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Adaptive Output-Feedback Stabilization and Disturbance Attenuation for Feedforward Systems with ISS Appended Dynamics

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***Abstract*— We propose an adaptive output-feedback control design technique for feedforward systems with Input-to-State Stable (ISS) appended dynamics and disturbance inputs. The design is based on our recent results on the application of dynamic high-gain scaling to state-feedback and output-feedback control of feedforward systems. Unlike previous approaches to the control of feedforward systems, the dynamic high-gain scaling technique provides robustness to additive disturbances and enabled the first output-feedback controller design for feedforward systems. In this paper, we further investigate the robustness properties of the dynamic high-gain scaling approach by introducing exogenous disturbance inputs and ISS appended dynamics coupled with all the system states and the input. The designed adaptive output-feedback controller achieves BIBS stability with respect to disturbance inputs and also provides a disturbance attenuation result. This provides the first results for feedforward systems with ISS appended dynamics and disturbance inputs.**

I. Introduction

We consider the class of systems given by

$$
\begin{aligned}
\dot{z}_i &= q_i(t, y, z, x_{i+2}, \dots, x_n, u, \omega) \ , \ \ i = 1, \dots, n-2 \\
\dot{z}_{n-1} &= q_{n-1}(t, y, z, u, \omega) \\
\dot{x}_1 &= \phi_{(1,2)}(y) x_2 + \phi_1(t, y, x_3, \dots, x_n, u, z_1, \dots, z_{n-1}, \omega) \\
\dot{x}_2 &= \phi_{(2,3)}(y) x_3 + \phi_2(t, y, x_4, \dots, x_n, u, z_2, \dots, z_{n-1}, \omega) \\
&\vdots \\
\dot{x}_{n-2} &= \phi_{(n-2,n-1)}(y) x_{n-1} + \phi_{n-2}(t, y, x_n, u, z_{n-2}, z_{n-1}, \omega) \\
\dot{x}_{n-1} &= \phi_{(n-1,n)}(y) x_n + \phi_{n-1}(t, y, u, z_{n-1}, \omega) \\
\dot{x}_n &= \mu(y) u \\
y &= [x_1, x_n]^T
\end{aligned}
\tag{1}
$$

where $x = [x_1, \dots, x_n]^T \in \mathcal{R}^n$ is the state of the system, $u \in \mathcal{R}$ is the input, $y \in \mathcal{R}^2$ is the measured output, and $z = [z_1^T, \dots, z_{n-1}^T] \in \mathcal{R}^{n_{z_1} + \dots + n_{z_{n-1}}}$ is the state of the appended dynamics. $\omega \in \mathcal{R}^{n_\omega}$ is a bounded disturbance input. μ and $\phi_{(i,i+1)}, i = 1, \dots, n-1$, are known continuous functions of y. $q_i, i = 1, \dots, n-1$, and $\phi_i, i = 1, \dots, n-1$, are uncertain time-varying functions[1] which are assumed to satisfy sufficient conditions (e.g., local Lipschitz property) for local existence and uniqueness of solutions of (1).

In [1,2], the dynamic high-gain scaling technique [3,4] was applied to feedforward systems to obtain state-feedback and output-feedback controllers. Previously available controller design techniques for feedforward systems in the literature include saturation-based designs [5–8] and forwarding [9,10]. Nested saturation designs rely on the use of small inputs and require the ϕ_i functions to involve only quadratic or higher powers in their arguments. Since the saturation levels are restricted to be sufficiently small, the scheme is sensitive to additive disturbances. Forwarding is a recursive passivation scheme which proceeds by adding one integrator at a time through the design of cross terms. However, forwarding is computationally complicated and the cross terms often need to be approximated numerically. A combination of forwarding and nested saturation was proposed in [11] to obtain weaker growth conditions. An adaptive state-feedback scaling-based design with the scaling governed by a switching logic was considered in [12]. However, due to lack of robustness to additive disturbances in these previous designs [5–12], the extension to the output feedback case was not feasible. In contrast, the dual high-gain approach in [1,2] provided a robustly stabilizing controller and enabled an output-feedback solution.

High-gain scaling is a popular technique for the control of strict-feedback systems. The basic adaptive high-gain controller given by $u = -ry, \dot{r} = y^2$ provides global stabilization under the assumption that the system is minimum-phase and of relative-degree one ([13–15] and references therein). Observer design based on static high-gain scaling (using observer gains $r, \dots, r^n$ with a constant r) was considered in [16,17] and semiglobal results were obtained. In [18], a high-gain observer and a backstepping based controller were designed with the dynamics of the scaling parameter r being of the form of a scalar Riccati equation. In [3], it was shown that the high-gain scaling in [18] essentially amplifies the *upper diagonal terms* ($\phi_{(i,i+1)}$) thus inducing the Cascading Upper Diagonal Dominance (CUDD) condition introduced in [19] (see Remark 2). The dual observer/controller dynamic high-gain scaling technique was introduced in [4] and utilized a high-gain observer and a high-gain controller with a single scaling parameter. Furthermore, the flexibility of the dynamic high-gain scaling technique was demonstrated in [4] through the introduction of uncertain terms dependent on all states and uncertain ISS appended dynamics with nonlinear gains from all the system states and the input (previous results allowed the ISS appended dynamics to have a nonzero gain only from the output). The dynamic high-gain scaling technique provides a unified framework for both strict-feedback [4] and feedforward [2] systems.

In this paper, we further investigate the robustness of the dynamic high-gain scaling based controllers as applied to feedforward systems. The system (1) contains the appended dynamics with states $z_1, \dots, z_{n-1}$. Note that the coupling between the x system and the appended dynamics involves all the states and the input. As in the strict-feedback case [4], we consider a triangular hierarchy of z_i subsystems coupled with portions of the state. As could be expected, the coupling in the feedforward case follows an upper triangular pattern. Uncertain parameters are allowed in both the bounds on the functions ϕ_i and in the appended dynamics (z). Furthermore, a disturbance input ω is introduced and appears in both the dynamics of x and the appended dynamics. The

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[1]While $q_i, i = 1, \dots, n-1$, and $\phi_i, i = 1, \dots, n-1$, can depend on all the states and the input, they are shown in (1) to depend only on subsets of the state to emphasize the state dependence of the bounds to be introduced.

observer and controller are similar to [2] with the novelty being in the form of the Lyapunov function and in the design of the high-gain scaling parameter. As in the results on ISS appended dynamics in the strict-feedback case [4], the dynamics of r are fashioned to make the derivative of the Lyapunov function negative if either the high-gain parameter or the derivative of the high-gain parameter is large (compared to functions of the states). The results in this paper further deepen the parallel between the feedforward and strict-feedback cases that was initiated in [2] and [4].

