Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

On the relation between analysis and synthesis conditions for discrete-time systems with saturation nonlinearities

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***Abstract*— This paper considers linear time-invariant (LTI) discrete-time systems with saturation and/or dead-zone nonlinearities, and proposes analysis and synthesis methods of a regional l_2 performance and/or a pole placement based on a quadratic Lyapunov function via a generalized sector and a polytopic approach. In particular, a new domain of l_2 performance is defined by a region of initial states of the system considering the l_2 performance and/or the pole placement. For analysis, the problems based on the two approaches can be recast as linear matrix inequality (LMI) optimization ones respectively, and in the special case of single saturation or single dead-zone nonlinearity, it is proved that the generalized sector approach is exactly the same as the polytopic approach. Similarly, for synthesis, the problem based on the generalized sector approach can be recast as an LMI optimization problem where the outputs of the nonlinearities are assumed to be available for control. Next, the relation is clarified that the analysis and synthesis conditions can be reduced to the corresponding conditions for the continuous-time systems as the sampling period goes to zero. Finally, it is pointed out that our LMI-based approach is helpful through a numerical example designing anti-windup control systems.**

I. INTRODUCTION

The actuator saturation has been recognized as a very important nonlinear element that could have a large impact on the control performance, and there is substantial body of literature on this subject [1]–[8]. Recently, non-conservative analysis conditions [4]–[6] have been derived from a quadratic Lyapunov function. These attractive results are based on the same idea with a new regional sector bound on the saturation and/or dead-zone nonlinearity: one is a multiplier (generalized sector) approach [5], [6] and the other is a polytopic one [4]. The relation between these two approaches has been clarified in the special case of stability analysis for the systems [7], however general relations still remain largely open to be solved.

This paper considers a regional l_2 performance analysis and synthesis for discrete-time systems with saturation and/or dead-zone nonlinearities based on the two approaches [7]. First, a new domain of l_2 performance for the systems is defined by a set of initial states with guaranteed an l_2 performance and a stability performance of a pole placement simultaneously to make it easy to apply the domain to actual control design problems, as will be seen as a numerical example in Subsection IV. *C*. Note that the domain in this paper is a general extension of the existing one in [7].

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For analysis, the analysis conditions of the domain can be recast as linear matrix inequality (LMI) conditions via the two approaches, respectively. Moreover, we point out that the conditions of the generalized sector approach are sufficient ones of the polytopic approach. In particular, the two analysis conditions are proved to be exactly the same in the case of single saturation or single dead-zone nonlinearity.

For synthesis, the problem based on the generalized sector approach can be recast as an LMI optimization problem where the outputs of the nonlinearities are assumed to be available for control [6]. Therefore, both a dynamic output feedback and an anti-windup controller can be simultaneously designed by our derived linear matrix inequalities (LMIs) where the anti-windup control system [1], [3], [6], [8] achieves a given domain of l_2 performance.

Next, this paper also clarifies the consistency of the analysis and/or the synthesis conditions between the discrete-time and the continuous-time systems where the conditions for the discrete-time systems can be reduced to the corresponding continuous-time case as the sampling period goes to zero.

Finally, the validity of our proposed approach is confirmed by a simple numerical example, and then this paper illustrates that the synthesis result of a saturating control input achieving a high control performance can be definitely obtained.

We use the following notation. The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n\times m}$. For a matrix M, M' denotes the transpose. For a vector x, x_i is the i^{th} entry of x. For vectors x and y, $x > y$ means that $x_i > y_i$ for all i, and similarly for $x \geq y$. For a symmetric matrix X, $\lambda_{max}(X)$ ($\lambda_{min}(X)$) denotes the maximum (minimum) eigenvalue. For a symmetric matrix X, $X > 0$ ($X \geq 0$) means that X is positive (semi)definite. For a square matrix Y, $\mathsf{He}(Y) := Y + Y'$. For a matrix M, $M^\perp \in \mathbb{R}^{(n-r)\times n}$ satisfies r = rank of M, $M^\perp M = 0$ and $M^\perp M^{\perp\prime} > 0$. For a vector x, $\|x\|$ means the Euclidean norm of x. For a sampling period h and a function $z \in l_2$, $\|z\|_{l_2} := \left(h\sum_{k=0}^{\infty} z'(kh)z(kh)\right)^{1/2}$. For a sampling period h and a function $z \in l_{2k}$, $\|z\|_{l_{2k}} := \left(h\sum_{\tau=0}^{k} z'(\tau h)z(\tau h)\right)^{1/2}$. For given vectors $w_1, ..., w_{\mathcal{I}}$ ($w_i \in \mathbb{R}^m$), a convex hull is defined by $\mathbf{Co}\{w_i : i \in [1, \mathcal{I}]\} := \left\{\sum_{i=1}^{\mathcal{I}} \alpha_i w_i : \ \sum_{i=1}^{\mathcal{I}} \alpha_i = 1, \ \alpha_i \geq 0\right\}$. Finally, for shift-operator $\mathbf{z}$, we use the "packed" notation

$$G(\mathbf{z}) =: \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

for a transfer function $G(\mathbf{z}) = C(\mathbf{z}I - A)^{-1}B + D$.

II. CONTROL SYSTEM WITH DEAD-ZONE NONLINEARITIES

First of all, let us start with the following generalized plant $\mathbf{G}(s)$ of a linear time-invariant (LTI) continuous-time system represented by

$$\left[\begin{array}{c}\dot{\boldsymbol{x}}(t)\\ \hdashline \boldsymbol{z}(t)\\ \boldsymbol{z}_p(t)\\ \hdashline \boldsymbol{y}(t)\end{array}\right]=\left[\begin{array}{c:cc:c}\mathbf{A} & \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3}\\ \hdashline \mathbf{C_1} & \mathbf{D_{11}} & \mathbf{D_{12}} & \mathbf{D_{13}}\\ \mathbf{C_2} & \mathbf{D_{21}} & \mathbf{D_{22}} & \mathbf{D_{23}}\\ \hdashline \mathbf{C_3} & \mathbf{D_{31}} & \mathbf{D_{32}} & 0\\ 0 & I & 0 & 0\end{array}\right]\left[\begin{array}{c}\boldsymbol{x}(t)\\ \hdashline \boldsymbol{w}(t)\\ \boldsymbol{w}_p(t)\\ \hdashline \boldsymbol{u}(t)\end{array}\right]. \quad (1)$$

The following $G(\mathbf{z})$ with shift-operator denotes the generalized plant of an LTI discrete-time system given by step invariant transformation of $\mathbf{G}(s)$ in (1) with zero-order-hold and ideal sampler. This paper considers the feedback control system in Figure 1 where the "disturbance" signal w is measurable, i.e., $G(\mathbf{z})$ is represented by

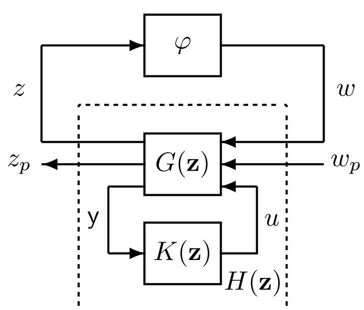

Fig. 1. Feedback control system with dead-zone nonlinearities.

