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# Generalised Filters and Stochastic Sampling Zeros

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***Abstract*— It is well-known that the zeros of sampled-data models for deterministic systems depend on the hold device used to generate the continuous-time system input. A dual result holds also for stochastic systems. In the latter case, the zeros of the sampled-data model depend on the anti-aliasing filter used before sampling. Generically, extra zeros appear in both deterministic and stochastic discrete-time models. These zeros have no continuous-time counterpart. This paper presents an anti-aliasing filter design, such that the *stochastic sampling zeros* are asymptotically assigned to the origin, for fast sampling rates. The design procedure relies only on the system relative degree.**

## I. Introduction

The sampling process is a key element when obtaining discrete-time to represent continuous-time systems [1]. Sampled-data models will depend not only on the underlying continuous-time system characteristics, but also on the *artifacts* of the sampling process itself, i.e., how the continuous-time input is generated, and how samples are taken from the continuous-time output.

For deterministic systems, it is well-known that the poles of the discrete-time pulse transfer function depend only on the sampling period and the poles of the underlying continuous-time system [2]. However, the relationship between the continuous and discrete-time zeros is much more involved. Furthermore, the sampled-data model is known to have *sampling zeros* with no continuous-time counterpart. For a system of relative degree $r$, there are exactly $r-1$ of these zeros. As the sampling rate increases, these zeros converge to specific locations in the complex plane. Moreover, the (asymptotic) zero locations depend on the *hold device* used to generate the continuous-time input to the system [3], [4], [5].

For stochastic systems, *i.e.*, systems with *white-noise* input, sampled-data models have, in general, relative degree 0. The poles depend only on the sampling period and the poles of the underlying continuous-time model. However, as for the deterministic case, the stochastic discrete-time model has extra zeros which have no continuous-time counterpart. These *stochastic sampling zeros* converge to particular locations in the complex plane as the sampling rate increases, depending on the system relative degree. Furthermore, the (asymptotic) location of these zeros is known to depend on the choice of the **anti-aliasing filter** used before sampling. In particular, three different cases have been discussed in the literature: *pure* instantaneous sampling (i.e., no pre-filter), rational filter, and *integrating* filter [6].

In this paper we show how to design the anti-aliasing filter to asymptotically assign the stochastic sampling zeros. In particular, we assign them to the origin leading to an output spectrum with no zeros. The anti-aliasing filter is characterised as a generalised sampling filter (GSF), defined by its *impulse response* [5]. The proposed GSF design procedure is independent of the specific plant model, relying only on knowledge of the system relative degree.

The authors have previously obtained similar results for the design of generalised hold devices to asymptotically assign the *deterministic* sampling zeros [7]. Other duality results between generalised hold devices and generalised sampling filters have been previously highlighted. In [5], for example, an optimal sampled-data control problem, using a zero-order hold input, was shown to be dual to an optimal state estimation problem using an integrating filter before sampling the system output.

The result presented in the current paper could be applied, for example, when estimating continuous-time autoregressive (CAR) models from sampled data. Replacing derivatives by divided differences in these models and using standard least squares estimation, may lead to very poor estimates of the model parameters [8]. In [9], a filtered least square algorithm is proposed by prefiltering the data using the stochastic sampling zeros polynomial. Alternatively, one could utilise the GSF described here to filter the system output *before* instantaneous sampling. This would lead to unbiased results without the need for any data prefiltering.

We also analyse the robustness of the GSF design to high frequency modelling errors.

## II. Sampled-data models for stochastic systems

Consider the following continuous-time system:

$$A(\rho)y(t) = B(\rho)\dot{v}(t) \tag{1}$$

where the input $\dot{v}(t)$ is a *continuous-time white noise* (CTWN) process with (constant) spectral density equal to 1, and:

$$A(\rho) = \rho^n + a_{n-1}\rho^{n-1} + \ldots + a_0 \tag{2}$$

$$B(\rho) = b_m\rho^m + b_{m-1}\rho^{m-1} + \ldots + b_0 \tag{3}$$

are polynomials in the derivative operator $\rho = \frac{d}{dt}$, and $n > m$. The continuous-time system (1) can be represented in state space form:

$$\rho x(t) = \frac{dx(t)}{dt} = Ax(t) + B\dot{v}(t) \tag{4}$$

$$y(t) = Cx(t) \tag{5}$$

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where $A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^n$.