II. Assumptions and Problem Statement

We consider the output-feedback stabilization of system (1). The control objective is to regulate the state x to the origin using dynamic output feedback with output $y = [x_1, x_n]^T$. A particular case in which only x_1 measurement is required is given in Remark 5.

Assumption A1: (*Controllability of system (1)*) A positive constant σ exists such that

$$|\phi_{(i,i+1)}(y)| \geq \sigma\ ,\ i = 1,\ldots,n-1 \qquad \forall y \in \mathcal{R}^2 \quad (2)$$

$$|\mu(y)| \geq \sigma \qquad \forall y \in \mathcal{R}^2. \quad (3)$$

Assumption A2: The functions $\phi_i, i = 1,\ldots,n-1$, can be bounded as[2]

$$\begin{aligned} |\phi_i| \leq\ & |\phi_{(1,2)}(y)|\gamma_1(x_n)\gamma_2\Big(\gamma_u(y)u\Big) \\ & \times\Bigg[\sum_{j=i+2}^{n-1}|x_j| + \theta|x_n| + \theta\gamma_u(y)|u|\Bigg] \\ & +\sqrt{|\phi_{(1,2)}(y)|\gamma_1(x_n)\gamma_2\Big(\gamma_u(y)u\Big)}\Bigg[\sum_{j=i}^{n-1}|z_j| + \chi_{x_i}(\omega)\Bigg] \\ & ,i = 1,\ldots,n-2 \end{aligned} \quad (4)$$

$$\begin{aligned} |\phi_{n-1}| \leq\ & \theta|\phi_{(1,2)}(y)|\gamma_1(x_n)\gamma_2\Big(\gamma_u(y)u\Big)\gamma_u(y)|u| \\ & +\sqrt{|\phi_{(1,2)}(y)|\gamma_1(x_n)\gamma_2\Big(\gamma_u(y)u\Big)}|z_{n-1}| \end{aligned} \quad (5)$$

where θ is an unknown positive parameter and γ_1, γ_2, γ_u, and $\chi_{x_i}, i = 1,\ldots,n-2$, are known continuous nonnegative functions. Furthermore, nonnegative constants p_1, p_2, and α_1 exist such that $\gamma_1(x_n) \leq p_1 + p_2|x_n|^{\alpha_1}$ for all $x_n \in \mathcal{R}$.

Assumption A3: Positive constants ρ_i and $\tilde{\rho}_i$, $i = 2,\ldots,n-1$, exist such that the functions $\phi_{(i,i+1)}(y)$ satisfy for all $y \in \mathcal{R}^2$ and $i = 2,\ldots,n-1$,

$$|\phi_{(i,i+1)}(y)| \leq \rho_i|\phi_{(i-1,i)}(y)|\ ;\ |\phi_{(i,i+1)}(y)| \geq \tilde{\rho}_i|\phi_{(i-1,i)}(y)|. \quad (6)$$

Assumption A4: A continuous nonnegative function $\gamma_o(x_n)$ exists such that for all $y \in \mathcal{R}^2$, $|\phi_{(n-1,n)}(y)|\gamma_u(y)/|\mu(y)| \leq \gamma_o(x_n)$ and nonnegative constants p_3, p_4, and α_2 exist such that $\gamma_o(x_n) \leq p_3 + p_4|x_n|^{\alpha_2}$ for all $x_n \in \mathcal{R}$.

Assumption A5: The $z_i, i = 1,\ldots,n-1$ subsystems are ISS with ISS Lyapunov functions V_{z_i} satisfying

$$\begin{aligned} \dot{V}_{z_i} \leq\ & -\alpha_{z_i}|z_i|^2 + |\phi_{(1,2)}(y)|\gamma_1(x_n)\gamma_2\Big(\gamma_u(y)\Big) \\ & \times\Bigg[\sum_{j=i+2}^{n-1}x_j^2 + \theta x_n^2 + \theta[\gamma_u(y)u]^2\Bigg] + \chi_{z_i}(\omega) \\ & ,i = 1,\ldots,n-2 \end{aligned} \quad (7)$$

$$\begin{aligned} \dot{V}_{z_{n-1}} \leq\ & -\alpha_{z_{n-1}}|z_{n-1}|^2 \\ & +|\phi_{(1,2)}(y)|\gamma_1(x_n)\gamma_2\Big(\gamma_u(y)\Big)\theta[\gamma_u(y)u]^2 \end{aligned} \quad (8)$$

with $\alpha_{z_i}, i = 1,\ldots,n-1$ being known positive constants and $\chi_{z_i}, i = 1,\ldots,n-2$ being known continuous nonnegative functions. Furthermore, positive constants $\overline{V}_{z_i}$ and $\underline{V}_{z_i}$ are known such that $\overline{V}_{z_i}|z_i|^2 \geq V_{z_i} \geq \underline{V}_{z_i}|z_i|^2, i = 1,\ldots,n-1$.

Remark 1: Note that the bounds imposed on ϕ_i in Assumption A2 provide a relaxation of the feedforward structure since ϕ_i are allowed to depend on the first state x_1. No magnitude bounds are required on the unknown parameter θ which is taken to be an aggregate of any unknown parameters in the system dynamics. For simplicity, without any loss of generality, γ_1, γ_2, γ_u, and θ are taken in Assumptions A2 and A5 to be the same for all the ϕ_i and V_{z_i}. Note that the introduction of $\phi_{(1,2)}$ in the bounds in Assumptions A2 and A5 does not impose a constraint (since $|\phi_{(1,2)}| \geq \sigma$) but is incorporated to make the assumptions weaker. The term $\sqrt{|\phi_{(1,2)}|\gamma_1\gamma_2(\gamma_u u)}$ in the bounds in Assumptions A2 and A5 does not involve any restriction and is equivalent to the term $\sqrt{|\phi_{(1,2)}|\tilde{\gamma}_1(x_n)\tilde{\gamma}_2(\gamma_u(y)u)}$ where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are nonnegative functions with $\tilde{\gamma}_1$ being polynomially bounded. The form of the bounds shown in Assumptions A2 and A5 is intended to simplify the expressions which arise in the Lyapunov stability analysis.