$$\left[\begin{array}{c}x((k+1)h)\\ \hdashline z(kh)\\ z_p(kh)\\ \hdashline \mathsf{y}(kh)\end{array}\right]=\left[\begin{array}{c:cc:c}A & B_1 & B_2 & B_3\\ \hdashline C_1 & D_{11} & D_{12} & D_{13}\\ C_2 & D_{21} & D_{22} & D_{23}\\ \hdashline C_3 & D_{31} & D_{32} & 0\\ 0 & I & 0 & 0\end{array}\right]\left[\begin{array}{c}x(kh)\\ \hdashline w(kh)\\ w_p(kh)\\ \hdashline z(kh)\end{array}\right],$$
$$w(kh)=\varphi(z(kh)) \quad (2)$$

where $x(kh)\in\mathbb{R}^{n_p}$, $u(kh)\in\mathbb{R}^{p}$, $\mathsf{y}(kh)\in\mathbb{R}^{q}$, $w_p(kh)\in\mathbb{R}^{k}$, $z_p(kh)\in\mathbb{R}^{l}$, $w(kh)\in\mathbb{R}^{m}$ and $z(kh)\in\mathbb{R}^{m}$, respectively, denote the discrete-time state, the discrete-time control input, the discrete-time measured output, the discrete-time exogenous input and output for evaluation of l_2 performance, and the discrete-time input and output of $\varphi:\mathbb{R}^m\to\mathbb{R}^m$ are dead-zone nonlinearities (equivalently, saturation nonlinearities $\phi:\mathbb{R}^m\to\mathbb{R}^m$), i.e.

$$\begin{array}{rcl} w & = & \varphi(z)\\ & \Leftrightarrow & w_i=\varphi_i(z_i)=\left\{\begin{array}{cl} z_i+\sigma_i & (\ z_i<-\sigma_i\ )\\ 0 & (\ |z_i|\le\ \sigma_i\ )\\ z_i-\sigma_i & (\ z_i>\ \sigma_i\ )\end{array}\right. \end{array} \quad (3)$$

where positive scalars $\sigma_i>0$ $(i=1,\cdots,m)$.

Note that the following relation with the discrete-time system (2) and the continuous-time system (1) on step invariant transformation:

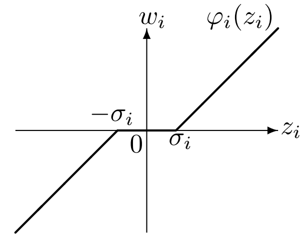

Fig. 2. Dead-zone nonlinearity $\varphi_i(z_i)$.

$$\begin{array}{rclrcl} A & = & e^{\mathbf{Ah}}, & \lim_{h\to0}A & = & I,\\ B_{\mathsf{j}} & = & \int_0^h e^{\mathbf{A}\tau}d\tau\mathbf{B}_{\mathsf{j}}, & \lim_{h\to0}B_{\mathsf{j}} & = & 0,\\ C_{\mathsf{i}} & = & \mathbf{C}_{\mathsf{i}}, & D_{\mathsf{ij}} & = & \mathbf{D}_{\mathsf{ij}},\\ & & & & & (\mathsf{i},\mathsf{j}=1,2,3)\end{array} \quad (4)$$

is well known where the discrete-time system (2) with the shift-operator does not become equal to the continuous-time system (1) as the sampling period h goes to 0. However, the relation between the discrete-time system (2) and the continuous-time system (1) holds as follows:

$$\lim_{h\to0}(A-I)/h=\mathbf{A},\quad \lim_{h\to0}B_{\mathsf{j}}/h=\mathbf{B}_{\mathsf{j}},\quad (\mathsf{j}=1,2,3). \quad (5)$$

One objective of this paper is to clarify whether analysis and synthesis conditions for the discrete-time system derived from below can be reduced to the corresponding conditions [6] for the continuous-time system as $h\to0$.

For given scalars $\theta\ge0$ and $\gamma>0$,

$$\mathcal{W}(\theta,\gamma):=\left\{w_p:\ \|w_p\|^2_{l_{2k}}\le\frac{1}{\gamma^2}\|z_p\|^2_{l_{2k}}+\theta^2\right\} \quad (6)$$

is defined. In this paper, consider the discrete-time exogenous input w_p with the bounded l_2 norm in $\mathcal{W}(\theta,\gamma)$. For this controlled object, consider a dynamic output feedback controller $u=K(\mathbf{z})\mathsf{y}$ with the discrete-time state $x_c(kh)\in\mathbb{R}^{n_c}$, i.e.

$$\left[\begin{array}{c}x_c((k+1)h)\\ u(kh)\end{array}\right]=\left[\begin{array}{cc}A_c & B_c\\ C_c & D_c\end{array}\right]\left[\begin{array}{c}x_c(kh)\\ \mathsf{y}(kh)\end{array}\right]. \quad (7)$$

Denote a discrete-time state vector of the closed-loop system by $\mathsf{x}(kh)\in\mathbb{R}^n$, i.e. $\mathsf{x}':=\left[\begin{array}{cc}x' & x_c'\end{array}\right]$ and $n:=n_p+n_c$. Let the closed-loop system $H(\mathbf{z})$ in Figure 1 be described by

$$\left[\begin{array}{c}\mathsf{x}((k+1)h)\\ \hdashline z(kh)\\ z_p(kh)\end{array}\right]=\left[\begin{array}{c:cc}\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2\\ \hdashline \mathcal{C}_1 & 0 & 0\\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22}\end{array}\right]\left[\begin{array}{c}\mathsf{x}(kh)\\ \hdashline w(k)\\ w_p(kh)\end{array}\right]$$
$$=:\left[\begin{array}{c:c}\mathcal{A} & \mathcal{B}\\ \hdashline \mathcal{C} & \mathcal{D}\end{array}\right]\left[\begin{array}{c}\mathsf{x}(kh)\\ \hdashline w(kh)\\ w_p(kh)\end{array}\right]. \quad (8)$$

For the use of the **the generalized sector condition** [4]–[6], $\mathcal{D}_{11}=0$, $\mathcal{D}_{12}=0$ are assumed. This paper analyzes and synthesizes the above control system.

III. DOMAIN OF l_2 PERFORMANCE

A. Preliminaries

In this subsection, we introduce a pole placement problem for LTI discrete-time systems. The objective here is to extend the existing domain of $\mathcal{L}_2$ performance [6] for continuous-time systems to a domain of l_2 performance for discrete-time systems, and to clarify the relation between the pole placement and an exponential stability performance.

We consider the following LTI discrete-time system:

$$\mathsf{x}((k+1)h) = \mathcal{A}\mathsf{x}(kh), \quad k = 0, 1, 2, \ldots. \tag{9}$$

where $x(kh) \in \mathbb{R}^n$ and h denote the state and the sampling period. As shown in Figure 3, we consider the problem such that eigenvalues $\lambda(\mathcal{A})$ of the system (9) are restricted by a circle area with the main coordinate δ and the radius ν. We introduce the following lemma:

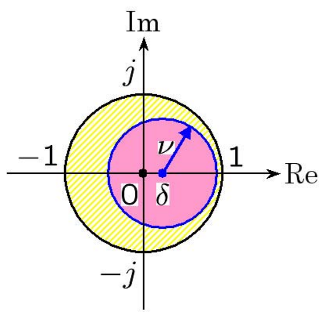

Fig. 3. Pole placement area.