*Remark 1:* The CTWN process that appears as input in (1) is a mathematical abstraction that does not exist in any meaningful sense [10], [11]. In fact, (4) should be considered as a stochastic differential equation (SDE):

$$dx(t) = Ax(t)dt + Bdv \tag{6}$$

where $v(t)$ is a Wiener process with zero mean and unitary incremental variance [5]:

$$E\{(dv)^2\} = E\{|v(t+dt) - v(t)|^2\} = dt \tag{7}$$

However, linear systems modelled by an SDE as in (6) can be formally analysed by the standard state-space model (4) [12]. □

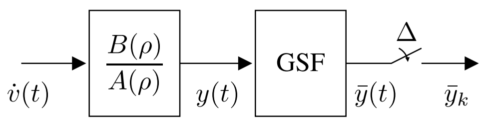


Fig. 1. Sampling scheme using a Generalised Sampling Filter.

We assume a sampling scheme as shown in Figure 1, where a Generalised Sampling Filter (GSF) is used prior to sampling the system output. The output sequence is obtained by instantaneously sampling the output of this filter:

$$\bar{y}_k = \bar{y}(k\Delta) = \frac{1}{\Delta}\int_{k\Delta-\Delta}^{k\Delta} y(\tau)h_g(\tau - k\Delta)d\tau \tag{8}$$

where $h_g(t)$ is the *impulse response* of the GSF defined on the interval $[0, \Delta)$.

*Remark 2:* The GSF defined in (8) can be understood as a generalisation of the, so called, *integrating filter*:

$$\bar{y}_k = \bar{y}(k\Delta) = \frac{1}{\Delta}\int_{k\Delta-\Delta}^{k\Delta} y(\tau)d\tau \tag{9}$$

This is also called *averaging filter*, and its impulse response in given by:

$$h_g(t) = \begin{cases} 1/\Delta & ; t \in [0,\Delta) \\ 0 & ; t \notin [0,\Delta) \end{cases} \tag{10}$$

□

The following result allows one to obtain a discrete-time description of the sampling scheme in Figure 1 in terms of the delta operator $\delta = \frac{q-1}{\Delta}$ [1]. This model is **exact** in the sense that its output sequence has the same second order properties as the continuous-time output *at the sampling instants*.

*Lemma 1:* Consider the sampling scheme in Figure 1. If the output of system (1) (or, equivalently, (4)–(5)) is prefiltered using a GSF with impulse response $h_g(t)$, then the exact discrete-time model is given by:

$$\delta x_k = A_\delta x_k + v_k \tag{11}$$
$$\bar{y}_{k+1} = C_g x_k + w_k \tag{12}$$

where:

$$A_\delta = \frac{e^{A\Delta} - I}{\Delta} \qquad C_g = \int_0^\Delta h_g(\tau)Ce^{A(\Delta-\tau)}d\tau \tag{13}$$

and $v_k$ and $w_k$ are white noise sequences such that:

$$E\left\{\begin{bmatrix} v_k \\ w_k\end{bmatrix}\begin{bmatrix} v_\ell \\ w_\ell\end{bmatrix}^T\right\} = \begin{bmatrix}\Omega_\delta & \Sigma_\delta \\ \Sigma_\delta^T & \Gamma_\delta\end{bmatrix}\frac{\delta_K[k-\ell]}{\Delta} \tag{14}$$

where $\delta_K$ represents the delta Kronecker function, and we have defined the matrices:

$$\begin{bmatrix}\Omega_\delta & \Sigma_\delta \\ \Sigma_\delta^T & \Gamma_\delta\end{bmatrix} \triangleq \frac{1}{\Delta}\int_0^\Delta M_g(\sigma)M_g(\sigma)^T d\sigma \tag{15}$$

$$M_g(\sigma) \triangleq \begin{bmatrix} e^{A\sigma}B \\ \Delta\int_0^\sigma h_g(\xi)Ce^{A(\sigma-\xi)}Bd\xi\end{bmatrix} \tag{16}$$

*Proof:* The proof is similar to that of Lemma 6.4.1 in [5]. Details can be found in [13]. □