Remark 2: Assumption A3 is instrumental in ensuring the solvability of a pair of coupled matrix Lyapunov inequalities that arise in the stability analysis and is the CUDD condition [3,19] applied to the nominal system obtained from (1) by dropping the appended dynamics and $\phi_i, i = 1,\ldots,n-1$. Let

$$D_o = \mathrm{diag}(n-1+b, n-2+b,\ldots,1+b,b) \quad (9)$$

$$D_c = \mathrm{diag}(n-1+b, n-2+b,\ldots,1+b,b,b_v+b) \quad (10)$$

$$C = [1,0\ldots,0] \quad (11)$$

with b and b_v being positive constants. Let $A_o(y)$ be the $n\times n$ matrix with $(i,j)^{th}$ element

$$\begin{aligned} A_{o_{(i,i+1)}} &= \phi_{(i,i+1)}\ ,\ i = 1,\ldots,n-1 \\ A_{o_{(i,1)}} &= g_i\ ,\ i = 1,\ldots,n \end{aligned} \quad (12)$$

with zeros everywhere else where $g_i, i = 1,\ldots,n$, are design freedoms and let $A_c(y)$ be the $(n+1)\times(n+1)$ matrix with $(i,j)^{th}$ element

$$\begin{aligned} & A_{c_{(i,i+1)}} = \phi_{(i,i+1)}\ , i=1,\ldots,n-1\ ;\ A_{c_{(n,n+1)}} = \rho_n\phi_{(n-1,n)} \\ & A_{c_{(n+1,j)}} = k_j\ ,\ j = 1,\ldots,n+1 \end{aligned} \quad (13)$$

with zeros everywhere else where $k_j, j = 1,\ldots,n+1$, are design freedoms and ρ_n is a positive constant. Using Theorem A1 in [3] and the reasoning in Remark 2 in [2], (2) and the first inequality in (6) are necessary and sufficient for existence of a constant matrix $P_o > 0$, positive constants ν_{1o}, ν_{1o}^* $\underline{\nu}_{2o}$, and $\overline{\nu}_{2o}$, and functions $g_1,\ldots,g_n$ such that the following coupled Lyapunov inequalities are satisfied:

$$\left.\begin{aligned} P_oA_o + A_o^TP_o \leq -\nu_{1o}|\phi_{(1,2)}|I - \nu_{1o}^*|\phi_{(1,2)}|C^TC \\ \underline{\nu}_{2o}I \leq P_oD_o + D_oP_o \leq \overline{\nu}_{2o}I \end{aligned}\right\} \quad (14)$$

Using the dual of Theorem A1 in [3], (2) and the second inequality in (6) are necessary and sufficient for existence of a constant matrix $P_c > 0$, positive constants ν_{1c}, $\underline{\nu}_{2c}$, and $\overline{\nu}_{2c}$, and functions $k_1,\ldots,k_{n+1}$ such that the coupled Lyapunov inequalities

$$P_cA_c + A_c^TP_c \leq -\nu_{1c}|\phi_{(1,2)}|I\ ;\ \underline{\nu}_{2c}I \leq P_cD_c + D_cP_c \leq \overline{\nu}_{2c}I \quad (15)$$

are satisfied for all $y \in \mathcal{R}^2$. Furthermore, from the construction in the proof of Theorem A1 in [3], the functions $g_1,\ldots,g_n$ can be chosen to be linear constant-coefficient combinations of $\phi_{(i,i+1)}, i = 1,\ldots,n-1$. Hence, using Assumption A3, a positive constant $\overline{g}$ exists such that

$$\sqrt{g_1^2(y) + g_2^2(y) + \ldots + g_n^2(y)} \leq \overline{g}|\phi_{(1,2)}(y)|. \quad (16)$$

[2]For notational convenience, we drop the arguments of functions when no confusion will result.

Remark 3: Assumption A4 is satisfied in two important special cases: (1) if $\gamma_u(y)$ and $\phi_{(n-1,n)}(y)$ are bounded by polynomial functions of x_n; (2) if $\gamma_u(y) = 0$, i.e., u does not appear in the bound on $|\phi_i|$. In case (2), γ_o can be taken to be identically zero and it is observed in Remark 6 that the polynomial boundedness assumption on γ_1 can be removed.

Remark 4: Additive class K functions of $z_{i+1}, \ldots, z_{n-1}$ can be allowed in (7). This case can be transformed back to the case considered here (under mild local order restrictions) by considering as z_i the collection $(z_i, \ldots, z_{n-1})$ and forming a new Lyapunov function V_{z_i} whose derivative satisfies (7). Furthermore, Assumption A5 can be relaxed to require Input-to-State practical Stability rather than Input-to-State Stability and Assumption A2 can be relaxed to include additive nonnegative constants in the bounds on ϕ_i. In that case, the technique in this paper provides practical stabilization results.

III. Observer Design

A full-order observer to estimate the unmeasured states $x_2, \ldots, x_{n-1}$ is designed as

$$\begin{aligned}\dot{\hat{x}}_i &= \phi_{(i,i+1)}(y)\hat{x}_{i+1} + r^{-i}g_i(y)(\hat{x}_1 - x_1), 1 \leq i \leq n-1 \\ \dot{\hat{x}}_n &= \mu(y)u + r^{-n}g_n(y)(\hat{x}_1 - x_1) \qquad (17)\end{aligned}$$

where $g_1(y), \ldots, g_n(y)$ are continuous functions of $y = [x_1, x_n]^T$ chosen as in Remark 2 and r is a dynamic high-gain scaling parameter whose dynamics to be designed later will ensure that $r(t) > 1$ for all $t \geq 0$. The observer errors e_i and the scaled observer errors ϵ_i are defined as

$$e_i = \hat{x}_i - x_i \;\; ; \;\; \epsilon_i = \frac{e_i}{r^{n-i}} \; , \; i = 1, \ldots, n. \qquad (18)$$

The dynamics of the scaled observer errors are given by, $i = 1, \ldots, n$,

$$\dot{\epsilon}_i = \frac{1}{r}\phi_{(i,i+1)}\epsilon_{i+1} - \frac{1}{r^{n-i}}\phi_i + \frac{1}{r}g_i\epsilon_1 - (n-i)\frac{\dot{r}}{r}\epsilon_i \qquad (19)$$

where $\phi_n \equiv 0$ and $\epsilon_{n+1} \equiv 0$ are dummy variables. In matrix form,

$$\dot{\epsilon} = \frac{1}{r}A_o\epsilon - \frac{\dot{r}}{r}(D_o - bI)\epsilon - \Phi \qquad (20)$$

where $\epsilon = [\epsilon_1, \ldots, \epsilon_n]^T$, b is a positive constant to be chosen later, $\Phi = [r^{1-n}\phi_1, \ldots, r^{-1}\phi_{n-1}, 0]^T$, and D_o and A_o are given by (9) and (12), respectively.