Lemma 1: Consider the system defined in (9). Let scalars $|\delta| < 1$ and $0 < \nu \le 1 - |\delta|$ be given. All eigenvalues of $\mathcal{A}$ are elements of the set

$$\Lambda := \left\{ \lambda \in \mathbb{C} : \begin{bmatrix} \bar{\lambda} \\ 1 \end{bmatrix}' \begin{bmatrix} -1 & \delta \\ \delta & \nu^2 - \delta^2 \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} > 0 \right\}$$

if and only if there exists a real matrix $\mathcal{P} > 0$ satisfying

$$\mathsf{He} \begin{bmatrix} h\mathcal{P}/2 & (\mathcal{A} - \delta I)\mathcal{P} \\ 0 & \nu^2 \mathcal{P}/2h \end{bmatrix} > 0. \tag{10}$$

Moreover, if there exists a real matrix $\mathcal{P} > 0$ in (10), then the initial state $\mathsf{x}(0)$ of system (9) satisfies

$$\|\mathsf{x}(kh)\| \le \sqrt{\frac{\lambda_{max}(\mathcal{P})}{\lambda_{min}(\mathcal{P})}} \|\mathsf{x}(0)\| (|\delta| + \nu)^k, \quad k > 0. \tag{11}$$

Proof: The proof is omitted for the convenience of space. ■

In the next subsection, we define the domain of l_2 performance related to the exponential stability performance by using Lemma 1, and derive sufficient conditions for analysis and synthesis of the domain of l_2 performance.

B. Analysis

For a given positive definite matrix $\mathcal{P} > 0$ and a given scalar $\eta \ge 0$, an ellipsoid is defined by

$$\mathcal{E}^{\mathcal{P}}(\eta) := \{\mathsf{x} \in \mathbb{R}^n \ : \ \mathsf{x}'\mathcal{P}^{-1}\mathsf{x} \le \eta\}. \tag{12}$$

For the control system with the dead-zone (equivalently saturation) nonlinearities in previous section, we convert the domain of $\mathcal{L}_2$ performance for continuous-time systems [6] into a **domain of l_2 performance** for discrete-time systems defined by the following in consideration of the exponential stability performance also:

Definition 1: Given a positive definite matrix $\mathcal{P} > 0$ and scalars $\alpha \ge 0$, $\beta \ge 0$, $\gamma > 0$, $|\delta| < 1$ and $0 < \nu \le 1 - |\delta|$. A domain of l_2 performance with level $(\alpha, \beta, \gamma, \delta, \nu)$ is defined by an initial state space region $\mathcal{E}^{\mathcal{P}}(\alpha^2)$ such that any state trajectory starting from a point in the region $\mathcal{E}^{\mathcal{P}}(\alpha^2)$ does not leave a region $\mathcal{E}^{\mathcal{P}}(\alpha^2 + \beta^2)$ for all time and l_2 performance, i.e.

$$\|z_p\|_{l_{2k}} \le \gamma(\|w_p\|_{l_{2k}} + \alpha), \quad {}^\forall k > 0 \tag{13}$$

is satisfied whenever $w_p \in \mathcal{W}(\beta, \gamma)$. In the special case where $w_p = 0$, the exponential stability performance, i.e.

$$\|\mathsf{x}(kh)\| \le \sqrt{\frac{\lambda_{max}(\mathcal{P})}{\lambda_{min}(\mathcal{P})}} \|\mathsf{x}(0)\| (|\delta| + \nu)^k, \quad {}^\forall k > 0 \tag{14}$$

is satisfied whenever $\mathsf{x}(0) \in \mathcal{E}^{\mathcal{P}}(\alpha^2 + \beta^2)$. For notational simplicity, we may just say "domain of l_2 performance with level $(\alpha, \beta, \gamma, \delta, \nu)$" by removing "with level $(\alpha, \beta, \gamma, \delta, \nu)$" if it can be inferred from the context.

We introduce the generalized sector approach [5], [6] and the polytopic approach [4]. These approaches based on quadratic Lyapunov functions enable us to estimate the domain of l_2 performance. In the special case of stability analysis for systems with single saturation, the polytopic approach derives a necessary and sufficient condition such that an ellipsoid becomes a domain of attraction [4]. We have clarified the relation between the generalized sector approach and the polytopic approach in the special case of stability analysis [7]. However, the general case of l_2 performance remains an open problem. Therefore, we analyze the relation between the generalized sector approach and the polytopic approach for the l_2 performance problem, and then we solve this open problem.

We derive sufficient conditions (conservatively) characterized by using the generalized sector approach [5], [6] and the polytopic approach [4] as follows such that the region of ellipsoid $\mathcal{E}^{\mathcal{P}}(\alpha^2)$ defined by (12) becomes the domain of l_2 performance for the system (2), (7) and (8). Below, $\mathcal{C}_1^i$ and R^i denote the i^{th} rows of $\mathcal{C}_1$ and R.

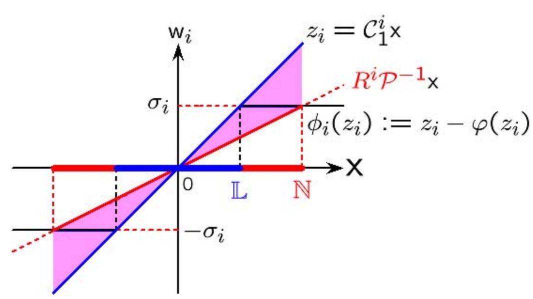

Fig. 4. Generalized sector bound for the saturation nonlinearity $\phi_i(z_i)$.

First, we introduce the generalized sector approach as follows. For a given matrix $\mathcal{N} \in \mathbb{R}^{m\times n}$, we define a local region

$$\mathcal{L}(\mathcal{N}) := \left\{ \mathsf{x} \in \mathbb{R}^n : \|\mathcal{N}^i \mathsf{x}\| \le \sigma_i, i \in [1,m] \right\}.$$

Proposition 1: [5], [6] Let matrices $\mathcal{C}_1$ and R, and a real nonsingular matrix $\mathcal{P}$ be given. Suppose that $\|R^i\mathcal{P}^{-1}\mathsf{x}\| \le \sigma_i$ for all $i \in [1,m]$, i.e. $\mathsf{x} \in \mathcal{L}(R\mathcal{P}^{-1})$. Then,

$$\varphi_i(\mathcal{C}_1^i\mathsf{x})'(\varphi_i(\mathcal{C}_1^i\mathsf{x}) - (\mathcal{C}_1^i - R^i\mathcal{P}^{-1})\mathsf{x}) \le 0.$$

Second, we introduce the polytopic approach as follows. Let $\mathcal{V}$ be the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. For example, if $m=2$, then,

$$\mathcal{V} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

There are 2^m elements in $\mathcal{V}$. Suppose that an each element of $\mathcal{V}$ is labeled as $E_j, j \in [1,2^m]$. Then,

$$\mathcal{V} = \left\{ E_j : j \in [1,2^m] \right\}.$$

Denote $E_j^- := I - E_j$. Clearly, E_j^- is also an element of $\mathcal{V}$ if $E_j \in \mathcal{V}$. Given two matrices $\mathcal{C}$, $H \in \mathbb{R}^{m\times n}$,

$$\left\{ E_j\mathcal{C} + E_j^- H : j \in [1,2^m] \right\}$$

is the set of matrices formed by choosing some rows from $\mathcal{C}$ and the rest from H.

Proposition 2: [4] Let matrices $\mathcal{C}_1$ and R, and a real nonsingular matrix $\mathcal{P}$ be given. Suppose that $\|R^i\mathcal{P}^{-1}\mathsf{x}\| \le \sigma_i$ for all $i \in [1,m]$, i.e. $\mathsf{x} \in \mathcal{L}(R\mathcal{P}^{-1})$. Then,

$$\varphi(\mathcal{C}_1\mathsf{x}) \in \mathbf{Co}\left\{ E_j^-(\mathcal{C}_1 - R\mathcal{P}^{-1})\mathsf{x} : j \in [1,2^m] \right\}.$$

First, the following theorem is obtained by the generalized sector approach in Proposition 1.

