimum-phase condition.

The cost-biased method has originally been introduced in [10] and applied to LQG switching control of ARX systems in [9]. In the cost-biased approach an extra term that favors parameters with smaller optimal LQG cost is added to the identification cost. This extra term is selected with a twofold objective. On the one hand, it should be strong enough so that the optimal cost associated with the estimated system is asymptotically not larger than the optimal cost for the true system. On the other hand, it should be mild enough so that self-tuning is preserved, hence the incurred cost is equal to the cost optimal cost for the estimated system. From this optimality of the adaptive control scheme follows.

The attenuating dither-noise method has been proposed in [20] in the context of standard adaptive LQG control, and applied to switching control of ARX systems in [21]. In the attenuating dither-noise method, an asymptotically vanishing noise (the 'dither noise'), is added to the control input so as to introduce sufficient excitation for achieving consistency without upsetting the system performance in the long run. A suitable bound on the growing rate of the input has to be ensured for the dither noise to effectively introduce the excitation needed for the parameter estimate consistency. In the adaptive LQG control scheme proposed in [22], the appropriate growing rate of the control input is obtained through an ad-hoc method. At certain time instants –adaptively selected on the basis of the growing rates of the input and output data– the control law is switched over the LQG expression, a minimum variance expression or is set equal to zero. In this way, consistency and optimality are both achieved. However, the restrictive minimum-phase assumption is required. In [20] and [23], by replacing the minimum-phase condition with the stability assumption, similar results are obtained by an analogous procedure. On the other hand, none of these two conditions seems to be natural in the LQG control problem, as stated in [22]. None of these conditions is required in [21].

This paper extends in a nontrivial way the results in [21] to the class of ARMAX systems. We propose a switching control scheme for uncertain ARMAX systems that is effective in guaranteeing stability and optimality according to an infinite horizon quadratic cost criterion. The adopted switching control methodology is estimator based, in that the switching logic relies on the Extended Least Squares (ELS) parameter estimation procedure ([22], [24]) for controller selection and switching. Self-optimality is achieved by means of the attenuating dither-noise method. An accurate analysis of the closed-loop properties of the ELS estimation when combined with switching control is involved in the proof of the self-optimality result.

It is important to note that the self-optimality result proven in the present paper does not require either the minimum-phase or the stability assumption.

The rest of the paper is organized as follows. In Section II, we precisely formulate the control problem addressed. In Section III we describe the proposed ELS-based switching control scheme and prove some closed-loop identification properties satisfied by the ELS algorithm when combined with switching control. The self-optimality result for the introduced switching control scheme is shown in Section IV. Some concluding remarks are drawn in Section V.

II. SWITCHING CONTROL PROBLEM

Consider an ARMAX system with input u and output y described by the stochastic difference equation

$$\mathscr{A}(\vartheta^\circ,z^{-1})y_{t+1} = \mathscr{B}(\vartheta^\circ,z^{-1})u_t + \mathscr{C}(\vartheta^\circ,z^{-1})w_{t+1}, \quad (1)$$

where the polynomials $\mathscr{A}(\vartheta^\circ,z^{-1}) = 1 - \sum_{i=1}^{n_s} a_i^\circ z^{-i}$, $\mathscr{B}(\vartheta^\circ,z^{-1}) = \sum_{i=1}^{m_s} b_i^\circ z^{-(i-1)}$, and $\mathscr{C}(\vartheta^\circ,z^{-1}) = 1 + \sum_{i=1}^{p_s} c_i^\circ z^{-i}$, depend on some unknown parameter vector $\vartheta^\circ = [a_1^\circ \dots a_{n_s}^\circ\ b_1^\circ \dots b_{m_s}^\circ\ c_1^\circ \dots c_{p_s}^\circ]^T$.

The signal $\{w_t\}$ is a stochastic disturbance satisfying the following standard assumption:

Assumption 1: $\{w_t\}$ is a martingale difference sequence with respect to a filtration $\{\mathscr{F}_t\}$, satisfying almost surely (a.s) the conditions:

1. $\sup_t E[|w_t|^\beta / \mathscr{F}_{t-1}] < \infty$, for some $\beta > 2$;
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} w_t^2 = \sigma^2 > 0$. ■

We suppose that some a-priori knowledge on the system parameter vector ϑ° is available. Specifically,

Assumption 2: ϑ° is an interior point of a known compact set $\Theta \subset \Re^{n_s+m_s+p_s}$. ■

The admissible models set for the system is then given by

$$\mathscr{A}(\vartheta,z^{-1})y_{t+1} = \mathscr{B}(\vartheta,z^{-1})u_t + \mathscr{C}(\vartheta,z^{-1})w_{t+1}, \quad \vartheta \in \Theta.$$

Consider a set of candidate controllers described by

$$\mathscr{R}(\gamma,z^{-1})u_t = \mathscr{S}(\gamma,z^{-1})y_t,$$

where $\mathscr{R}(\gamma;q^{-1}) = 1 - \sum_{i=1}^{m_c} r_i q^{-i}$ and $\mathscr{S}(\gamma;q^{-1}) = \sum_{i=0}^{n_c} s_i q^{-i}$, depend on $\gamma = [r_1 \dots r_{m_c}^\circ\ s_0 \dots s_{n_c}]^T \in \Gamma = \Re^{n_c+m_c+1}$.