IV. Controller Design

The control input is designed as

$$u = \frac{\phi_{(n,n+1)}(y)\xi_{n+1}}{\mu(y)r} \qquad (21)$$

where $\phi_{(n,n+1)}(y) = \rho_n\phi_{(n-1,n)}(y)$ with ρ_n being a positive constant. ξ_{n+1} is a new state variable with the dynamics

$$\dot{\xi}_{n+1} = v - b_v\frac{\dot{r}}{r}\xi_{n+1} \qquad (22)$$

with v being the new control input and b_v being a design parameter which can be picked to be any positive constant. The control input transformation given by (21) and (22) corresponds to a dynamic extension of the state so that in the extended system, the uncertain functions ϕ_i are bounded by a function of the states x and ξ_{n+1} and do not involve the new input v. Defining

$$\xi_i = \frac{\hat{x}_i}{r^{n-i}} \; , \; i = 1, \ldots, n, \qquad (23)$$

the dynamics of ξ_i, $i = 1, \ldots, n$, are given by

$$\dot{\xi}_i = \frac{1}{r}\phi_{(i,i+1)}\xi_{i+1} - (n-i)\frac{\dot{r}}{r}\xi_i + \frac{1}{r}g_i\epsilon_1 \qquad (24)$$

and the dynamics of ξ_{n+1} are given in (22). The control input v is picked to be

$$v = r^{-1}[k_1(y), k_2(y), k_3(y), \ldots, k_{n+1}(y)]\xi \qquad (25)$$

where $\xi = [\xi_1, \ldots, \xi_{n+1}]^T$ and $k_1(y), \ldots, k_{n+1}(y)$ are continuous functions of the output y chosen as in Remark 2. With the control law (25), the dynamics of ξ are

$$\dot{\xi} = \frac{1}{r}A_c\xi - \frac{\dot{r}}{r}(D_c - bI)\xi + \frac{1}{r}g\epsilon_1 \qquad (26)$$

where $g = [g_1, \ldots, g_n, 0]^T$ and D_c and A_c are given by (10) and (13), respectively.

V. Stability Analysis

The functions $g_1, \ldots, g_n, k_1, \ldots, k_{n+1}$ are chosen as in Remark 2 so that the Lyapunov inequalities (14) and (15) are satisfied with some symmetric positive-definite matrices P_o and P_c and some positive constants ν_{1o}^*, ν_{1o}, $\underline{\nu}_{2o}$, $\overline{\nu}_{2o}$, ν_{1c}, $\underline{\nu}_{2c}$, and $\overline{\nu}_{2c}$. Consider an *observer Lyapunov function* V_o and a *controller Lyapunov function* V_c given by

$$V_o = \epsilon^T P_o\epsilon \; , \; V_c = \xi^T P_c\xi. \qquad (27)$$

The dynamics of r will be designed such that $r(t)$ is monotonically nondecreasing and greater than 1 for all time t. Using (14) and (15), the derivatives of V_o and V_c can be bounded as

$$\begin{aligned}\dot{V}_o \leq& -\frac{\nu_{1o}}{r}|\phi_{(1,2)}||\epsilon|^2 - \frac{\nu_{1o}^*}{r}|\phi_{(1,2)}|\epsilon_1^2 - \frac{\dot{r}}{r}\epsilon^T[P_oD_o + D_oP_o]\epsilon \\ &+2b\frac{\dot{r}}{r}\epsilon^T P_o\epsilon - 2\epsilon^T P_o\Phi \qquad (28)\end{aligned}$$

$$\begin{aligned}\dot{V}_c \leq& -\frac{\nu_{1c}}{r}|\phi_{(1,2)}||\xi|^2 - \frac{\dot{r}}{r}\xi^T[P_cD_c + D_cP_c]\xi \\ &+2b\frac{\dot{r}}{r}\xi^T P_c\xi + \frac{2}{r}\xi^T P_c g\epsilon_1. \qquad (29)\end{aligned}$$

Using Assumption A2, a bound on Φ can be obtained as

$$\begin{aligned}|\Phi| \leq& \frac{1}{r^2}|\phi_{(1,2)}(y)|\gamma(x_n)\gamma_2\Big(\gamma_u(y)u\Big)\Big\{n[|\xi| + |\epsilon|] \\ &+n^{\frac{1}{2}}\theta[|\xi_n| + |\xi_{n+1}| + |\epsilon_n|]\Big\} \\ &+\sqrt{\phi_{(1,2)}(y)|\gamma_1(x_n)\gamma_2\Big(\gamma_u(y)u\Big)}\Big\{n\sum_{j=1}^{n-1}\frac{|z_j|}{r^{n-j}} + \frac{\chi_x(\omega)}{r^2}\Big\} \qquad (30)\end{aligned}$$

where $\chi_x(\omega) = \sum_{i=1}^{n-2}\chi_{x_i}(\omega)$ and $\gamma(x_n) = \gamma_1(x_n)[1 + \rho_n\gamma_o(x_n)]$. Hence,

$$\begin{aligned}|2\epsilon^T P_o\overline{\Phi}| \leq& \frac{5n}{r^2}\lambda_{max}(P_o)|\phi_{(1,2)}(y)|\gamma(x_n)\gamma_2(\gamma_u(y)u)[|\xi|^2 + |\epsilon|^2] \\ &+\frac{3\theta^2}{r^2}\lambda_{max}(P_o)|\phi_{(1,2)}(y)|\gamma(x_n)\gamma_2(\gamma_u(y)u)[\xi_n^2 + \xi_{n+1}^2 + \epsilon_n^2] \\ &+n\lambda_{max}(P_o)\Big[n^2\sum_{j=1}^{n-1}\frac{|z_j|^2}{r^{2(n-j-1)}} + \frac{\chi_x^2(\omega)}{r^2}\Big]. \qquad (31)\end{aligned}$$

A composite Lyapunov function for the x subsystem is defined as

$$V_x = \frac{1}{r^{2b}}[cV_o + V_c] = \frac{1}{r^{2b}}[c\epsilon^T P_o\epsilon + \xi^T P_c\xi] \qquad (32)$$

where c is a positive constant such that $c \geq \frac{4}{\nu_{1o}^*\nu_{1c}}\lambda_{max}^2(P_c)\overline{g}^2$. The constant c in (32) is used to handle the cross term $\frac{2}{r}\xi^T P_c g\epsilon_1$ in (29). The term $\frac{1}{r^{2b}}$ in (32) is introduced to cancel the terms $2b\frac{\dot{r}}{r}\epsilon^T P_o\epsilon$ and $2b\frac{\dot{r}}{r}\xi^T P_c\xi$ in (28) and (29). The motivation for using a constant $b > 0$ is to ensure that D_o and D_c are positive-definite making the Lyapunov inequalities (14) and (15) feasible. Differentiating (32) and using (28), (29), and (31),

$$\begin{aligned}\dot{V}_x \leq& -\frac{c\nu_{1o}}{r^{1+2b}}|\phi_{(1,2)}||\epsilon|^2 - \frac{\nu_{1c}}{2r^{1+2b}}|\phi_{(1,2)}||\xi|^2 \\ &-\frac{c\dot{r}}{r^{1+2b}}\epsilon^T[P_oD_o + D_oP_o]\epsilon - \frac{\dot{r}}{r^{1+2b}}\xi^T[P_cD_c + D_cP_c]\xi \\ &+\frac{5cn}{r^{2+2b}}|\phi_{(1,2)}|\lambda_{max}(P_o)\gamma(x_n)\gamma_2\Big(\gamma_u(y)u\Big)[|\epsilon|^2 + |\xi|^2] \\ &+\frac{3c\theta^2}{r^{2+2b}}|\phi_{(1,2)}|\lambda_{max}(P_o)\gamma(x_n)\gamma_2\Big(\gamma_u(y)u\Big)[\xi_n^2 + \xi_{n+1}^2 + \epsilon_n^2] \\ &+\frac{cn^3}{r^{2b}}\lambda_{max}(P_o)\sum_{j=1}^{n-1}\frac{|z_j|^2}{r^{2(n-j-1+b)}} + \frac{cn}{r^{2+2b}}\lambda_{max}(P_o)\chi_x^2(\omega). \qquad (33)\end{aligned}$$