Theorem 1: Consider the feedback connection of $\mathcal{H}(\mathbf{z}) := \mathcal{C}(\mathbf{z}I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$ and dead-zone nonlinearities φ ($\sigma_i > 0$ for all $i = 1,\dots,m$) defined in (3) and (8). Let a real matrix $\mathcal{P} > 0$, and scalars $\alpha \ge 0$, $\beta \ge 0$, $\gamma > 0$, $|\delta| < 1$ and $0 < \nu \le 1 - |\delta|$ be given. Then $\mathcal{E}^{\mathcal{P}}(\alpha^2)$ is a domain of l_2 performance with level $(\alpha,\beta,\gamma,\delta,\nu)$ if there exist a matrix R and a diagonal matrix S such that

$$\mathsf{He}\underbrace{\begin{bmatrix} h\mathcal{P}/2 & (\mathcal{A}-\delta I)\mathcal{P} & \mathcal{B}_1 S & \mathcal{B}_2 & 0 \\ 0 & \nu^2\mathcal{P}/2h & 0 & 0 & 0 \\ 0 & R - \mathcal{C}_1\mathcal{P} & S & 0 & 0 \\ 0 & 0 & 0 & I/2 & 0 \\ 0 & \mathcal{C}_2\mathcal{P} & \mathcal{D}_{21}S & \mathcal{D}_{22} & \gamma^2 I/2 \end{bmatrix}}_{\Pi_G} > 0, \tag{15}$$

$$\begin{bmatrix} \mathcal{P} & \rho R^{i\prime} \\ \rho R^i & \sigma_i^2 \end{bmatrix} \ge 0, \quad \rho := \sqrt{\alpha^2+\beta^2}$$
$$^\forall i = 1,\cdots,m. \tag{16}$$

Proof: The proof is omitted for the convenience of space. ∎

Next, the following theorem is obtained by the polytopic approach in Proposition 2.

Theorem 2: Consider the feedback connection of $\mathcal{H}(\mathbf{z}) := \mathcal{C}(\mathbf{z}I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$ and dead-zone nonlinearities φ ($\sigma_i > 0$ for all $i = 1,\dots,m$) defined in (3) and (8). Let a real matrix $\mathcal{P} > 0$, and scalars $\alpha \ge 0$, $\beta \ge 0$, $\gamma > 0$, $|\delta| < 1$ and $0 < \nu \le 1 - |\delta|$ be given. Then $\mathcal{E}^{\mathcal{P}}(\alpha^2)$ is a domain of l_2 performance with level $(\alpha,\beta,\gamma,\delta,\nu)$ if there exist a matrix R such that

$$\mathsf{He}\underbrace{\begin{bmatrix} h\mathcal{P}/2 & (\mathcal{A}-\delta I)\mathcal{P} + \mathcal{B}_1 W & \mathcal{B}_2 & 0 \\ 0 & \nu^2\mathcal{P}/2h & 0 & 0 \\ 0 & 0 & I/2 & 0 \\ 0 & \mathcal{C}_2\mathcal{P} + \mathcal{D}_{21}W & \mathcal{D}_{22} & \gamma^2 I/2 \end{bmatrix}}_{\Pi_P} > 0, \tag{17}$$

$$\begin{bmatrix} \mathcal{P} & \rho R^{i\prime} \\ \rho R^i & \sigma_i^2 \end{bmatrix} \ge 0, \quad W := E_j^-(\mathcal{C}_1\mathcal{P} - R),$$
$$\rho := \sqrt{\alpha^2+\beta^2} \quad ^\forall i = 1,\dots,m \quad ^\forall j \in [1,2^m]. \tag{18}$$

Proof: The proof is omitted for the convenience of space. ∎

C. Main result of analysis

The following is the main result of analysis that is the inclusion relation between Theorem 1 (the generalized sector approach) and Theorem 2 (the polytopic approach).

Theorem 3: Given a symmetric matrix $\mathcal{P}$. If the statement in Theorem 1 holds, then the statement in Theorem 2 holds. Thus, the domain of l_2 performance in Theorem 1 is included in that in Theorem 2. Moreover, in the special case of the system (8) with single saturation nonlinearity: $m = 1$, the statement in Theorem 1 is equivalent to the statement in Theorem 2. Thus, the domain of l_2 performance in Theorem 1 is exactly the same as that in Theorem 2.

Proof: (Theorem 1 ⇒ Theorem 2) Fix $\mathcal{P}$ and suppose that the statement in Theorem 1 holds. The following relation:

$$\mathsf{He}\,\mathcal{T}_j\Pi_G\mathcal{T}_j' > 0, \quad \mathcal{T}_j := \begin{bmatrix} I & 0 & -\mathcal{B}_1E_j^- & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & -\mathcal{D}_{21}E_j^- & 0 & I \end{bmatrix}$$
$$\Leftrightarrow \quad \mathsf{He}\,\Pi_P > 0 \quad ^\forall j \in [1,2^m]$$

holds where $E_jSE_j^- = 0$ (this is trivial from the diagonal matrix S and the definition of E_j and E_j^-). Hence, we see that the statement in Theorem 2 holds for the variable R and S.

(Theorem 2 ⇒ Theorem 1) Here we consider the special case of $m = 1$, fix $\mathcal{P}$ and suppose that the statement in Theorem 2 ($m = 1$) holds as follows:

(Congruence transformation)

$$
\begin{aligned}
&{}^{\exists}R \quad \text{s.t.} \quad (18) \quad \text{and} \\
&\mathsf{He}\begin{bmatrix} \mathcal{P}^{-1}/2h & V & \mathcal{P}^{-1}\mathcal{B}_2/h & 0 \\ 0 & \nu^2\mathcal{P}^{-1}/2h & 0 & 0 \\ 0 & 0 & I/2 & 0 \\ 0 & \mathcal{C}_2+\mathcal{D}_{21}\mathsf{W} & \mathcal{D}_{22} & \gamma^2 I/2 \end{bmatrix} > 0, \\
&V := \mathcal{P}^{-1}(\mathcal{A}-\delta I+\mathcal{B}_1\mathsf{W})/h, \quad \mathsf{W} := \alpha(\mathcal{C}_1 - R\mathcal{P}^{-1}) \\
&{}^{\forall}\alpha \quad \text{s.t.} \quad 0 \le \alpha \le 1.
\end{aligned}
$$

$\Leftrightarrow$ (Schur complement of (1,1) and (4,4) brock matrices)

$$
\begin{aligned}
&{}^{\exists}R \quad \text{s.t.} \quad (18) \quad \text{and} \quad T\Omega T' > 0, \\
&\Omega := -\begin{bmatrix} \mathcal{A}'-\delta I \\ \mathcal{B}_1' \\ \mathcal{B}_2' \end{bmatrix} \mathcal{P}^{-1}/h \begin{bmatrix} \mathcal{A}'-\delta I \\ \mathcal{B}_1' \\ \mathcal{B}_2' \end{bmatrix}' \\
&\qquad + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \nu^2\mathcal{P}^{-1}/h \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}' + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}' \\
&\qquad - \begin{bmatrix} \mathcal{C}_2' \\ \mathcal{D}_{21}' \\ \mathcal{D}_{22}' \end{bmatrix} I/\gamma^2 \begin{bmatrix} \mathcal{C}_2' \\ \mathcal{D}_{21}' \\ \mathcal{D}_{22}' \end{bmatrix}', \\
&T := \begin{bmatrix} I & (\mathcal{C}_1' - \mathcal{P}^{-1}R')\alpha & 0 \\ 0 & 0 & I \end{bmatrix} \quad {}^{\forall}\alpha \quad \text{s.t.} \quad 0 \le \alpha \le 1.
\end{aligned}
$$