In order to allow the switching scheme to obtain a certain control performance level, we require the candidate controller set to be 'sufficiently rich' in a sense that we shall make precise next.

Denote by $Cl(\vartheta,\gamma)$ the closed-loop system where the model with parameter ϑ is controlled by the controller with parameter γ. Suppose that the control performance of $Cl(\vartheta,\gamma)$ is measured by some (positive) cost criterion $J(\vartheta,\gamma)$.

Let $\Sigma : \Theta \to \Gamma$ be the map associating to the model with parameter $\vartheta \in \Theta$ its optimal controller according to the cost criterion J: $\Sigma(\vartheta) = \arg\min_{\gamma\in\Gamma} J(\vartheta,\gamma)$. The optimal performance achievable for the model with parameter $\vartheta \in \Theta$ is then given by $J^\star(\vartheta) := J(\vartheta,\Sigma(\vartheta))$. We assume that the map Σ is continuous over Θ and that the candidate controllers set guarantees an adequate performance level $\bar{J}(<\infty)$ over the admissible models class:

Assumption 3: $Cl(\vartheta,\Sigma(\vartheta))$ is asymptotically stable and $J^\star(\vartheta) \le \bar{J}$, $\forall \vartheta \in \Theta$. ■

Our objective is designing a switching control scheme that guarantee self-optimality with respect to the J cost. We consider in particular the case when J is the infinite horizon quadratic cost $\limsup_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1}[y_t^2 + \alpha u_t^2]$, where the control weighting coefficient α is strictly positive.

The control objective can then be written as

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1}[y_t^2 + \alpha u_t^2] = J^\star(\vartheta^\circ), \tag{2}$$

i.e., the actual cost incurred by the switched controlled system should be equal to the cost obtained by applying the optimal controller for the system from the very beginning.

We study the case when $m_s \ge 1$ and $\max\{n_s,p_s\} \ge 1$. As a matter of fact, if $m_s = 0$, then the control input u_t cannot affect the system and the problem is not well-posed, whereas if $\max\{n_s,p_s\} = 0$, then the problem is trivially solved by setting $u_t = 0$, $t \ge 0$.

III. ELS-BASED SWITCHING CONTROL SCHEME

In this section, we introduce an estimator-based switching control scheme for system (1) that is self-optimizing according to (2), under the assumptions on the system and the candidate controllers set described in Section II.

The proposed switching control law can be expressed as

$$u_t = u_t^e + v_t, \tag{3}$$

where u_t^e is the certainty equivalence control input

$$u_t^e = \mathscr{S}(\Sigma(\vartheta_t),z^{-1})y_t + [1-\mathscr{R}(\Sigma(\vartheta_t),z^{-1})]u_t \tag{4}$$

tuned to the model with parameter ϑ_t, and $\{v_t\}$ is an asymptotically vanishing dither noise. The switching mechanism is incorporated in the definition of ϑ_t, which is in fact a piecewise constant signal changing value only at certain adaptively selected switching times.

Dither noise $\{v_t\}$: Letting $\{d_t\}_{t\ge0}$ be a sequence of independent and identically distributed (i.i.d.) random variables with continuous distribution, independent of $\{w_t\}_{t\ge1}$ and satisfying $E[d_t] = 0$, $E[d_t^2] = 1$, $|d_t| \le K$, $K > 0$, the dither noise $\{v_t\}_{t\ge0}$ is given by

$$v_t = \frac{d_t}{(t+1)^\zeta},\ \zeta \in (0, \frac{1}{4(\max\{n_s,m_s,p_s\}+n_s)}).$$

$\{v_t\}$ satisfies the following condition ([22])

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=0}^{t-1} v_i^2 = 0, \tag{5}$$

which is needed for proving the optimality result in (2). Without loss of generality, in the sequel we shall assume that the family of σ-algebra $\{\mathscr{F}_t\}_{t\ge0}$ introduced in Assumption 1 is rich enough such that both w_t and v_t are $\mathscr{F}_t$-measurable.