*Remark 3:* The exact discrete-time model (11)–(12) can equivalently be rewritten using the shift operator $q$ as:

$$q\, x_k = x_{k+1} = A_q x_k + \tilde{v}_k \tag{17}$$
$$\bar{y}_k = C_g x_k + w_k \tag{18}$$

where $A_q = I + \Delta A_\delta = e^{A\Delta}$, and the input noise sequence is $\tilde{v}_k = \Delta v_k$. As a consequence:

$$E\left\{\begin{bmatrix} \tilde{v}_k \\ w_k\end{bmatrix}\begin{bmatrix} \tilde{v}_\ell \\ w_\ell\end{bmatrix}^T\right\} = \begin{bmatrix}\Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q\end{bmatrix}\delta_K[k-\ell] \tag{19}$$

where $\Omega_q = \Delta\Omega_\delta$, $\Sigma_q = \Sigma_\delta$, and $\Gamma_q = \frac{1}{\Delta}\Gamma_\delta$.

□

Even though Lemma 1 gives an exact sampled-data model for the system in Figure 1, this discrete-time description depends on two noise sequences as inputs. The following lemma allows one to express $\bar{y}_k$ as the output of a system with a single (scalar) white noise input, i.e., it is in *innovations form* [14].

*Lemma 2:* The state-space model (17)–(18) and the following innovations model are equivalent in the sense that their outputs share the same second order properties:

$$z_{k+1} = A_q z_k + K_q e_k \tag{20}$$
$$\bar{y}_k = C_g z_k + e_k \tag{21}$$

where $e_k$ is a white noise sequence with covariance matrix

$$E\{e_k^2\} = \Gamma_q + C_g P C_g^T \tag{22}$$

The Kalman gain, $K_q$, is given by:

$$K_q = (A_q P C_g^T + \Sigma_q)(\Gamma_q + C_g P C_g^T)^{-1} \tag{23}$$

and $P$ is the state covariance matrix given by the discrete-time Riccati equation:

$$A_q P A_q^T - P - K_q(\Gamma_q + C_g P C_g^T)K_q^T + \Omega_q = 0 \tag{24}$$

□

A consequence of the innovations form in Lemma 2 is that the sequence of output samples $\bar{y}_k$ can be exactly described by the model:

$$\bar{y}_{k+1} = H(q)\, e_k \tag{25}$$
$$H(q) = C_g(qI - A_q)^{-1}K_q + 1 \tag{26}$$

*Remark 4:* Equation (26) clearly shows that the discrete-time poles depend only on $A_q$, i.e., on the continuous-time system matrix $A$ and the sampling period $\Delta$. However, the zeros of the model depend on $C_q$ and $K_q$, and, thus, on the GSF impulse response $h_g(t)$. □

Given a GSF, Lemmas 1 and 2 provide a systematic way of obtaining a sampled-data model for a given continuous-time system. However, given a desired location for the sampling zeros, is not clear how to obtain a GSF that fulfils these requirements. In particular, the procedure would require the solution of the Riccati equation (24) in terms of the GSF impulse response $h_g(t)$. In the next section we will see that a more basic approach can be followed to design a GSF to achieve a designated numerator polynomial of the output spectrum.

## III. Sampling Zeros of the output spectrum.

In this section we analyse how the zeros of the spectral density of $\bar{y}_k$ depend on the choice of the GSF. We argue that the role of the impulse response $h_g(t)$ in the output spectrum can be described more directly.

*Lemma 3:* Given the discrete-time model (17)–(19), the discrete-time output spectrum, expressed in the complex variable $z = e^{j\omega\Delta}$, is given by:

$$\Phi_{\bar{y}}(z) = \Delta \begin{bmatrix} C_g(zI_n - A_q)^{-1} & 1 \end{bmatrix} \times \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} \begin{bmatrix} (z^{-1}I_n - A_q^T)^{-1}C_g^T \\ 1 \end{bmatrix} \tag{27}$$

*Proof:* From (17)–(18), we note that the discrete-time output can be expressed as:

$$\bar{y}_{k+1} = \begin{bmatrix} C_g(qI_n - A_q)^{-1} & 1 \end{bmatrix} \begin{bmatrix} \tilde{v}_k \\ w_k \end{bmatrix} = \mathbf{H}(q)\, \mathbf{n}_k \tag{28}$$