$\Leftrightarrow$ (Finsler's theorem)

$$
\begin{aligned}
&{}^{\exists}\mu > 0,\ R \quad \text{s.t.} \quad (18) \quad \text{and} \\
&\Omega - H\underbrace{\begin{bmatrix} -\alpha & 1 \end{bmatrix}'(-\mu)\begin{bmatrix} -\alpha & 1 \end{bmatrix}}_{\Theta} H' > 0, \\
&H := \begin{bmatrix} \mathcal{C}_1 - R\mathcal{P}^{-1} & 0 & 0 \\ 0 & I & 0 \end{bmatrix}' \quad {}^{\forall}\alpha \quad \text{s.t.} \quad 0 \le \alpha \le 1.
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
&{}^{\exists}\Theta,\ R \quad \text{s.t.} \quad (18) \quad \text{and} \quad \Omega - H\Theta H' > 0, \\
&\begin{bmatrix} 1 & \alpha \end{bmatrix}\Theta\begin{bmatrix} 1 & \alpha \end{bmatrix}' \ge 0 \quad {}^{\forall}\alpha \quad \text{s.t.} \quad 0 \le \alpha \le 1.
\end{aligned}
$$

$\Leftrightarrow$ (Loss-less S-procedure)

$$
\begin{aligned}
&{}^{\exists}T > 0,\ R \quad \text{s.t.} \quad (18) \quad \text{and} \\
&\Omega - H\begin{bmatrix} 0 & T \\ T & -2T \end{bmatrix} H' > 0.
\end{aligned}
$$

$\Leftrightarrow$ (Schur complement of $\mathcal{P}^{-1}/h$ and I/γ^2 terms)

$$
\begin{aligned}
&{}^{\exists}S := T^{-1} > 0,\ R \quad \text{s.t.} \quad (18) \quad \text{and} \\
&\mathsf{He}\begin{bmatrix} \mathcal{P}^{-1}/2h & \mathcal{P}^{-1}(\mathcal{A}-\delta I)/h & \mathcal{P}^{-1}\mathcal{B}_1/h & \mathcal{P}^{-1}\mathcal{B}_2/h & 0 \\ 0 & \nu^2\mathcal{P}^{-1}/2h & 0 & 0 & 0 \\ 0 & S^{-1}(R\mathcal{P}^{-1}-\mathcal{C}_1) & S^{-1} & 0 & 0 \\ 0 & 0 & 0 & I/2 & 0 \\ 0 & \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} & \gamma^2 I/2 \end{bmatrix} > 0.
\end{aligned}
$$

From a congruence transformation, we see that the statement in Theorem 1 via the generalized sector approach ($m = 1$) holds for the variables S and R. ■

Theorem 3 points out that the analysis based on the generalized sector approach is a sufficient condition of that based on the polytopic approach in the case of multi-dead-zone (equivalently, saturation) nonlinearities, and an equivalent condition of that based on the polytopic approach in the case of single dead-zone (equivalently, saturation) nonlinearity: $m = 1$. Note that in the special case of $m = 1$, $h \to 0$, $\beta = 0$, $\gamma \to \infty$, $\delta = 0$ and $\nu = 1$, Theorem 1 based on the generalized sector approach becomes a necessary and sufficient condition of stability analysis on the domain of attraction with a quadratic Lyapunov function [4].

Next, we shall indicate the relation between the analysis condition in Theorem 1 and that in [6]. Below, the analysis condition with shift-operator in Theorem 1 can be reduced to the corresponding analysis condition for the continuous-time system in [6] as $h \to 0$. We assume that $\delta := 0$ and $\nu := e^{-\varepsilon h}$ to make the transformation easy.

$$
\mathsf{He}\mathcal{T}_{d2c}\begin{bmatrix} -h\mathcal{P}/2 & -(\mathcal{A}-\delta I)\mathcal{P} & -\mathcal{B}_1 S & -\mathcal{B}_2 & 0 \\ 0 & -\nu^2\mathcal{P}/2h & 0 & 0 & 0 \\ 0 & -R+\mathcal{C}_1\mathcal{P} & -S & 0 & 0 \\ 0 & 0 & 0 & -I/2 & 0 \\ 0 & -\mathcal{C}_2\mathcal{P} & -\mathcal{D}_{21}S & -\mathcal{D}_{22} & -\gamma^2 I/2 \end{bmatrix}\mathcal{T}_{d2c}' < 0,
$$

$$
\mathcal{T}_{d2c} := \begin{bmatrix} -I/h & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & -I \\ -hI & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$\Leftrightarrow$

$$
\mathsf{He}\left[\begin{array}{cccc:c} \left((\mathcal{A}-I)/h+(1-\nu^2 I)/2h\right)\mathcal{P} & \mathcal{B}_1 S/h & \mathcal{B}_2/h & 0 & 0 \\ \mathcal{C}_1\mathcal{P}-R & -S & 0 & 0 & 0 \\ 0 & 0 & -I/2 & 0 & 0 \\ \mathcal{C}_2\mathcal{P} & \mathcal{D}_{21}S & \mathcal{D}_{12} & -\gamma^2 I/2 & 0 \\ \hdashline h(\mathcal{A}-I)\mathcal{P} & h\mathcal{B}_1 S & h\mathcal{B}_2 & 0 & -h^3\mathcal{P}/2 \end{array}\right] < 0. \tag{19}
$$

Here $H(\mathbf{z})$ denotes the closed loop system by the step invariant transformation of $\mathbb{H}(s)$ represented by

$$
\left[\begin{array}{c} \dot{\mathsf{x}}(t) \\ \hdashline z(t) \\ z_p(t) \end{array}\right] = \left[\begin{array}{c:cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hdashline \mathbb{C}_1 & 0 & 0 \\ \mathbb{C}_2 & \mathbb{D}_{21} & \mathbb{D}_{22} \end{array}\right]\left[\begin{array}{c} \mathsf{x}(t) \\ \hdashline w(t) \\ w_p(t) \end{array}\right].
$$

It is clarified that the analysis conditions (16) and (19) can be reduced to the analysis conditions [6] for the continuous-time system as $h \to 0$.

The control system (2) in the section II is regard as a linear system within a region $|z_i| \le \sigma_i$ ($\sigma_i = 1, \cdots, m$) from the definition of (3). Hence we can get "linear" analysis conditions for the domain of l_2 performance based on the linear analysis [2], [3], [8], within the linear region, by choosing $R = \mathcal{C}_1\mathcal{P}$ and $S \to 0$ in Theorem 1. Clearly, Theorem 1 includes the linear analysis conditions as the above special case. Moreover, Theorem 1 includes the circle analysis [2], [3], [8] as a special case where $R = K\mathcal{C}_1$ (K: a free parameter of diagonal matrix) [5], [6]. Thus, the achievable performance level given by Theorem 1 is no worse than that given by the linear analysis and the circle criterion due to the freedom in R and S.