The positive z_i-dependent terms in (33) must be handled by incorporating the ISS Lyapunov functions of the z_i subsystems. Noting the ISS Lyapunov inequalities assumed in Assumption A5, the overall Lyapunov function is defined as

$$V = V_x + \sum_{j=1}^{n-1}\left\{\frac{1}{\alpha_{z_j}}\left(cn^3\lambda_{max}(P_o)+\kappa_z^*\right)\frac{V_{z_j}}{r^{2(n-j-1+b)}}\right\} + \frac{1}{2c_\theta r^{2b+2b_2}}(\hat{\theta}-\theta^*)^2 \quad (34)$$

where κ_z^*, c_θ, and b_2 are positive design freedoms and $\theta^* = \theta + \theta^2$. The scaling $1/r^{2b+2b_2}$ in the term quadratic in the adaptation error $(\hat{\theta}-\theta^*)$ is required to induce a time-scale separation between $|\xi|$ and $\hat{\theta}$ and will be seen to be crucial in the stability analysis below.

Differentiating (34) and using (14), (15), and (33),

$$\begin{aligned}\dot{V} \le& -\frac{c\nu_{1o}}{r^{1+2b}}|\phi_{(1,2)}||\epsilon|^2 - \frac{\nu_{1c}}{2r^{1+2b}}|\phi_{(1,2)}||\xi|^2 \\ &-\kappa_z^*\sum_{j=1}^{n-1}\frac{|z_j|^2}{r^{2(n-j-1+b)}} - \frac{c\dot{r}\underline{\nu}_{2o}}{r^{1+2b}}|\epsilon|^2 - \frac{\dot{r}\underline{\nu}_{2c}}{r^{1+2b}}|\xi|^2 \\ &+\frac{1}{r^{2+2b}}w_1(x_n,\gamma_u(y)u)|\phi_{(1,2)}|[|\epsilon|^2+|\xi|^2] \\ &+\frac{1}{r^{2+2b}}w_2(x_n,\gamma_u(y)u)|\phi_{(1,2)}|\theta^*[\xi_n^2+\xi_{n+1}^2+\epsilon_n^2] \\ &+\frac{cn}{r^{2+2b}}\lambda_{max}(P_o)\chi_x^2(\omega)+\sum_{j=1}^{n-2}\frac{(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}r^{2+2b}}\chi_{z_j}(\omega) \\ &+\frac{1}{c_\theta r^{2b+2b_2}}(\hat{\theta}-\theta^*)\dot{\hat{\theta}} \end{aligned} \quad (35)$$

where

$$w_1(x_n,\gamma_u(y)u) = \left[5cn\lambda_{max}(P_o) + \sum_{j=1}^{n-2}\frac{2(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}}\right]\overline{\gamma}(x_n)\gamma_2(\gamma_u(y)u) \quad (36)$$

$$w_2(x_n,\gamma_u(y)u) = \left[3c\lambda_{max}(P_o) + \sum_{j=1}^{n-1}\frac{2(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}}\right]\overline{\gamma}(x_n)\gamma_2(\gamma_u(y)u) \quad (37)$$

$$\overline{\gamma}(x_n) = \gamma_1(x_n)[1+\rho_n\gamma_o(x_n)+\rho_n^2\gamma_o^2(x_n)]. \quad (38)$$

By Assumptions A2 and A4, γ_1 and γ_o are polynomially bounded implying that $\overline{\gamma}(x_n)$ can be bounded by a function of x_n/r^b as

$$\overline{\gamma}(x_n) \le \tilde{\gamma}\left(\frac{x_n}{r^b}\right)r^{b(\alpha_1+2\alpha_2)} \quad (39)$$

$$\tilde{\gamma}\left(\frac{x_n}{r^b}\right) = \left[p_1+p_2\left|\frac{x_n}{r^b}\right|^{\alpha_1}\right]\left[1+\rho_n\left(p_3+p_4\left|\frac{x_n}{r^b}\right|^{\alpha_2}\right) + \rho_n^2\left(p_3+p_4\left|\frac{x_n}{r^b}\right|^{\alpha_2}\right)^2\right] \quad (40)$$

if $r \ge 1$. Hence,

$$w_i(x_n,\gamma_u(y)u) \le \tilde{w}_i\left(\frac{x_n}{r^b},\gamma_u(y)u\right)r^{b(\alpha_1+2\alpha_2)}\ ,\ i=1,2 \quad (41)$$

$$\tilde{w}_1\left(\frac{x_n}{r^b},\gamma_u(y)u\right) = \left[5cn\lambda_{max}(P_o) + \sum_{j=1}^{n-2}\frac{2(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}}\right]\tilde{\gamma}\left(\frac{x_n}{r^b}\right)\gamma_2(\gamma_u(y)u) \quad (42)$$

$$\tilde{w}_2\left(\frac{x_n}{r^b},\gamma_u(y)u\right) = \left[3c\lambda_{max}(P_o) + \sum_{j=1}^{n-1}\frac{2(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}}\right]\tilde{\gamma}\left(\frac{x_n}{r^b}\right)\gamma_2(\gamma_u(y)u). \quad (43)$$

The dynamics of the parameter estimator $\hat{\theta}$ are designed as

$$\dot{\hat{\theta}} = c_\theta\Big[-\frac{\sigma_\theta}{r}\hat{\theta}+\frac{w_2(x_n,\gamma_u(y)u)|\phi_{(1,2)}(y)|}{r^{2-2b_2}}[\xi_n^2+\xi_{n+1}^2+\epsilon_n^2]\Big] \quad (44)$$

where σ_θ is a nonnegative design parameter representing the σ-modification which is required to prevent parameter drift instability in the presence of the exogenous disturbance ω. The factor $1/r$ incorporated into the σ-modification term dynamically reduces the bandwidth of the dynamics of $\hat{\theta}$ with increasing r and is required to integrate the σ-modification feature into the high-gain control design framework for feedforward systems. $\hat{\theta}$ is initialized to be positive. By (44), $\hat{\theta}$ remains positive for all time (however, $\hat{\theta}(t)$ could go to zero asymptotically as $t \to \infty$).