Thus, the output spectrum can be obtained as:

$$\Phi_{\bar{y}}(z) = \mathbf{H}(z)\, \mathbf{\Phi_n} \left[\mathbf{H}(z^{-1})\right]^T \tag{29}$$

where $\mathbf{\Phi_n}$ is the spectral density of the two noise sources vector $\mathbf{n}_k$. From (19), we have that:

$$\mathbf{\Phi_n} = \mathcal{F}_d\{E\{\mathbf{n}_{\ell+k}(\mathbf{n}_\ell)^T\}\} = \Delta \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} \tag{30}$$

Note that we have used the discrete-time Fourier transform as in [5], with a scaling factor $\Delta$ in front, which allows a clearer connection to the continuous-time case. □

The result in Lemma 3 is closely related to Lemma 2, noting that the output spectrum of the innovations model (20)–(21) is given by:

$$\Phi_{\bar{y}}(z) = H(z)H(z^{-1})\Phi_e \tag{31}$$

where the spectral factor $H(z)$ is given by (26) and $\Phi_e = \Delta(\Gamma_q + C_gPC_g^T)$ is the (constant) spectral density of the innovations sequence.

As pointed out above, the zeros of $H(z)$ in (26) depend on the choice of the GSF impulse response $h_g(t)$ (matrices $C_g$ and $K_q$). This is apparent also in (27), where $C_g$, $\Sigma_q$, and $\Gamma_q$ are functions of the GSF. However, in the first case, determining $K_q$ involves the solution of the Riccati equation (24). Thus, we can compute the output spectrum as in (27), and then we can obtain a (stable) spectral factor $H(z)$ such that (31) holds.

In the next section, we follow this approach to assign the stochastic sampling zeros of $\Phi_{\bar{y}}(z)$ (and, thus, the sampling zeros of the spectral factor $H(z)$) by choosing the coefficients of an appropriate parametrisation of the GSF impulse response $h_g(t)$.

## IV. Generalised filters to assign the asymptotic sampling zeros

In this section we turn to the key focus of the current paper, namely, how to design a GSF such that the stochastic sampling zeros converge asymptotically to specific locations in the complex plane, as the sampling period goes to zero. In particular, we are interested in assigning the sampling zeros to the origin. Equivalently, we seek an output spectrum with no stochastic sampling zeros.

The choice of the GSF to assign the sampling zeros is not unique. Thus, we restrict ourselves to a class of filters whose impulse response satisfies the following constraint.

*Assumption 1:* Given a system relative degree $r$, we consider a GSF such that its impulse response can be parametrised as:

$$h_g(t) = \begin{cases} \frac{1}{\Delta}(h_0 + \sum_{\ell=1}^{r} h_\ell \phi_\ell(t)) & ; t \in [0, \Delta) \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{32}$$

where $h_0, \dots, h_r \in \mathbb{R}$, and the functions $\phi_\ell(t)$ satisfy the following condition:

$$\int_0^\Delta \phi_\ell(t)dt = 0 \tag{33}$$

□

Note that we have introduced the scaling factor $\frac{1}{\Delta}$ in (32) to resemble the *averaging* idea of the integrating filter (10). In fact, the integrating filter corresponds to the choice $h_0 = 1$ and $h_\ell = 0$, for $\ell = 1 \dots r$. We will see that the condition (33) simplifies some of the calculations required to obtain the output spectrum (27).

*Remark 5:* Assumption 1 guarantees that, once the functions $\phi_\ell(t)$ in (32) are chosen, the $r + 1$ coefficients $h_0, \dots, h_r$ provide enough *degrees of freedom* to assign the $r$ sampling zeros and the noise variance, if required. □

The design procedure presented in this section is based on a key limiting argument used in contemporary results regarding asymptotic behaviour of the sampling zeros [3], [6], [15]. Specifically, for fast sampling rates, any linear system (with a rational transfer function) of relative degree $r$ evolves, at high frequencies, as an $r$-order integrator.

Thus, in the following sections, we consider the first and second order integrator cases and we show how different GSFs can be designed to assign their asymptotic sampling zeros. We also show that when using the obtained GSFs for arbitrary systems of relative degree 1 and 2, similar stochastic sampling zeros are obtained when using fast sampling rates.