D. Synthesis

In this subsection, we characterize the domain of l_2 performance achievable by the class of dynamic output feedback controllers $K(\mathbf{z})$ and we use the analysis conditions given in Theorem 1. For synthesis problems, we characterize a **restricted domain of l_2 performance** [**?**], [3], [6] which is the domain of l_2 performance to be the set of initial states $\mathsf{x}(0)$ such that $x_c(0) = 0$. Clearly, the restricted domain of l_2 performance is a subset of the domain of l_2 performance; For a given matrix $\mathcal{P} \in \mathbb{R}^{n\times n}$, if $\mathcal{E}^{\mathcal{P}}(\alpha^2)$ in (12) is a domain of $l_2$performance for the closed loop system, then

$$\mathcal{E}_0^X(\alpha^2) := \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^n : x'Xx \le \alpha^2, \quad x \in \mathbb{R}^{n_p} \right\} \quad (20)$$

is a restricted domain of l_2 performance where $X \in \mathbb{R}^{n_p\times n_p}$ is the upper left block matrix of $\mathcal{P}$. Conversely, for a given matrix $X \in \mathbb{R}^{n_p\times n_p}$, the above $\mathcal{E}_0^X(\alpha^2)$ of (20) is a restricted domain of l_2 performance if there exists a matrix $\mathcal{P} \in \mathbb{R}^{n\times n}$ such that $\mathcal{E}^{\mathcal{P}}$ is a domain of l_2 performance and

$$\mathcal{P} = \begin{bmatrix} X & * \\ * & * \end{bmatrix}$$

holds for some block matrices $*$.

The synthesis conditions for the w-measurement control synthesis problem [3], [6], [8] can be reduced to LMIs where the output w of dead-zone (saturation) nonlinearities is measurable.

Theorem 4: Consider the closed-loop system (2), dead-zone nonlinearities φ ($\sigma_i > 0$ for all $i = 1, \dots, m$) defined in (3) and a given controller (7). Let a real matrix $X > 0$, and scalars $\alpha \ge 0$, $\beta \ge 0$, $\gamma > 0$, $|\delta| < 1$ and $0 < \nu \le 1-|\delta|$ be given. Define the set $\mathcal{E}_0^X(\alpha^2)$ by (20). Then $\mathcal{E}_0^X(\alpha^2)$ is an achievable restricted domain of l_2 performance with level $(\alpha, \beta, \gamma, \varepsilon, \delta, \nu)$ if there exist a symmetric matrix Y, a diagonal matrix S, and matrices F, J, L, M, $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfying

$$\begin{bmatrix} D_{11}S + D_{13}J & D_{12} + D_{13}MD_{32} \end{bmatrix} = 0, \quad (21)$$

$$\mathsf{He} \underbrace{\begin{bmatrix} hX/2 & 0 & H_{11} - \delta X & H_{14} & 0 \\ hI & hY/2 & H_{21} - \delta I & H_{24} & 0 \\ 0 & 0 & \nu^2 X/2h & 0 & 0 \\ 0 & 0 & 0 & I/2 & 0 \\ 0 & 0 & H_{41} & H_{44} & \gamma^2 I/2 \end{bmatrix}}_{\mathbb{X}} > 0, \quad (22)$$

$$\mathsf{He} \underbrace{\begin{bmatrix} hY/2 & H_{21} - \delta I & H_{22} - \delta Y & H_{23} & H_{24} & 0 \\ 0 & \nu^2 X/2h & 0 & 0 & 0 & 0 \\ 0 & \nu^2 I/h & \nu^2 Y/2h & 0 & 0 & 0 \\ 0 & \mathcal{R}_1 - H_{31} & \mathcal{R}_2 - H_{32} & S & 0 & 0 \\ 0 & 0 & 0 & 0 & I/2 & 0 \\ 0 & H_{41} & H_{42} & H_{43} & H_{44} & \gamma^2 I/2 \end{bmatrix}}_{\mathbb{Y}} > 0, \quad (23)$$

$$\begin{bmatrix} X & I & \rho \mathcal{R}_1^{i\,\prime} \\ I & Y & \rho \mathcal{R}_2^{i\,\prime} \\ \rho \mathcal{R}_1^i & \rho \mathcal{R}_2^i & {\sigma_i}^2 \end{bmatrix} \ge 0, \quad \begin{matrix} {}^\forall i = 1, ..., m, \\ \rho := \sqrt{\alpha^2 + \beta^2}, \end{matrix} \quad (24)$$

$$\begin{aligned} H_{11} &:= XA + LC_3, & H_{14} &:= XB_2 + LD_{32}, \\ H_{21} &:= A + B_3MC_3, & H_{22} &:= AY + B_3F, \\ H_{23} &:= B_1S + B_3J, & H_{24} &:= B_2 + B_3MD_{32}, \\ H_{31} &:= C_1 + D_{13}MC_3, & H_{32} &:= C_1Y + D_{13}F, \\ H_{41} &:= C_2 + D_{23}MC_3, & H_{42} &:= C_2Y + D_{23}F, \\ H_{43} &:= D_{21}S + D_{23}J, & H_{44} &:= D_{22} + D_{23}MD_{32}, \end{aligned}$$

where $\mathcal{R}_1^i$ and $\mathcal{R}_2^i$ are the i^{th} rows of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively.

Proof: The proof is omitted for the convenience of space. ■

Next, we shall indicate the relation between the synthesis conditions (22) and (23) in Theorem 3 and those in [6]. Below, the synthesis conditions with shift-operator in Theorem 3 can be reduced to the corresponding synthesis conditions for the continuous-time system in [6] as $h \to 0$. We assume that $\delta := 0$ and $\nu := e^{-\varepsilon h}$ to make the transformation easy.

$$\mathsf{He}\, \mathcal{T}_1(-\mathbb{X})\mathcal{T}_1' < 0, \qquad \mathsf{He}\, \mathcal{T}_2(-\mathbb{Y})\mathcal{T}_2' < 0,$$

$$\mathcal{T}_1 := \begin{bmatrix} -I/h & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & -I \\ -hI & 0 & 0 & 0 & 0 \\ 0 & -hI & 0 & 0 & 0 \end{bmatrix}, \mathcal{T}_2 := \begin{bmatrix} 0 & hI & 0 & 0 & 0 & 0 \\ -I/h & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I \\ -hI & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow$$

$$\mathsf{He} \left[\begin{array}{ccc|cc} (H_{11} - X)/h + (1-\nu^2)X/2h & H_{14}/h & 0 & 0 & 0 \\ 0 & -I/2 & 0 & 0 & 0 \\ H_{41} & H_{44} & -\gamma^2 I/2 & 0 & 0 \\ \hline h(H_{11} - X) & hH_{14} & 0 & -h^3 X/2 & 0 \\ h(H_{21} - I) & hH_{24} & 0 & -h^3 I & -h^3 Y/2 \end{array}\right] < 0, \quad (25)$$

$$\mathsf{He} \left[\begin{array}{c|cccc|c} -h\nu^2 X/2 & 0 & 0 & 0 & 0 & 0 \\ \hline H_{21} - \nu^2 I & (H_{22} - Y)/h + (1-\nu^2)Y/2h & H_{23}/h & H_{24}/h & 0 & 0 \\ h(H_{31} - \mathcal{R}_1) & H_{32} - \mathcal{R}_2 & -S & 0 & 0 & 0 \\ 0 & 0 & 0 & -I/2 & 0 & 0 \\ hH_{41} & H_{42} & H_{43} & H_{44} & -\gamma^2 I/2 & 0 \\ \hline h^2 H_{21} & h(H_{22} - Y) & hH_{23} & hH_{24} & 0 & -h^3 Y/2 \end{array}\right] < 0. \quad (26)$$

It is clarified that the synthesis conditions (24), (25) and (26) can be reduced to the synthesis conditions for the continuous-time system [6] as $h \to 0$.