Parameter estimate ϑ_t: Denote by $\hat{\vartheta}_t$ the parameter estimate obtained through the standard ELS algorithm.

Algorithm 1 (ELS estimation algorithm):

$$\begin{aligned}
&\hat{\vartheta}_{t+1} = \hat{\vartheta}_t + (1+\varphi_t^T P_t \varphi_t)^{-1} P_t \varphi_t (y_{t+1} - \varphi_t^T \hat{\vartheta}_t)\\
&P_{t+1} = P_t - (1+\varphi_t^T P_t \varphi_t)^{-1} P_t \varphi_t \varphi_t^T P_t\\
&\varphi_t := [y_t \dots y_{t-n_s+1}\ u_t \dots u_{t-m_s+1}\ \hat{w}_t \dots \hat{w}_{t-p_s+1}]^T\\
&\hat{w}_t := y_t - \varphi_{t-1}^T \hat{\vartheta}_t
\end{aligned}$$

initialized with $P_0 = \beta_0 I > 0$ and $\hat{\vartheta}_0 \in \Theta$. ■

The parameter estimate ϑ_t in (4) is constructed based on a suitable a-posteriori modification of the ELS estimate that forces ϑ_t to belong to the admissible parameters set Θ, while preserving the identification properties of the ELS algorithm.

Let $D_t(\vartheta) = (\hat{\vartheta}_t - \vartheta)^T P_t^{-1}(\hat{\vartheta}_t - \vartheta)$. Then, ϑ_t is computed through the following algorithm:

$$\vartheta_t = \begin{cases} \arg\min_{\vartheta\in\Theta} D_t(\vartheta), & \text{if } t = t_i,\ i = 0,1,2,\dots \\ \vartheta_{t-1}, & \text{otherwise,} \end{cases} \tag{6}$$

where $\{t_i\}$ is the switching times sequence that is obtained by the recursive equation $t_{i+1} = t_i + T_i$ initialized with $t_0 = 0$, T_i being the dwell time interval.

We now specify how T_i is adaptively selected.
Consider the autonomous closed-loop system where the model with parameter ϑ is in closed-loop with the controller with parameter γ:

$$\begin{cases} \mathscr{A}(\vartheta,z^{-1})y_{t+1} = \mathscr{B}(\vartheta,z^{-1})u_t \\ \mathscr{R}(\gamma,z^{-1})u_t = \mathscr{S}(\gamma,z^{-1})y_t \end{cases} \tag{7}$$

By letting $x_t := [y_t \dots y_{t-n+1}\, u_{t-1} \dots u_{t-m+1}]^T$ where $n := \max\{n_s, n_c+1\}$ and $m := \max\{m_s, m_c+1\}$, system (7) can be given the state space representation

$$\begin{cases} x_{t+1} = A(\vartheta)x_t + B(\vartheta)u_t \\ u_t = L(\gamma)x_t, \end{cases}$$

where

$$A(\vartheta) = \left[\begin{array}{cccc|cccc}
a_1 & \dots & a_{n-1} & a_n & b_2 & \dots & b_{m-1} & b_m \\
1 & 0 & \dots & & 0 & \dots & & 0 \\
 & \ddots & \ddots & & & \ddots & & 0 \\
 & & 1 & 0 & & & & 0 \\
\hline
0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \\
0 & \dots & \dots & 0 & 1 & 0 & & \\
 & \ddots & \ddots & & & \ddots & \ddots & \\
 & & 0 & 0 & & & 1 & 0
\end{array}\right]$$

$$B(\vartheta) = [b_1\, 0 \cdots 0\,|\,1\, 0 \cdots 0]^T,$$

$$L(\gamma) = [s_0 \cdots s_{n-1}\, s_n\,|\,r_1 \cdots r_{m-1}\, r_m],$$

with $a_i = 0$ if $i > n_s$, $s_i = 0$ if $i > n_c$, $b_i = 0$ if $i > m_s$, $r_i = 0$ if $i > m_c$, thus leading to

$$x_{t+1} = F(\vartheta,\gamma)x_t,$$

where $F(\vartheta,\gamma)=A(\vartheta)+B(\vartheta)L(\gamma)$.

The dwell time interval T_i is chosen so as to stabilize the autonomous estimated system with parameter ϑ_t. Precisely,

$$T_i=\inf\{\tau\in\mathbb{N}:\|F(\vartheta_{t_i},\Sigma(\vartheta_{t_i}))^\tau\|\le\mu\},\quad i=0,1,\ldots,$$

where $0<\mu<1$ is a contraction constant.