### A. First order integrator

We consider the first order integrator $\frac{B(\rho)}{A(\rho)} = \rho^{-1}$. The matrices of the corresponding state space representation (4)–(5) are given by $A = 0$, $B = 1$, and $C = 1$. Note that, for any given sampling period $\Delta$, $A_q = 1$.

*Example 1 (Integrating filter):* This is one of the filters considered in [6], where asymptotic results are obtained for fast sampling rates. The impulse response, in this case, is defined in (10). For this choice we have:

$$C_g = 1 \qquad \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} = \begin{bmatrix} \Delta & \frac{\Delta}{2} \\ \frac{\Delta}{2} & \frac{\Delta}{3} \end{bmatrix} \tag{34}$$

which, replacing in (27), yields:

$$\Phi_{\bar{y}}(z) = \frac{\Delta^2}{3!} \frac{(z + 4 + z^{-1})}{(z-1)(z^{-1}-1)} \tag{35}$$

This is consistent with the asymptotic result in [6, Theorem 3.2], and we can readily obtain a sampled-data model by spectral factorisation as in (31):

$$H(z) = \frac{\Delta(\sqrt{3}+1)}{2\sqrt{3}} \frac{(z + 2 - \sqrt{3})}{(z-1)} \tag{36}$$

□

*Example 2 (Piecewise constant GSF):* This GSF has the same kind of impulse response filter as the generalised hold functions considered in [7]. Here, however we parameterise $h_g(t)$ in a slightly different way:

$$h_g(t) = \begin{cases} \frac{1}{\Delta}(h_0 + h_1) & ; t \in [0, \frac{\Delta}{2}) \\ \frac{1}{\Delta}(h_0 - h_1) & ; t \in [\frac{\Delta}{2}, \Delta) \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{37}$$

where $h_0, h_1 \in \mathbb{R}$. For such a choice we have:

$$C_g = h_0 \tag{38}$$

$$\begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} = \begin{bmatrix} \Delta & \frac{\Delta}{2}\left(h_0 + \frac{1}{4}h_1\right) \\ \frac{\Delta}{2}\left(h_0 + \frac{1}{4}h_1\right) & \frac{\Delta}{3}\left(h_0^2 + \frac{3}{4}h_0h_1 + \frac{1}{4}h_1^2\right) \end{bmatrix} \tag{39}$$

which, on substituting into (27), gives:

$$\Phi_{\bar{y}}(z) = h_0^2 \frac{\Delta^2}{3!} \frac{\left(z + 4 + z^{-1}\right)}{(z-1)(z^{-1}-1)} + \frac{h_1^2}{2} \frac{\Delta^2}{3!} \frac{\left(-z + 2 - z^{-1}\right)}{(z-1)(z^{-1}-1)} \tag{40}$$

If we now choose, for example, $h_0 = 1$ and $h_1 = \sqrt{2}$, we obtain a sampled spectrum with **no** zeros, or, equivalently, a stable spectral factor with zeros at the origin:

$$\Phi_{\bar{y}}(z) = \frac{\Delta^2}{(z-1)(z^{-1}-1)} \quad \Rightarrow \quad H(z) = \frac{\Delta z}{(z-1)} \tag{41}$$

□

*Example 3 (Sinusoidal GSF):* Another simple GSF impulse response which satisfies Assumption 1 is given by:

$$h_g(t) = \begin{cases} \frac{1}{\Delta}\left(h_0 + h_1 \pi \sin\left(\frac{2\pi}{\Delta}t\right)\right) & ; t \in [0, \Delta) \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{42}$$

where $h_0, h_1 \in \mathbb{R}$, and the constant $\pi$ has been introduced only as a scaling factor. For this choice we have:

$$C_g = h_0 \tag{43}$$

$$\begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} = \begin{bmatrix} \Delta & \frac{\Delta}{2}(h_0 + h_1) \\ \frac{\Delta}{2}(h_0 + h_1) & \frac{\Delta}{3}\left(h_0^2 + \frac{3}{2}h_0h_1 + \frac{9}{8}h_1^2\right) \end{bmatrix} \tag{44}$$

Upon substituting into (27), we obtain:

$$\Phi_{\bar{y}}(z) = \frac{\Delta^2}{3!}\left[\frac{h_0^2\left(z + 4 + z^{-1}\right)}{(z-1)(z^{-1}-1)} + \frac{9h_1^2}{4}\frac{\left(-z + 2 - z^{-1}\right)}{(z-1)(z^{-1}-1)}\right] \tag{45}$$