The dynamics of the scaling parameter r are designed to be of the form

$$\dot{r}=\lambda\Big(R\Big(\frac{x_n}{r^b},\gamma_u(y)u,\frac{\hat{\theta}}{r^{b+b_2}}\Big)-r\Big)\Omega(r,y,\gamma_u(y)u,\hat{\theta});r(0)>1 \quad (45)$$

$$\begin{aligned}R\Big(\frac{x_n}{r^b},\gamma_u(y)u,\frac{\hat{\theta}}{r^{b+b_2}}\Big) = \max\Bigg\{&\overline{R}, \\ &\left[\max\left(\frac{2}{c\nu_{1o}},\frac{4}{\nu_{1c}}\right)\tilde{w}_1\left(\frac{x_n}{r^b},\gamma_u(y)u\right)\right]^{\frac{1}{1-b(\alpha_1+2\alpha_2)}} \\ &\left[\max\Big(\frac{2}{c\nu_{1o}},\frac{4}{\nu_{1c}}\Big)\tilde{w}_2\Big(\frac{x_n}{r^b},\gamma_u(y)u\Big)\frac{\hat{\theta}}{r^{b+b_2}}\right]^{\frac{1}{1-b(1+\alpha_1+2\alpha_2)-b_2}}\Bigg\}\end{aligned} \quad (46)$$

$$\Omega(r,y,\gamma_u(y)u,\hat{\theta}) = \max\left\{\overline{\Omega},\max\left(\frac{1}{c\underline{\nu}_{2o}},\frac{1}{\underline{\nu}_{2c}}\right)|\phi_{(1,2)}(y)| \times\frac{(w_1(x_n,\gamma_u(y)u)+w_2(x_n,\gamma_u(y)u)\hat{\theta})}{r}\right\} \quad (47)$$

with $\overline{R}$ and $\overline{\Omega}$ being nonnegative constants free to be picked by the designer. λ is chosen to be any nonnegative continuous function such that $\lambda(\pi)=1$ for $\pi > 0$ and $\lambda(\pi)=0$ for $\pi < -\epsilon_r$ with ϵ_r being some positive constant. r is initialized to be bigger than 1. By (45), r is monotonically nondecreasing and $r(t)$ remains greater than 1 for all time t. The design of the function R as shown in (46) is obtained by requiring that the positive terms in the third and fourth lines of (35) should be dominated by the negative terms in the first line of (35) if $r \ge R$ and $\theta^* = \hat{\theta}$. R is designed in terms of x_n/r^b and $\hat{\theta}/r^{b+b_2}$ rather than simply in terms of x_n and $\hat{\theta}$ since boundedness of the Lyapunov function V in (34) directly only guarantees boundedness of x_n/r^b and $\hat{\theta}/r^{b+b_2}$ but not necessarily boundedness of x_n and $\hat{\theta}$. Hence, as will be shown below, the design of R in (46) and the dynamics of r in (45) enable boundedness of r to be inferred from boundedness of the Lyapunov function V. To ensure that the exponents in the second and third terms in (46) are positive, b and b_2 will be picked such that

$$0 \le b \le \tfrac{1}{\alpha_1+2\alpha_2} \quad (48)$$

$$0 \le b_2 \le 1-b(1+\alpha_1+2\alpha_2). \quad (49)$$

To satisfy (49), (48) must be strengthened to

$$0 \le b \le \tfrac{1}{1+\alpha_1+2\alpha_2}. \quad (50)$$

The definition of Ω in (47) is obtained by requiring that the positive terms in the third and fourth lines of (35) should be dominated by the negative terms in the second line of (35) if $\dot{r} = \Omega$ and $\theta^* = \hat{\theta}$. The form of the dynamics of r in (45) involving the functions λ, R, and Ω is similar to the strict-feedback case[4] and ensures that $\dot{r}$ is large ($\dot{r} = \Omega$) until r becomes large (i.e., until $r \ge (R+\epsilon_r)$). Considering the two cases: (a) $r \geq R$ and (b) $r < R$, and using (35), (44), (46), and (47), it follows that in either case,

$$\begin{aligned}\dot{V} \leq & -\frac{c\nu_{1o}}{2r^{1+2b}}|\phi_{(1,2)}||\epsilon|^2 - \frac{\nu_{1c}}{4r^{1+2b}}|\phi_{(1,2)}||\xi|^2 \\ & -\kappa_z^* \sum_{j=1}^{n-1} \frac{|z_j|^2}{r^{2(n-j-1+b)}} - \frac{\sigma_\theta}{2r^{1+2b+2b_2}}(\hat{\theta}-\theta^*)^2 \\ & +\frac{cn}{r^{2+2b}}\lambda_{max}(P_o)\chi_x^2(\omega) \\ & +\sum_{j=1}^{n-2}\frac{(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}r^{2+2b}}\chi_{z_j}(\omega)+\sigma_\theta\frac{{\theta^*}^2}{2r^{1+2b+2b_2}} \quad (51) \\ \leq & -\frac{1}{r}\sigma_V V + \frac{cn}{r^{2+2b}}\lambda_{max}(P_o)\chi_x^2(\omega) \\ & +\sum_{j=1}^{n-2}\frac{(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}r^{2+2b}}\chi_{z_j}(\omega)+\sigma_\theta\frac{{\theta^*}^2}{2r^{1+2b+2b_2}} \quad (52)\end{aligned}$$

where

$$\begin{aligned}\sigma_V \quad = \quad & \min\left\{\frac{\sigma\nu_{1o}}{2\lambda_{max}(P_o)}, \frac{\sigma\nu_{1c}}{4\lambda_{max}(P_c)}, \sigma_\theta c_\theta, \right. \\ & \left. \frac{\kappa_z^*\alpha_{z_j}}{\overline{V}_{z_j}(cn^3\lambda_{max}(P_o)+\kappa_z^*)}\right\}. \quad (53)\end{aligned}$$

Since the disturbance ω is a bounded signal, the Lyapunov inequality (52) guarantees that $\dot{V}$ is negative if

$$\begin{aligned}V \geq & \frac{1}{\sigma_V}\left\{cn\lambda_{max}(P_o)\sup_t \chi_x^2(\omega(t)) \right. \\ & \left. +\sum_{j=1}^{n-2}\frac{(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}}\sup_t \chi_{z_j}(\omega(t))+\sigma_\theta\frac{{\theta^*}^2}{2}\right\}. \quad (54)\end{aligned}$$