Note that the introduced state space representation is non-minimal but, because of the block triangular matrix structure of $A(\vartheta)$, the added eigenvalues are all identically equal to zero. This, jointly with Assumption 3 and the continuity of the Σ map, implies that $\max\{|\lambda_{\max}(F(\vartheta,\Sigma(\vartheta)))|:\vartheta\in\Theta\}\le\lambda$, for some $\lambda\in(0,1)$. This property and the adaptive mechanism for choosing the dwell time interval allow one to prove the following result.

Proposition 1: The autonomous estimated system

$$x_{t+1}=F(\vartheta_t,\Sigma(\vartheta_t))x_t$$

is a.s. exponentially stable, uniformly in time: $\|x_t\|\le c\nu^{t-t^\star}\|x_{t^\star}\|$, $0\le t^\star\le t$, a.s., where $\nu\in(0,1)$ and $c>0$ are appropriate (random) constants. Also, the dwell time interval T_i is bounded: $\sup_{i\ge0}T_i\le T$, for some appropriate constant T. ■

The proof of Proposition 1 is similar to that of Proposition 3.1 in [9], hence it is omitted.

Let $\phi_t^T:=[y_t\ldots y_{t-n_s+1}\ u_t\ldots u_{t-m_s+1}\ w_t\ldots w_{t-p_s+1}]$. System (1) can then be expressed as $y_{t+1}=\phi_t^T\vartheta^\circ+w_{t+1}$. By adding and subtracting to the right-hand-side of this equation $\phi_t^T\vartheta_t$, we obtain $y_{t+1}=\phi_t^T\vartheta_t+e_t+w_{t+1}$, where $e_t:=\phi_t^T(\vartheta^\circ-\vartheta_t)$ is the 'estimation error'.

Based on Assumption 4 below, we next derive some properties of the ϑ_t estimate that will be fundamental for proving that the estimation error e_t is small.

Assumption 4: $\mathscr{C}^{-1}(\vartheta^\circ,e^{i\omega})+\mathscr{C}^{-1}(\vartheta^\circ,e^{-i\omega})>1$, $\forall\omega\in[0,2\pi]$. ■

Theorem 1: Suppose that u_t is $\mathscr{F}_t$-measurable. Then, $\sum_{i=0}^{t_i-1}(\phi_i^T(\vartheta^\circ-\vartheta_{t_i}))^2=O(\log\lambda_{max}(\sum_{i=0}^{t_i-1}\phi_i\phi_i^T))$, a.s.

Proof: Let $\hat{\vartheta}_t':=\arg\min D_t(\vartheta)$. Then,

$$(\hat{\vartheta}_t-\hat{\vartheta}_t')^TP_t^{-1}(\hat{\vartheta}_t-\hat{\vartheta}_t')\le(\hat{\vartheta}_t-\vartheta^\circ)^TP_t^{-1}(\hat{\vartheta}_t-\vartheta^\circ).$$

Since u_t is $\mathscr{F}_t$-measurable and the conditions in Assumptions 1 and 4 are satisfied, by [22, Theorem 4.1]

$$(\hat{\vartheta}_t-\vartheta^\circ)^TP_t^{-1}(\hat{\vartheta}_t-\vartheta^\circ)=O(\log\lambda_{\max}(P_t^{-1})),\ \text{a.s.}$$

Therefore, $(\hat{\vartheta}_t-\hat{\vartheta}_t')^TP_t^{-1}(\hat{\vartheta}_t-\hat{\vartheta}_t')=O(\log\lambda_{\max}(P_t^{-1}))$, a.s. Then,

$$\begin{aligned}&(\hat{\vartheta}_t'-\vartheta^\circ)^TP_t^{-1}(\hat{\vartheta}_t'-\vartheta^\circ)\\&\le2(\hat{\vartheta}_t'-\hat{\vartheta}_t)^TP_t^{-1}(\hat{\vartheta}_t'-\hat{\vartheta}_t)+2(\hat{\vartheta}_t-\vartheta^\circ)^TP_t^{-1}(\hat{\vartheta}_t-\vartheta^\circ)\\&=O(\log\lambda_{\max}(P_t^{-1})),\ \text{a.s.}\end{aligned}$$

Given that $P_t^{-1}=\sum_{i=0}^{t-1}\varphi_i\varphi_i^T+\frac{1}{\beta_0}I$ and $\beta_0>0$, we have

$$\sum_{i=0}^{t-1}\big(\varphi_i^T(\hat{\vartheta}_t'-\vartheta^\circ)\big)^2=O\big(\log\lambda_{\max}(P_t^{-1})\big),\ \text{a.s.}\qquad(8)$$