- **Synthesis problem maximizing a rejectable l_2 disturbance level β** [3], [8]

Given a positive definite matrix $X_0 > 0$ and scalars $\alpha \ge 0$, $\beta \ge 0$, $\gamma > 0$, $|\delta| < 1$, $0 < \nu \le 1 - |\delta|$. $\mathcal{E}_0^X(\alpha^2)$ is a restricted domain of l_2 performance. It is desired to find β that gives the "maximum" rejectable l_2 disturbance level. First, for these given $X_0 > 0$ and $\alpha \ge 0$, a set of initial state vectors is defined by

$$\mathcal{E}_0^{X_0}(\alpha^2) := \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^n : x'X_0x \le \alpha^2, \ x \in \mathbb{R}^{n_p} \right\}. \quad (27)$$

Since $\rho > 0$, an appropriate congruent transformation leads to

$$(24) \Leftrightarrow \begin{bmatrix} X & I & \mathcal{R}_1^{i\,\prime} \\ I & Y & \mathcal{R}_2^{i\,\prime} \\ \mathcal{R}_1^i & \mathcal{R}_2^i & {\sigma_i}^2/\rho^2 \end{bmatrix} \ge 0, \quad {}^\forall i = 1, ..., m. \quad (28)$$

Then, because of $\mathcal{E}_0^{X_0}(\alpha^2) \subseteq \mathcal{E}_0^{X}(\alpha^2)$, we have the following optimization problem:

$$\begin{aligned} &\min_{X,Y,F,J,L,M,\mathcal{R}_1,\mathcal{R}_2,S,1/\rho^2} \quad 1/\rho^2 \\ &\text{subject to } X \leq X_0,\ (21)-(23),\ (28) \end{aligned}$$

where S is restricted to be diagonal. We can calculate $\beta = \sqrt{\rho^2 - \alpha^2}$ by using the solution of the above optimization problem, while there exists no solution if $\rho^2 < \alpha^2$ holds.

Here we propose an algorithm for determining controller parameters which achieve the performance levels, after solving the optimization problems, where we can use the solutions X, Y, F, J, L, M, $\mathcal{R}_1$, $\mathcal{R}_2$, S and β.

Using the projection lemma, we can see that the conditions (22) and (23) are equivalent to

$$\mathsf{He}\begin{bmatrix} hX/2 & 0 & H_{11}-\delta X & Q-\delta I & H_{13} & H_{14} & 0 \\ hI & hY/2 & H_{21}-\delta I & H_{22}-\delta Y & H_{23} & H_{24} & 0 \\ 0 & 0 & \nu^2 X/2h & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu^2 I/h & \nu^2 Y/2h & 0 & 0 & 0 \\ 0 & 0 & \mathcal{R}_1 - H_{31} & \mathcal{R}_2 - H_{32} & S & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I/2 & 0 \\ 0 & 0 & H_{41} & H_{42} & H_{43} & H_{44} & \gamma^2 I/2 \end{bmatrix} > 0, \quad (29)$$

$$H_{13} := (XB_1 + LD_{31} + U)S$$

where we have two additional variables Q and U. Then, using these solutions Q, U of (29) and

$$W = JS^{-1} - MD_{31}, \qquad (30)$$

state space matrices of the controller $K(\mathbf{z})$ in (7) are given by

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} X - Y^{-1} & XB_3 \\ 0 & I \end{bmatrix}^{-1} \times \left[\begin{array}{c:cc} Q - XAY & L & U \\ \hdashline F & M & W \end{array}\right] \begin{bmatrix} -Y & 0 & 0 \\ C_3 Y & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1}.$$

IV. RESULT FOR CONTROL INPUT SATURATION

A. Problem Formulation

Here we consider a control system synthesis problem for the special case where the nonlinearities are

- $\phi := I - \varphi$
- in the control input ports.

The nonlinearities ϕ in Figure 4 are saturation functions. Hence, this problem formulation is important due to a saturating control system synthesis problem. Below, consider the control system synthesis problem taking into account this ϕ in Figure 5. From the reference [3], we can use the anti-

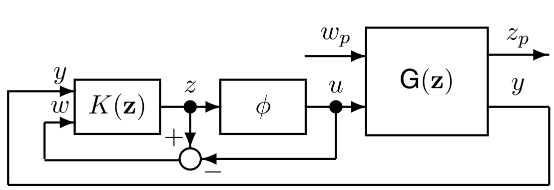

Fig. 5. Anti-windup control system.

windup controller [1], [3], [6], [8] in Figure 5 without loss of generality.

Consider the LTI generalized plant $\mathsf{G}(\mathbf{z})$ with the $m-$control input represented by the following state space realization:

$$\begin{bmatrix} x((k+1)h) \\ z_p(kh) \\ y(kh) \end{bmatrix} = \begin{bmatrix} \mathsf{A} & \mathsf{B}_1 & \mathsf{B}_2 \\ \mathsf{C}_1 & \mathsf{D}_{11} & \mathsf{D}_{12} \\ \mathsf{C}_2 & \mathsf{D}_{21} & 0 \end{bmatrix} \begin{bmatrix} x(kh) \\ w_p(kh) \\ u(kh) \end{bmatrix}, \quad (31)$$

where $x(kh) \in \mathbb{R}^{n_p}$, $u(kh) \in \mathbb{R}^p$, $y(kh) \in \mathbb{R}^q$, $w_p(kh) \in \mathbb{R}^k$ and $z_p(kh) \in \mathbb{R}^l$, respectively, denote the state, the control input, the measured output, the exogenous input and output for evaluation of the l_2 performance.

In this case, we have the closed-loop system shown in Figure 1 as the following state space realization:

$$\left[\begin{array}{c} x((k+1)h) \\ \hdashline z(kh) \\ z_p(kh) \\ \hdashline y(kh) \\ w(kh) \end{array}\right] = \left[\begin{array}{c:cc:c} \mathsf{A} & -\mathsf{B}_2 & \mathsf{B}_1 & \mathsf{B}_2 \\ \hdashline 0 & 0 & 0 & I \\ \mathsf{C}_1 & -\mathsf{D}_{12} & \mathsf{D}_{11} & \mathsf{D}_{12} \\ \hdashline \mathsf{C}_2 & 0 & \mathsf{D}_{21} & 0 \\ 0 & I & 0 & 0 \end{array}\right] \left[\begin{array}{c} x(kh) \\ \hdashline w(kh) \\ w_p(kh) \\ \hdashline z(kh) \end{array}\right],$$

$$w = \varphi(z), \qquad (32)$$

$$K(\mathbf{z}) = \left(\begin{array}{c|cc} A_c & B_{c_1} & B_{c_2} \\ \hline C_c & N\mathsf{D}_{21}^{\perp} & 0 \end{array}\right) \qquad (33)$$

where N is an arbitrary variable matrix of free parameter.

B. Solvability Condition

The following corollary is derived from Theorem 3.