Hence, $V(t)$ is a bounded signal on the maximal interval of existence of solutions $[0, t_f)$. From the definition of V in (34), this implies that $|\epsilon|/r^b$, $|\xi|/r^b$, $V_{z_j}/r^{2(n-j-1+b)}, j = 1,\ldots,n-1$, and $\hat{\theta}/r^{b+b_2}$ are bounded implying boundedness of $\frac{x_n^2}{r^{2b}} \leq \frac{2(\epsilon_n^2+\xi_n^2)}{r^{2b}}$. Using (21) and Assumption A4,

$$|\gamma_u(y)u| \leq \left[p_3 + p_4\left|\frac{x_n}{r^b}\right|^{\alpha_2}\right]\left|\rho_n\frac{\xi_{n+1}}{r^b}\right| r^{b\alpha_2+b-1} \quad (55)$$

if $r \geq 1$. If b is smaller than $1/(1+\alpha_2)$, it follows from (55) that the boundedness of x_n/r^b and ξ_{n+1}/r^b implies the boundedness of $\gamma_u(y)u$. Hence, picking b such that

$$0 \leq b \leq \min\left(\tfrac{1}{1+\alpha_1+2\alpha_2}, \tfrac{1}{1+\alpha_2}\right), \quad (56)$$

boundedness of V implies boundedness of $R(x_n/r^b, \gamma_u(y)u, \hat{\theta}/r^{b+b_2})$. By the scaling parameter dynamics (45), $\dot{r}$ is zero if $r \geq (R+\epsilon_r)$ so that boundedness of R implies boundedness of r. With the boundedness of V and r established, the boundedness of all closed-loop signals follows by routine signal-chasing. Hence, $t_f = \infty$ and solutions exist for all time.

Theorem 1: Under Assumptions A1-A5, given any initial conditions $(x(0), z(0))$ for the plant state and $(r(0), \hat{\theta}(0), \hat{x}(0))$ for the controller state with $r(0) > 1$ and $\hat{\theta}(0) \geq 0$, the closed-loop system formed by (1), (17), (21), (44), and (45) possesses a unique solution on the time interval $[0,\infty)$ and all closed-loop signals are uniformly bounded on $[0,\infty)$. Furthermore, by an appropriate choice of the controller parameters, x_n can be regulated to an arbitrarily small neighbourhood of the origin.

Proof of Theorem 1: The existence and uniqueness of solutions and uniform boundedness of all closed-loop signals were proved in the foregoing analysis. Using the Comparison Lemma, it follows from (52) that in the limit as $t \to \infty$, the closed-loop solutions tend to the compact set in which

$$\begin{aligned}\frac{1}{r}\sigma_V V \leq & \frac{1}{r^{\min(1,2b_2)}}\frac{1}{r^{1+2b}}\left\{cn\lambda_{max}(P_o)\sup_t \chi_x^2(\omega(t)) \right. \\ & \left. +\sum_{j=1}^{n-2}\frac{(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}}\sup_t \chi_{z_j}(\omega(t))+\sigma_\theta\frac{{\theta^*}^2}{2}\right\}. \quad (57)\end{aligned}$$

Noting that $V \geq \frac{1}{r^{2b}}\xi^T P_c \xi \geq \frac{1}{r^{2b}}\lambda_{min}(P_c)\xi_n^2$, it follows that in the compact set (57), ξ_n satisfies

$$\begin{aligned}\xi_n^2 \leq & \frac{1}{r^{\min(1,2b_2)}}\frac{1}{\sigma_V\lambda_{min}(P_c)}\left\{cn\lambda_{max}(P_o)\sup_t \chi_x^2(\omega(t)) \right. \\ & \left. +\sum_{j=1}^{n-2}\frac{(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}}\sup_t \chi_{z_j}(\omega(t))+\sigma_\theta\frac{{\theta^*}^2}{2}\right\}. \quad (58)\end{aligned}$$

Given a positive constant ϵ_ξ, ξ_n can be regulated to the compact set $[-\epsilon_\xi, \epsilon_\xi]$ by picking $\overline{\Omega} > 0$ and picking $\overline{R}$ to be

$$\begin{aligned}\overline{R} \geq & \left[\frac{1}{\epsilon_\xi^2\sigma_V\lambda_{min}(P_c)}\left\{cn\lambda_{max}(P_o)\sup_t \chi_x^2(\omega(t)) \right.\right. \\ & \left.\left. +\sum_{j=1}^{n-2}\frac{(cn^3\lambda_{max}(P_o)+\kappa_z^*)}{\alpha_{z_j}}\sup_t \chi_{z_j}(\omega(t))+\sigma_\theta\frac{{\theta^*}^2}{2}\right\}\right]^{\frac{1}{\min(1,2b_2)}} \quad (59)\end{aligned}$$

With $\Omega > 0$, it follows from (45) that r becomes larger than $\overline{R}$ within a finite time. Hence, the convergence of ξ_n to the compact set (58) proved above implies convergence of ξ_n to the set $[-\epsilon_\xi, \epsilon_\xi]$. $\diamond$

Theorem 2: Under the assumptions of Theorem 1, if, furthermore, $\chi_{x_i} \equiv 0, i = 1,\ldots,n-2$ and $\chi_{z_i} \equiv 0, i = 1,\ldots,n-2$, then picking $\sigma_\theta = 0$, asymptotic regulation of the state and input to the origin is achieved, i.e., $\lim_{t\to\infty} x_i(t) = 0, i = 1,\ldots,n$, $\lim_{t\to\infty} \hat{x}_i(t) = 0, i = 1,\ldots,n$, $\lim_{t\to\infty}|z_i(t)| = 0, i = 1,\ldots,n-1$, and $\lim_{t\to\infty} u(t) = 0$.

Proof of Theorem 2: With $\sigma_\theta = 0$, the foregoing analysis on existence and uniqueness of solutions must be modified slightly since σ_V defined in (53) reduces to zero when $\sigma_\theta = 0$. However, the existence and uniqueness of solutions and boundedness of all closed-loop signals can be inferred from (51) by noting that with $\sigma_\theta = 0$, $\chi_{x_i} \equiv 0, i = 1,\ldots,n-2$, and $\chi_{z_i} \equiv 0, i = 1,\ldots,n-2$, we have

$$\dot{V} \quad \leq \quad -\frac{1}{r}\tilde{\sigma}_V V \quad (60)$$

$$\begin{aligned}\tilde{\sigma}_V \quad = \quad & \min\left\{\frac{\sigma\nu_{1o}}{2\lambda_{max}(P_o)}, \frac{\sigma\nu_{1c}}{4\lambda_{max}(P_c)}, \right. \\ & \left. \frac{\kappa_z^*\alpha_{z_j}}{\overline{V}_{z_j}(cn^3\lambda_{max}(P_o)+\kappa_z^*)}\right\}. \quad (61)\end{aligned}$$

The conclusions of Theorem 2 follow directly from (60).

Remark 5: From the stability analysis, it is seen that only x_1 measurement is required (i.e., measurement of x_n not necessary) in the special case in which $\phi_{(i,i+1)}, i = 1,\ldots,n-1$, and μ depend only on x_1, γ_1 and γ_o are bounded functions, and θ is known. In this case, it is not necessary to build the estimator $\hat{\theta}$ and the term $\hat{\theta}$ in (45) can be replaced by θ^2.