Observe now that

$$\begin{aligned}&\sum_{i=0}^{t-1}\big(\phi_i^T(\hat{\vartheta}_t'-\vartheta^\circ)\big)^2\\&\le2\sum_{i=0}^{t-1}\big(\varphi_i^T(\hat{\vartheta}_t'-\vartheta^\circ)\big)^2+O\big(\sum_{i=0}^{t-1}\|\phi_i-\varphi_i\|^2\big)\end{aligned}\qquad(9)$$

Equation (4.64) in [22] written according to our notations provides a bound on the last term of (9)

$$\sum_{i=0}^{t-1}\|\phi_t-\varphi_t\|^2=O\big(\log\lambda_{\max}(P_t^{-1})\big),\ \text{a.s.}\qquad(10)$$

By plugging equations (8) and (10) in (9), we have

$$\sum_{i=0}^{t-1}\big(\phi_i^T(\hat{\vartheta}_t'-\vartheta^\circ)\big)^2=O\big(\log\lambda_{\max}(P_t^{-1})\big),\ \text{a.s.}$$

By equation (4.66) in [22]

$$\sum_{i=0}^{t-1}\big(\phi_i^T(\hat{\vartheta}_t'-\vartheta^\circ)\big)^2=O\big(\log(\lambda_{\max}(\sum_{i=0}^{t-1}\phi_i\phi_i^T+\frac{1}{\beta_0}I)\big),\ \text{a.s.}$$

Based on Assumption 1 and the fact that $\max\{n_s,p_s\}\ge1$, it is easily proven that $N=O(\lambda_{\max}(\sum_{t=0}^{N-1}\phi_i\phi_i^T))$, a.s. Hence the right-hand-side of the equation above is equal to $O(\log(\lambda_{\max}(\sum_{i=0}^{t-1}\phi_i\phi_i^T))$, a.s. This jointly with the fact that $\vartheta_{t_i}=\hat{\vartheta}_{t_i}$ and the boundedness of the dwell time interval in Proposition 1 concludes the proof. ■

It is important to note that the proof of Theorem 1 does not rely on the presence of the dither noise and holds irrespectively of the excitation conditions (*closed-loop identification property*). Based on Theorem 1 and the fact that $\{T_i,i=0,1,\ldots\}$ and $\{\|\vartheta^\circ-\vartheta_{t_i}\|,i=0,1,\ldots\}$ are uniformly bounded, one can prove that the estimation error generated within the control loop is 'small' compared to the signals involved in the loop (cf. Proposition 3.3 in [9]).

Proposition 2:

$$\sum_{t=0,t\notin\mathscr{B}_{N-1}}^{N-1}e_t^2\le o\Big(\sum_{t=0}^{N-1}\|\phi_t\|^2\Big),\ \text{a.s.}\qquad(11)$$

where $\mathscr{B}_{N-1}$ is a set of instant points which depends on N, whose cardinality is bounded: $|\mathscr{B}_{N-1}|\le c_B,\forall N$.

IV. SELF-OPTIMALITY ANALYSIS

We now prove that the switched control system

$$\begin{cases}\mathscr{A}(\vartheta^\circ,z^{-1})y_{t+1}=\mathscr{B}(\vartheta^\circ,z^{-1})u_t+\mathscr{C}(\vartheta^\circ,z^{-1})w_{t+1}\\\mathscr{R}(\Sigma(\vartheta_t),z^{-1})u_t=\mathscr{S}(\Sigma(\vartheta_t),z^{-1})y_t+v_t\end{cases}\qquad(12)$$

has the same performance as the optimal control system

$$\begin{cases}\mathscr{A}(\vartheta^\circ,z^{-1})y_{t+1}^\circ=\mathscr{B}(\vartheta^\circ,z^{-1})u_t^\circ+\mathscr{C}(\vartheta^\circ,z^{-1})w_{t+1}\\\mathscr{R}(\Sigma(\vartheta^\circ),z^{-1})u_t^\circ=\mathscr{S}(\Sigma(\vartheta^\circ),z^{-1})y_t^\circ,\end{cases}\qquad(13)$$

in the sense that $\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\big[y_t^2+\alpha u_t^2\big]=\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\big[y_t^{\circ2}+\alpha u_t^{\circ2}\big]=J^\star(\vartheta^\circ)$, a.s. (self-optimality result). The desired performance level $\bar{J}$ specified in Assumption 3 is then guaranteed.

For this purpose, we first need to prove that system (12) is L^2 stable: $\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}[y_t^2+u_t^2]<\infty$, a.s.