If we now choose, for example, $h_0 = 1$ and $h_1 = 2/3$, we obtain again a sampled spectrum (and a stable spectral factor) as in (41). □

The GSFs obtained in Examples 2 and 3 allow one to assign the sampling zeros of the discrete-time model of a stochastic first order integrator. Specifically, we chose the *weighting* coefficients $h_\ell$ to assign these zeros to the origin. However, this GSF can also be used, for fast sampling rates, with *any* system of relative degree 1 to obtain asymptotic sampling zeros *near* the origin. We illustrate this principle in the following example.

*Example 4:* We consider the continuous-time system:

$$\frac{B(\rho)}{A(\rho)} = \frac{1}{\rho + 2} \tag{46}$$

We fix the sampling period to be $\Delta = 0.1$, which corresponds to a sampling frequency around one decade above the model bandwidth.

If we use the piecewise GSF obtained in Example 2, we obtain the following stable spectral factor of the output spectrum:

$$H(z) = \frac{0.287(z - 2.489 \cdot 10^{-4})}{(z - e^{-0.2})} \tag{47}$$

On the other hand, if we use the *sinusoidal* GSF obtained in Example 3, we obtain:

$$H(z) = \frac{0.287(z - 6.590 \cdot 10^{-5})}{(z - e^{-0.2})} \tag{48}$$

Note that, as expected, for both cases the only sampling zero is basically at the origin. □

### B. Second order integrator

For a second order integrator, associated with any system of relative degree 2, the expressions that allow one to describe the stochastic sampling zero are more involved. However, the design procedure outlined above can be readily adapted as we show next.

Thus, consider the second order integrator $\frac{B(\rho)}{A(\rho)} = \rho^{-2}$. The state space representation (4)–(5) is given by:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{49}$$

Given sampling period $\Delta$, we have that:

$$A_q = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} \tag{50}$$

*Example 5 (Integrating filter):* This filter is given by (10). In this case we have that:

$$C_g = \begin{bmatrix} 1 & \frac{\Delta}{2} \end{bmatrix} \quad \begin{bmatrix} \Omega_q & \Sigma_q \\ \Sigma_q^T & \Gamma_q \end{bmatrix} = \left[\begin{array}{cc|c} \frac{\Delta^3}{3} & \frac{\Delta^2}{2} & \frac{\Delta^3}{8} \\ \frac{\Delta^2}{2} & \Delta & \frac{\Delta^2}{6} \\ \hline \frac{\Delta^3}{8} & \frac{\Delta^2}{6} & \frac{\Delta^3}{20} \end{array}\right] \tag{51}$$

which, substituting into (27), gives:

$$\Phi_{\bar{y}}^{IF}(z) = \frac{\Delta^4}{5!} \frac{(z^2 + 26z + 66 + 26z^{-1} + z^{-2})}{(z-1)^2(z^{-1}-1)^2} \tag{52}$$

Which, again, is consistent with the asymptotic result [6, Theorem 3.2].

□

*Example 6 (Piecewise constant GSF):* We consider a GSF defined by its impulse response:

$$h_g(t) = \begin{cases} \frac{1}{\Delta}(h_0 + h_1 + h_2) & ; t \in [0, \frac{\Delta}{4}) \\ \frac{1}{\Delta}(h_0 + h_1 - h_2) & ; t \in [\frac{\Delta}{4}, \frac{\Delta}{2}) \\ \frac{1}{\Delta}(h_0 - h_1 + h_2) & ; t \in [\frac{\Delta}{2}, \frac{3\Delta}{4}) \\ \frac{1}{\Delta}(h_0 - h_1 - h_2) & ; t \in [\frac{3\Delta}{4}, \Delta) \\ 0 & ; t \notin [0, \Delta) \end{cases} \tag{53}$$

where $h_0, h_1, h_2 \in \mathbb{R}$. For this choice we have:

$$C_g = \begin{bmatrix} h_0 & \frac{\Delta}{2}\left(h_0 + \frac{1}{2}h_1 + \frac{1}{4}h_2\right) \end{bmatrix} \tag{54}$$