Remark 6: The polynomial boundedness of functions γ_1 and γ_o occuring in the bounds in Assumptions A2, A4, and A5 can be relaxed if the bounds in Assumptions A2 and A5 do not explicitly involve u, i.e., if any dependence on the control input is bounded. In this case, the dynamic extension

ξ_{n+1} introduced in Section IV is not required. Instead, the control input u designed in (21) incorporates an additional term $-\frac{b_2}{\mu}\frac{\dot{r}}{r}x_n$. The assumption $\gamma_u \equiv 0$ (i.e., that the bounds in Assumptions A2 and A5 do not involve u) is crucial for the introduction of this term to ensure that $\dot{r}$ does not appear in the bounds on ϕ_i. The extra term in u obviates the need for the additional scaling $\frac{1}{r^{2b}}$ in the Lyapunov function definition (32). The removal of this additional scaling implies that boundedness of x_n follows directly from boundedness of V. Thus, it is not necessary to appeal to a polynomial boundedness assumption on γ_1 and γ_o as was done in the foregoing stability analysis. The stability and convergence of the closed-loop system can be proved by using the techniques in [2] along with the new scaling techniques in this paper. The details are omitted here for brevity.

VI. An Illustrative Example

Consider the seventh-order system

$$\begin{aligned}
\dot{z}_1 &= -z_1 + \theta_1 x_4 x_1 x_3 \\
\dot{z}_2 &= -(1+z_3^2)z_2 + \omega + x_4^2 x_1 \\
\dot{z}_3 &= -2z_3 + x_1 x_4 u \\
\dot{x}_1 &= (1+x_1^2+x_4^2)x_2 + x_3 x_1^2 + \theta_2 x_1 x_4^2 + x_1 z_1 + x_4\omega + x_4 z_2 \\
\dot{x}_2 &= (1 + x_1 x_4 + x_1^2 + x_4^2)x_3 + \theta_3 x_4^5 u + x_1\omega + x_4^2 z_3 \\
\dot{x}_3 &= (2 + x_1^2 + x_4^2 + 0.5x_4^2\sin(x_1))x_4 + \theta_4 x_1 x_4 u + x_4^3 z_3 \\
\dot{x}_4 &= (1+x_1^2)u \\
y &= [x_1, x_4]^T
\end{aligned} \tag{62}$$

with $\theta_1, \ldots, \theta_4$ being unknown parameters with no known magnitude bounds. The system (62) is of the form (1) with $n = 4$, $q_1 = -z_1 + \theta_1 x_4 x_1 x_3$, $q_2 = -(1+z_3^2)z_2 + \omega + x_4^2 x_1$, $q_3 = -2z_3 + x_1 x_4 u$, $\phi_{(1,2)} = 1 + x_1^2 + x_4^2$, $\phi_{(2,3)} = 1 + x_1 x_4 + x_1^2 + x_4^2$, $\phi_{(3,4)} = 2 + x_1^2 + x_4^2 + 0.5x_4^2\sin(x_1)$, $\mu = 1 + x_1^2$, $\phi_1 = x_3 x_1^2 + \theta_2 x_1 x_4^2 + x_1 z_1 + x_4\omega + x_4 z_2$, $\phi_2 = \theta_3 x_4^5 u + x_1\omega + x_4^2 z_3$, and $\phi_3 = \theta_4 x_1 x_4 u + x_4^3 z_3$. Assumptions A1 and A3 are satisfied with $\sigma = 1$, $\rho_2 = 3/2$, $\tilde{\rho}_2 = 1/2$, $\rho_3 = 4$, and $\tilde{\rho}_3 = 1/3$. Assumptions A2, A4, and A5 are also satisfied with $\gamma_1 = 1 + x_4^2 + x_4^4$, $\gamma_2 = \gamma_u = 1$, $\chi_{x_1} = \chi_{x_2} = |\omega|$, $\chi_{x_3} = 0$, $\theta = \max(1, \frac{1}{2}\theta_1^2, \frac{1}{2}|\theta_2|, |\theta_3|, \frac{1}{2}|\theta_4|)$, $\gamma_o = 2 + \frac{3}{2}x_4^2$, $V_{z_i} = \frac{1}{2}z_i^2, i = 1, 2, 3$, $\alpha_{z_1} = \alpha_{z_2} = \frac{1}{2}$, $\alpha_{z_3} = \frac{3}{2}$, $\chi_{z_2} = \omega^2$, and $\chi_{z_1} = \chi_{z_3} = 0$.

Thus, Assumptions A1-A5 are satisfied by system (62) and hence, the output-feedback control scheme proposed in this paper is applicable to the system (62). In this case, the observer and controller are given by

$$\begin{aligned}
\dot{\hat{x}}_1 &= (1+x_1^2+x_4^2)\hat{x}_2 + \frac{g_1(y)}{r}(\hat{x}_1 - x_1) \\
\dot{\hat{x}}_2 &= (1 + x_1 x_4 + x_1^2 + x_4^2)\hat{x}_3 + \frac{g_2(y)}{r^2}(\hat{x}_1 - x_1) \\
\dot{\hat{x}}_3 &= (2 + x_1^2 + x_4^2 + 0.5x_4^2\sin(x_1))\hat{x}_4 + \frac{g_3(y)}{r^3}(\hat{x}_1 - x_1) \\
\dot{\hat{x}}_4 &= (1+x_1^2)u + \frac{g_4(y)}{r^4}(\hat{x}_1 - x_1) \\
u &= \frac{\rho_4(2 + x_1^2 + x_4^2 + 0.5x_4^2\sin(x_1))\xi_5}{(1+x_1^2)r} \\
\dot{\xi}_5 &= \sum_{i=1}^{4}\frac{k_i(y)}{r^{5-i}}\hat{x}_i + \frac{k_5(y)}{r}\xi_5 - b_v\frac{\dot{r}}{r}\xi_5
\end{aligned} \tag{63}$$

with ρ_4 and b_v being any positive constants and $g_1, \ldots, g_4, k_1, \ldots, k_5$ being functions chosen as in Remark 2. The overall controller is given by (63), (44), and (45).

VII. Conclusion

We proposed an adaptive output-feedback control design technique for feedforward systems with ISS appended dynamics and disturbance inputs. Prior work on both ISS appended dynamics and disturbance attenuation has been restricted to systems of the strict-feedback form. Furthermore, previous results required the ISS appended dynamics to be driven only by the output of the system and the dynamic high-gain scaling technique based controller in [4] provided the first results for state-level coupling with ISS appended dynamics. The results in this paper confirm the expectation noted in [2] that the new paradigm of dynamic high-gain scaling should allow extensions for feedforward systems along various lines that have been hitherto investigated only for strict-feedback systems.

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