Recall the system representation

$$y_{t+1} = \phi_t^T \vartheta_t + e_t + w_{t+1},$$

where $e_t = \phi_t^T(\vartheta^\circ - \vartheta_t)$ is the estimation error.
Based on this equation, by considering the estimation error e_t as if it were an exogenous input, the closed-loop switched system (12) can be expressed as a variational system with respect to the estimated system with parameter ϑ_t as follows

$$\begin{cases} \mathscr{A}(\vartheta_t, z^{-1})y_{t+1} = \mathscr{B}(\vartheta_t, z^{-1})u_t + \mathscr{C}(\vartheta_t, z^{-1})w_{t+1} + e_t \\ \mathscr{R}(\Sigma(\vartheta_t), z^{-1})u_t = \mathscr{S}(\Sigma(\vartheta_t), z^{-1})y_t + v_t. \end{cases} \tag{14}$$

The L^2 stability of the switched control system can then be proven based on the uniform exponential stability of the autonomous closed-loop estimated system (cf. Proposition 1); and on the fact that by switching to the controller designed for the best ELS model, one keeps the estimation error e_t 'small' (cf. Proposition 2).

Theorem 2: The switched control system (12) is L^2-stable.
Proof: Let $x_t := [y_t \dots y_{t-n+1}\, u_{t-1} \dots u_{t-m+1}]^T$ where $n := \max\{n_s, n_c+1\}$ and $m := \max\{m_s, m_c+1\}$. With reference to (14) the switched control system can be given the representation

$$\begin{cases} x_{t+1} = A(\vartheta_t)x_t + B(\vartheta_t)u_t + C[e_t + \mathscr{C}(\vartheta_t, z^{-1})w_t] \\ u_t = L(\Sigma(\vartheta_t))x_t + v_t, \end{cases}$$

where the A, B and L matrices have been defined in Section III, whereas $C = [1\, 0 \cdots 0]^T$. This corresponds to

$$x_{t+1} = F(\vartheta_t, \Sigma(\vartheta_t))x_t + B(\vartheta_t)v_t + C[e_t + \mathscr{C}(\vartheta_t, z^{-1})w_{t+1}],$$

where $F(\vartheta, \gamma) = A(\vartheta) + B(\vartheta)L(\gamma)$. Alternatively, with reference to (12) the switched control system can be given the representation

$$x_{t+1} = F^\circ(\Sigma(\vartheta_t))x_t + B(\vartheta^\circ)v_t + C[\mathscr{C}(\vartheta^\circ, z^{-1})w_{t+1}],$$

where $F^\circ(\gamma) = A(\vartheta^\circ) + B(\vartheta^\circ)L(\gamma)$.

Fix a time instant $N > 0$.
Consider set $\mathscr{B}_{N-1}$ introduced in Proposition 2.

For the following derivations, it is convenient to use both the representations derived above for the switched control system: the latter for the time instants $t \in \mathscr{B}_{N-1}$, and the former for $t \notin \mathscr{B}_{N-1}$. Since ϑ_t belongs to the compact set Θ and $\Sigma(\vartheta)$ is a continuous function of ϑ, $\vartheta \in \Theta$, we then have that $\|F^\circ(\Sigma(\vartheta_t))\|$ is uniformly bounded. From this fact, the uniform exponential stability of the autonomous system $x_{t+1} = F(\vartheta_t, \Sigma(\vartheta_t))x_t$ (Proposition 1), and the fact that $|\mathscr{B}_{N-1}| \leq c_B\ \forall N$ (see Proposition 2), it is easy to show that the state vector x_t can be bounded as follows:

$$\|x_t\| \leq k_1 \Big\{ \sum_{i=1}^{t} \nu^{t-i}|w_i| + \sum_{i=0}^{t-1} \nu^{t-i}|v_i| + \sum_{i=0, i\notin\mathscr{B}_{N-1}}^{t-1} \nu^{t-i}|e_i| \Big\},$$

$t \leq N$, where k_1 is a suitable constant. Note that here we assume for simplicity zero initial conditions ($u_{i-1} = y_i = w_i = 0$ for $i \leq 0$). The proof could be easily generalized to account for non zero initial conditions. Based on this equation, by some cumbersome computations we obtain

$$\|x_t\|^2 \leq \frac{4k_1^2}{1-\nu} \Big[\sum_{i=1}^{t} \nu^{t-i}w_i^2 + \sum_{i=0}^{t-1} \nu^{t-i}v_i^2 + \sum_{i=0, i\notin\mathscr{B}_{N-1}}^{t-1} \nu^{t-i}e_i^2 \Big],$$