Computing the noise spectrum (30) and replacing in (27), we obtain:

$$\Phi_{\bar{y}}(z) = h_0^2 \Phi_{\bar{y}}^{IF}(z) + h_1^2 \Phi_{\bar{y}}^1(z) + h_2^2 \Phi_{\bar{y}}^2(z) + h_1 h_2 \Phi_{\bar{y}}^3(z) \tag{55}$$

where $\Phi_{\bar{y}}^{IF}(z)$ is spectrum (52) obtained in Example 5, and $\Phi_{\bar{y}}^{\ell}(z)$ ($\ell = 1, 2, 3$) are other spectra that do not depend on the GSF parameters. Solving numerically for the parameters, we obtain that any of the following choices:

$$h0 = 1 \qquad h1 = \mp 9.891 \qquad h2 = \pm 23.782 \tag{56}$$

$$\text{or} \quad h0 = 1 \qquad h1 = \mp 4.691 \qquad h2 = \pm 5.382 \tag{57}$$

leads us to a sampled spectrum with **no** zeros:

$$\Phi_{\bar{y}}(z) = \frac{\Delta^4}{5!} \frac{K}{(z-1)^2(z^{-1}-1)^2} \tag{58}$$

□

*Example 7 (Sinusoidal GSF):* Here we restrict the GSF impulse response to the form:

$$h_g(t) = \begin{cases} \frac{1}{\Delta}\left(h_0 + h_1 \pi \sin\left(\frac{2\pi}{\Delta}t\right) + h_2 \pi \sin\left(\frac{4\pi}{\Delta}t\right)\right) \\ \qquad\qquad\qquad\qquad ; t \in [0, \Delta) \\ 0 \quad ; t \notin [0, \Delta) \end{cases} \tag{59}$$

where $h_0, h_1, h_2 \in \mathbb{R}$. For this choice we have:

$$C_g = \begin{bmatrix} h_0 & \frac{\Delta}{2}\left(h_0 + h_1 + \frac{1}{2}h_2\right) \end{bmatrix} \tag{60}$$

Computing the noise spectrum (30) and substituting in (27), gives:

$$\Phi_{\bar{y}}(z) = h_0^2 \Phi_{\bar{y}}^{IF}(z) + h_1^2 \Phi_{\bar{y}}^1(z) + h_2^2 \Phi_{\bar{y}}^2(z) + h_1 h_2 \Phi_{\bar{y}}^3(z) \tag{61}$$

where $\Phi_{\bar{y}}^{IF}(z)$ is the spectrum (52) obtained in Example 5, and $\Phi_{\bar{y}}^{\ell}(z)$ ($\ell = 1, 2, 3$) are other spectra that do not depend on the GSF parameters. Solving numerically for the parameters, we see that for any of the following choices:

$$h_0 = 1 \qquad h_1 = \pm 3.902 \qquad h_2 = \mp 9.804 \tag{62}$$

$$\text{or} \quad h_0 = 1 \qquad h_1 = \pm 1.902 \qquad h_2 = \mp 1.804 \tag{63}$$

yields a sampled spectrum with **no** zeros, as in (58). □

The GSFs described above can be used to assign the stochastic sampling zeros of general linear models of relative degree 2 *close* to the origin, when using *fast* sampling rates. In the following example we illustrate the use of the GSFs obtained in Examples 6 and 7 for a general system of relative degree 2.

*Example 8:* Consider the following second order system:

$$\frac{B(\rho)}{A(\rho)} = \frac{2}{(\rho + 2)(\rho + 1)} \tag{64}$$

We first use the piecewise GSF obtained in Example 6. In particular, in (53) we choose $h0 = 1$, $h1 = 4.691$, and $h2 = -5.382$. For a sampling period $\Delta = 0.1$, we obtain the following stable spectral factor:

$$H(z) = \frac{5.269 \cdot 10^{-2}(z - z_1)(z - z_1^*)}{(z - e^{-0.1})(z - e^{-0.2})} \tag{65}$$

where $z_1 = -0.014 + j0.081$, and where $^*$ denotes complex conjugation.