Corollary 1: Consider the closed-loop system (8) with a given generalized plant (32), dead-zone nonlinearities φ ($\sigma_i > 0$ for all $i = 1, \ldots, m$) defined in (3) and a given controller (33). Let a real matrix $X > 0$, and scalars $\alpha \geq 0$, $\beta \geq 0$, $\gamma > 0$, $|\delta| < 1$ and $0 < \nu \leq 1 - |\delta|$ be given. Define the set $\mathcal{E}_0^X(\alpha^2)$ by (20). Then $\mathcal{E}_0^X(\alpha^2)$ is an achievable restricted domain of l_2 performance with level $(\alpha, \beta, \gamma, \varepsilon, \delta, \nu)$ if there exist a symmetric matrix Y, a diagonal matrix S, and matrices E, F, N, $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfying

$$\mathsf{He}\begin{bmatrix} hX/2 & 0 & X\mathsf{A} + E\mathsf{C}_2 - \delta X & X\mathsf{B}_1 + E\mathsf{D}_{21} & 0 \\ hI & hY/2 & \mathsf{A} + \mathsf{B}_2 N\mathsf{D}_{21}^{\perp}\mathsf{C}_2 - \delta I & \mathsf{B}_1 & 0 \\ 0 & 0 & \nu^2 X/2h & 0 & 0 \\ 0 & 0 & 0 & I/2 & 0 \\ 0 & 0 & \mathsf{C}_1 + \mathsf{D}_{12} N \mathsf{D}_{21}^{\perp}\mathsf{C}_2 & \mathsf{D}_{11} & \gamma^2 I/2 \end{bmatrix} > 0, \qquad (34)$$

$$\mathsf{He}\begin{bmatrix} hY/2 & \mathsf{A} + \mathsf{B}_2 N\mathsf{D}_{21}^{\perp}\mathsf{C}_2 - \delta I & \mathsf{A}Y + \mathsf{B}_2 F - \delta Y & -\mathsf{B}_2 S & \mathsf{B}_1 & 0 \\ 0 & \nu^2 X/2h & 0 & 0 & 0 & 0 \\ 0 & \nu^2 I/h & \nu^2 Y/2h & 0 & 0 & 0 \\ 0 & \mathcal{R}_1 - N\mathsf{D}_{21}^{\perp}\mathsf{C}_2 & \mathcal{R}_2 - F & S & 0 & 0 \\ 0 & 0 & 0 & 0 & I/2 & 0 \\ 0 & \mathsf{C}_1 + \mathsf{D}_{12}N\mathsf{D}_{21}^{\perp}\mathsf{C}_2 & \mathsf{C}_1 Y + \mathsf{D}_{12}F & -\mathsf{D}_{11}S & \mathsf{D}_{11} & \gamma^2 I/2 \end{bmatrix} > 0, \qquad (35)$$

$$\begin{bmatrix} X & I & \rho\mathcal{R}_1^{i\,\prime} \\ I & Y & \rho\mathcal{R}_2^{i\,\prime} \\ \rho\mathcal{R}_1^i & \rho\mathcal{R}_2^i & {\sigma_i}^2 \end{bmatrix} \geq 0, \quad \begin{array}{l} {}^\forall i = 1, \ldots, m, \\ \rho := \sqrt{\alpha^2 + \beta^2} \end{array} \qquad (36)$$

where $\mathcal{R}_1^i$ and $\mathcal{R}_2^i$ are the i^{th} rows of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively.

For example, in order to find the maximal rejectable l_2 disturbance level β characterized in Corollary 1, we may want to minimize $1/\rho^2$ subject to constrains $X \leq X_0$, (34), (35), (36) over the variables X, Y, E, F, N, $\mathcal{R}_1$, $\mathcal{R}_2$, S and $1/\rho^2$. Note that this feasible set is convex.

C. Numerical Example

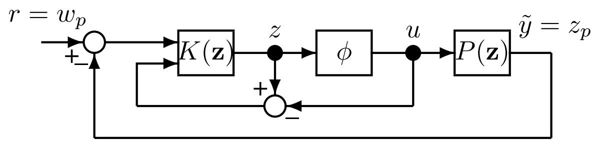

Fig. 6. An example of anti-windup control system.

Here we consider the example of anti-windup control system in Figure 6 with a discrete-time plant $P(\mathbf{z})$ corresponding to a continuous-time plant $10/s(s+1)$, $\sigma = 1$, $\alpha = 0$, $\gamma = 1.05$, $\nu = e^{-0.11h}$ in [6]. This system is represented by (31) in Figure 5 with

$$\begin{bmatrix} \mathsf{A} & \mathsf{B}_1 & \mathsf{B}_2 \\ \mathsf{C}_1 & \mathsf{D}_{11} & \mathsf{D}_{12} \\ \mathsf{C}_2 & \mathsf{D}_{21} & 0 \end{bmatrix} = \left[\begin{array}{cc|c|c} 1 & 1-e^{-h} & 0 & e^{-h}+h-1 \\ 0 & e^{-h} & 0 & 1-e^{-h} \\ \hline 10 & 0 & 0 & 0 \\ \hline -10 & 0 & 1 & 0 \end{array}\right].$$

The level of a rejectable l_2 performance: β is maximized by using Corollary 1. As a result, the maximum β approaches the maximum β in the case of continuous-time as $h \to 0$. In

TABLE I

MAXIMUM β FOR EACH h VS. CONTINUOUS-TIME

sampling period h	disturbance level β
1.0×10^{-2}	1.1199
1.0×10^{-3}	1.2001
1.0×10^{-4}	1.2083
1.0×10^{-5}	1.2091
continuous	1.2092

the case of $h = 0.01$, $\beta = 1.1199$, the optimal anti-windup controller

$$K(\mathbf{z}) = \left[\frac{1.127\mathbf{z}-1.115}{\mathbf{z}^2-0.9509\mathbf{z}+5.596\times10^{-4}} \quad \frac{0.03917\mathbf{z}+5.596\times10^{-4}}{\mathbf{z}^2-0.9509\mathbf{z}+5.596\times10^{-4}} \right] \tag{37}$$

and the following responses of the system in Figure 6 are obtained.

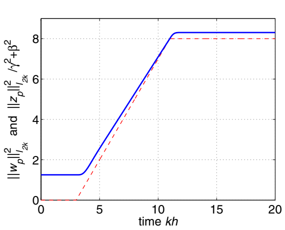

Fig. 7. A bound for $\|w_p\|_{l_{2k}}$.

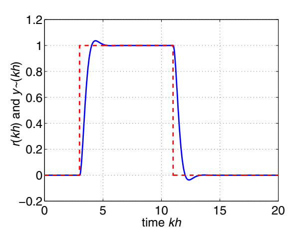

Fig. 8. $r(kh)$ and $\tilde{y}(kh)$ vs. kh.

Incidentally, we obtain the synthesis result: $\beta = 0.3212$ for $h = 0.01$ which is maximized by the linear analysis similar to the reference [2], [3], [8]. Therefore, it is confirmed that the result derived from the proposed method in this paper is better than that from the linear analysis. Note that we propose the framework of LMI-based l_2 performance analysis and synthesis for LTI control systems with saturation/dead-zone nonlinearities where the anti-windup compensator is definitely helpful.

V. CONCLUSION

This paper has defined a new domain of l_2 performance considering a pole placement for discrete-time control systems with saturation and/or dead-zone nonlinearities. For analysis, the analysis conditions of the domain can be recast as LMI conditions via the generalized sector and the polytopic approach, respectively. Moreover, we have pointed out that the conditions of the generalized sector approach are sufficient ones of the polytopic approach. In particular, the two analysis conditions have been proved to be exactly the same in the case of single saturation or single dead-zone nonlinearity. Similarly, for synthesis, the w-measurement control synthesis condition of the domain can be recast as an LMI condition via the generalized sector approach. Therefore, our results indicate that the special synthesis condition without considering the l_2 performance and the pole placement becomes a necessary and sufficient condition in the case of a stabilizing synthesis problem with saturating control based on a quadratic Lyapunov function. Next, the paper has clarified the consistency of the analysis and/or the synthesis conditions between the discrete-time and the continuous-time systems where the conditions for the discrete-time systems can be reduced to the corresponding continuous-time case as the sampling period goes to zero. Finally, it has been confirmed through a numerical example that the domain of l_2 performance is helpful designing anti-windup control systems.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge helpful discussions with Professor M. Ikeda and Associate Professor T. Asai at Osaka University.

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