$t \leq N$, a.s., from which we finally have

$$\frac{1}{N}\sum_{t=0}^{N}\|x_t\|^2 \leq k_2 \Big[\frac{1}{N}\sum_{t=1}^{N} w_i^2 + \frac{1}{N}\sum_{t=0}^{N-1} v_i^2 + \frac{1}{N}\sum_{t=0, t\notin\mathscr{B}_{N-1}}^{N-1} e_t^2 \Big], \text{ a.s.,}$$

k_2 being a suitable constant.
We next bound the terms in the right-hand-side of this inequality.
By Assumption 1, $\frac{1}{N}\sum_{t=1}^{N} w_t^2 = O(1)$, a.s. By (5), $\frac{1}{N}\sum_{t=0}^{N-1} v_t^2 = o(1)$, a.s. Term $\frac{1}{N}\sum_{t=0, t\notin\mathscr{B}_{N-1}}^{N-1} e_t^2$ is immediately bounded by means of (11). Hence, the final bound for $\frac{1}{N}\sum_{t=0}^{N}\|x_t\|^2$ is obtained:

$$\frac{1}{N}\sum_{t=0}^{N}\|x_t\|^2 \leq \frac{1}{N} o\Big(\sum_{t=0}^{N-1}\|\phi_t\|^2\Big) + O(1), \text{ a.s.}$$

By the definition of ϕ_t and Assumption 1, $\frac{1}{N}\sum_{t=0}^{N-1}\|\phi_t\|^2 \leq \frac{1}{N}\sum_{t=0}^{N}\|x_t\|^2 + O(1)$, a.s. We then have that $\frac{1}{N}\sum_{t=0}^{N}\|\phi_t\|^2$ remains bounded, hence, the system is L^2-stable. ■

We are now in a position to derive the self-optimality result. The proof is based on Theorem 6.2 in [22] (reformulated below according to our notations), which requires the following assumption.

Assumption 5: $\mathscr{A}(\vartheta^\circ, z^{-1})$, $\mathscr{B}(\vartheta^\circ, z^{-1})$, and $\mathscr{C}(\vartheta^\circ, z^{-1})$ are coprime and a_{n_s}, b_{m_s}, c_{p_s} are not all identically zero. ■

Theorem 3: Suppose that the diminishing excited control (3) is applied to system (1) satisfying Assumptions 1 and 5. If the input u_t^e satisfies $\frac{1}{t}\sum_{i=0}^{t-1}(u_i^e)^2 = O(1)$, a.s., then there exists a constant $\bar{c} > 0$ such that

$$\lambda_{min}\Big(\sum_{i=0}^{t-1}\phi_i\phi_i^T\Big) \geq \bar{c}\, t^{\frac{1}{2}},$$

for t large enough, a.s. ■

Theorem 4: The switched control system (12) is self-optimizing.
Proof: Due to space limitations we only give a sketch of the proof.
The growing rate condition on $u_t^e = u_t - v_t$ required for Theorem 3 to hold immediately follows from the L^2 stability result in Theorem 2 and (5). Then, by combining Theorem 3 with Theorem 1, the L^2 stability result and Assumption 1, consistency of the parameter estimate ϑ_t is guaranteed: $\lim_{t\to\infty}\vartheta_t = \vartheta^\circ$, a.s. Due to the continuity of Σ, this, in turn, entails that $\lim_{t\to\infty}\Sigma(\vartheta_t) = \Sigma(\vartheta^\circ)$, i.e., the controller placed in closed-loop with the system is the optimal one for it, at least asymptotically. This generally does not imply self-optimality. The proof of the self-optimality relies on the representation of system (12) as a variational system with respect to system (13):

$$\begin{cases} \mathscr{A}(\vartheta^\circ, z^{-1})y_{t+1} = \mathscr{B}(\vartheta^\circ, z^{-1})u_t + \mathscr{C}(\vartheta^\circ, z^{-1})w_{t+1} \\ \mathscr{R}(\Sigma(\vartheta^\circ), z^{-1})u_t = \mathscr{S}(\Sigma(\vartheta^\circ), z^{-1})y_t + \Delta L_t^\circ x_t + v_t \end{cases} \tag{15}$$

where $x_t := [y_t \dots y_{t-n+1}\, u_{t-1} \dots u_{t-m+1}]^T$ and $\Delta L_t^\circ := L(\Sigma(\vartheta^\circ)) - L(\Sigma(\vartheta_t))$, with $n := \max\{n_s, n_c+1\}$ and $m := \max\{m_s, m_c+1\}$ and vector L defined in Section III. Let $s_t := [y_t \;\; \sqrt{\alpha}\, u_{t-1}]^T$ and $s_t^\circ := [y_t^\circ \;\; \sqrt{\alpha}\, u_{t-1}^\circ]^T$.
We now describe the key steps involved in proving that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (\|s_t\|^2 - \|s_t^\circ\|^2) = 0. \qquad (16)$$