We also use the *sinusoidal* GSF obtained in Example 7. The sampling period is fixed to $\Delta = 0.01$. In (59) we choose $h0 = 1$, $h1 = 1.902$, and $h2 = -1.804$. The sampled-data model is then given by:

$$H(z) = \frac{0.22 \cdot 10^{-10}(z - z_1)(z - z_2)}{(z - e^{-0.01})(z - e^{-0.02})} \tag{66}$$

where $z_1 = -1.0435 \cdot 10^{-3}$ and $z_1 = -1.0439 \cdot 10^{-3}$.

Note that, as in Example 4, both GSFs design successfully assign the sampling zeros very close to the origin, as expected. □

## V. Robustness Issues

The design procedure describe in this paper relies only on knowledge of the relative degree of the continuous-time system. However, the presence of high frequency poles and/or zeros can make the relative degree difficult to define, especially when considering fast sampling. Moreover, a continuous-time white noise process is only a mathematical abstraction that can only be approximated, in practice, by standard processes with broad-band spectra [16]. The latter observation implies also potential high frequency modelling errors in the nominal continuous-time system description.

The issues raised above stress the fact that any GSF design of the type described in this paper should be applied only within a **bandwidth of validity** for the continuous-time model. This means that one has to restrict the sampling frequency to the bandwidth where one can rely on the relative degree assumption. The authors have previously highlighted this issue both for the design of generalised holds for deterministic systems [7], and for CAR system identification [17].

The following example illustrates the use of the GSF designed in Example 2, in the presence of unmodelled high frequency dynamics.

*Example 9:* Consider the presence of an unmodelled fast pole in the continuous-time system defined (46), i.e., the true system is then given by:

$$\frac{B(\rho)}{A(\rho)} = \frac{1}{(\rho+2)(\frac{1}{\omega_u}\rho+1)} \tag{67}$$

We use the piecewise constant GSF obtained in Example 2 for *nominal* systems of relative degree 1.

We assume an unmodelled fast pole located at $\omega_u = 200$[rad/s], and we consider the following two cases for the sampling period:

(i) $\Delta = 0.1$: This corresponds to a sampling frequency $\omega_s \approx 60$[rad/s]. In this case, the unmodelled pole lies well beyond the sampling frequency, so we expect no considerable effect on the sampled-data model. We obtain the spectral factor:

$$H(z) = \frac{1.1 \cdot 10^{-3}(z+1.2\cdot 10^{-2})(z-5.3\cdot 10^{-3})}{(z-e^{-0.2})(z-e^{-20})} \tag{68}$$

We can see that the sampling zeros and the (unmodelled) fast discrete-time pole are located close the origin. Thus, the system can be roughly approximated as in (47).

In the above case, we have chosen a sampling frequency *within the bandwidth* where the assumption on the relative degree is justified.

(ii) $\Delta = 0.01$: We now increase the sampling frequency to $\omega_s \approx 600$[rad/s]. The unmodelled pole, in this case, should ideally be considered in the GSF design. However, assuming that the presence of the high frequency pole is unknown, and using the same GSF as above, we obtain the following sampled-data model:

$$H(z) = \frac{3.8 \cdot 10^{-5}(z+2.1\cdot 10^{-1})(z-3.3\cdot 10^{-2})}{(z-e^{-0.02})(z-e^{-2})} \tag{69}$$

We see that, in this case, the slowest sampling zero is far from the origin. The reason for this outcome is understandable since the relative degree assumption on the *nominal* model is no longer valid at this sampling rate. □

The previous example confirms the heuristic notion that the *system relative degree* and our design procedure, should be considered in terms of a *bandwidth of validity* for the nominal model of the continuous-time system.

## VI. Conclusions

In this paper we have developed a procedure to design an anti-aliasing (generalised) filter which assigns the stochastic sampling zeros asymptotically to the origin, as the sampling period goes to zero.

The design procedure relies only on knowledge of the system relative degree. The proposed methodology is based on the observation that, at high frequencies (i.e., for fast sampling rates), any linear system of relative degree $r$ evolves as an $r$ order integrator.

The design procedure has been illustrated for systems of relative degree 1 and 2. Nonetheless, the general design principle can be adapted for any system having higher relative degree.

Finally, we have made an important observation regarding *nominal* continuous-time models when using fast sampling rates. In particular, relative degree may be an ill-defined quantity in continuous-time because of the presence of high frequency poles or zeros. Thus, the use of the proposed methodology should be considered within a *bandwidth of validity* where one can rely on the relative degree assumption.

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