Indeed, equation (16) implies the self-optimality result, since $\lim_{N\to\infty}\{\frac{1}{N}\sum_{t=0}^{N-1}[y_t^2 + \alpha u_t^2] - \frac{1}{N}\sum_{t=0}^{N-1}[(y_t^\circ)^2 + \alpha (u_t^\circ)^2]\}$ $= \lim_{N\to\infty}\{\frac{1}{N}\sum_{t=1}^{N}(\|s_t\|^2 - \|s_t^\circ\|^2) - \frac{1}{N}(y_N^2 - (y_N^\circ)^2)\}$, where $\frac{1}{N}(y_N^2 - (y_N^\circ)^2)$ can be shown to tend to zero in view of the L^2 stability of systems (12) and (13).

The time evolution of vectors s_t and s_t° is governed by the following equations, which can be straightforwardly derived from equations (15) and (13):

$$s_{t+1} = M(\vartheta^\circ, z^{-1})\, s_t + \left[\begin{array}{c} w_{t+1} \\ \sqrt{\alpha}\Delta L_t^\circ x_t + \sqrt{\alpha} v_t \end{array}\right], \qquad (17)$$

$$s_{t+1}^\circ = M(\vartheta^\circ, z^{-1})\, s_t^\circ + \left[\begin{array}{c} w_{t+1} \\ 0 \end{array}\right], \qquad (18)$$

where

$$M(\vartheta, z^{-1}) := \left[\begin{array}{cc} [1-\mathscr{A}(\vartheta, z^{-1})]z & \frac{1}{\sqrt{\alpha}}\mathscr{B}(\vartheta, z^{-1})z \\ \sqrt{\alpha}\mathscr{S}(\vartheta, z^{-1}) & [1-\mathscr{R}(\vartheta, z^{-1})]z \end{array}\right].$$

Now, $\left|\frac{1}{N}\sum_{t=1}^{N}(\|s_t\|^2 - \|s_t^\circ\|^2)\right| \leq [\frac{1}{N}\sum_{t=1}^{N}(\|s_t\| + \|s_t^\circ\|)^2]^{\frac{1}{2}} [\frac{1}{N}\sum_{t=1}^{N}(\|s_t\| - \|s_t^\circ\|)^2]^{\frac{1}{2}}$, where the first factor on the right-hand-side is bounded because of the L^2 stability of (12) and (13). As for the second factor, we have

$$\frac{1}{N}\sum_{t=1}^{N}(\|s_t\| - \|s_t^\circ\|)^2 \leq \frac{1}{N}\sum_{t=1}^{N}\|s_t - s_t^\circ\|^2.$$

Hence, if we show that the right-hand-side of this inequality tends to zero, (16) is finally proven, which concludes the proof of the theorem.

The time evolution of vector $s_t - s_t^\circ$ is governed by

$$(s_{t+1} - s_{t+1}^\circ) = M(\vartheta^\circ, z^{-1})\,(s_t - s_t^\circ) + \sqrt{\alpha}\left[\begin{array}{c} 0 \\ \Delta L_t^\circ x_t + v_t \end{array}\right],$$

obtained by subtracting (18) from (17). This system with $\Delta L_t^\circ x_t + v_t$ considered as an exogenous signal can be shown to be uniformly exponential stable. Then, one can prove that

$$\frac{1}{N}\sum_{t=1}^{N}\|s_t - s_t^\circ\|^2 \leq h\{\frac{1}{N}\sum_{t=0}^{N-1}(\Delta L_t^\circ x_t)^2 + \frac{1}{N}\sum_{t=0}^{N-1} v_t^2\},$$

where h is a suitable constant. By the property that $\|\Delta L_t^\circ\| \to 0$ (due to the consistency result) and the L^2 stability of (12), it is easily seen that $\frac{1}{N}\sum_{t=0}^{N-1}(\Delta L_t^\circ x_t)^2 \to 0$. The term $\frac{1}{N}\sum_{t=0}^{N-1} v_t^2$ asymptotically vanishes as well (see (5)). Therefore, $\frac{1}{N}\sum_{t=1}^{N}\|s_t - s_t^\circ\|^2 \to 0$, thus completing the proof. ■

V. CONCLUDING REMARKS

In this paper, we have described an adaptive control scheme for uncertain ARMAX systems that combines switching control with the attenuating dither-noise method. Self-optimality has been assessed with reference to an infinite horizon quadratic control cost criterion.

It is worth noticing that the self-optimality result has been proven without requiring that the uncertain system to be controlled is minimum-phase. Also, it could be extended to other control cost criteria.

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