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Estimation under uncontrolled and controlled communications in Networked Control Systems

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***Abstract*— An LTI estimation framework is proposed for networked control systems (NCS), in which local Kalman filter estimates are sent to the remote estimator. Both controlled and uncontrolled data communications are considered. For uncontrolled communication, minimum rate requirements are given for stochastic moment stability, which depend only on the least stable poles. For controlled communication, sufficient stability conditions are formulated. The framework also makes it possible to improve the trade-off between estimation performance and communication cost.**

I. Introduction

We address an estimation problem for linear plants in networked control systems. The continuous-time plant under consideration is

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{w} \\ \mathbf{y} &= C\mathbf{x} + \mathbf{v} \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times n}$, and $(C,\ A)$ is an observable pair. The Gaussian white disturbance $\mathbf{w} \in \mathbb{R}^n$ and noise $\mathbf{v} \in \mathbb{R}^l$ are mutually independent and zero-mean with covariance matrices $\Sigma_w > 0 \in \mathbb{R}^{n\times n}$ and $\Sigma_v > 0 \in \mathbb{R}^{l\times l}$.

The estimation scheme is motivated by the burgeon of *smart sensors*, which incorporate both computation and communication units. A smart sensor is co-located with the plant, Fig. 1. The sensor has a *Kalman filter* to estimate the plant state, which is sampled and sent to a remote estimator.

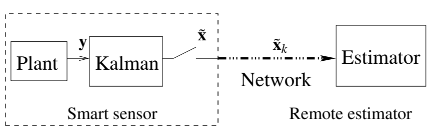

Fig. 1. The estimation scheme: the Kalman filter state is sent with controlled or uncontrolled communications over a lossy network to the remote estimator.

Significant research efforts have been devoted to the problem of determining the *minimum bit rate* that is needed to stabilize a system through feedback [1], [2], [3], which is of great theoretical interest. However, in most digital networks, data is transmitted in atomic units called *packets*, and sending a single bit or several hundred bits consumes the same amount of network resources. For example, an Ethernet IEEE 802.3 frame has a 112 or 176-bit header and a data field that must be at least 368-bit long, and each Bluetooth time slot carries 625 bits leaving at least 499 bits for data payload. This observation leads to an alternative view of band-limited channels, in which communication is measured as *packet rate*. Smaller packet rate results in shorter queuing delay and fewer dropouts due to time-out in the communication systems. It was shown in [4], [5] that the packet rate is reduced by only transmitting data when the remote estimation error becomes large. This is also related to the concept of Lebesgue Sampling [6].

The flow of data to the remote estimator is mediated by the smart sensor and the network. The former decides when to send data, whereas the latter may drop some of the data. In view of this, the data flow has both controlled (by the sensor) and uncontrolled (due to the network) components.

For uncontrolled communication, we model the times at which data is sent to the remote estimator by a Poisson process with a constant rate λ. For controlled communication, we model data sending times as jumps of an integer random process, whose jump 'intensity' depends on the estimation error. Typically, a smart sensor would increase the message sending rate when the estimation error increases.

This paper provides necessary and sufficient conditions for stability in the m^{th} moment of the remote estimation error. For uncontrolled communication, there is a minimum Poisson rate below which the statistical moments of the remote estimates are unbounded and above which they are bounded. This rate depends on the order of the moment, the unstable eigenvalues of the plant, and the probability of data loss. For controlled communication, the proposed polynomial data sending rates guarantee moment stability of any order. The results obtained are contrasted with related work in the literature.

The remote estimation of linear plants is investigated by many authors. In the references below, *raw measurements* are sent to the remote time-varying Kalman filter (TVKF) via a lossy network in an *uncontrolled* fashion.

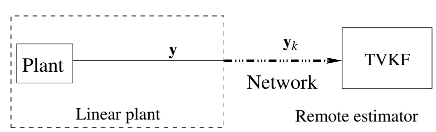

Fig. 2. The scheme in the cited references: the measurements are sent over a lossy network to a remote time-varying Kalman filter.

In their seminal paper [7], Sinopoli *et. al.* model *intermittent observations* as a Bernoulli process with success probability $1-p$, where p is the data loss probability. They

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consider the discrete-time LTI

$$\begin{aligned} \mathbf{x}^+ &= A_d\mathbf{x} + \mathbf{w} \\ \mathbf{y} &= C_d\mathbf{x} + \mathbf{v} \end{aligned} \tag{2}$$

where the matrices are similarly defined as in (1), and $\mathbf{w}$ and $\mathbf{v}$ are *i.i.d.* Gaussian random processes. They show that there exists a critical value p_c such that if $p < p_c$ the expected error covariance is finite. Section III-E contains more discussion of this paper. In [8], an estimator is explicitly constructed for a one-dimensional unstable system over a noisy binary communication link.

Matveev and Savkin [9] study an optimal estimation problem for a plant similar to (2), in which partial observations are sent from different sensors via independent time-delayed lossy networks. Packet losses are modeled as infinite network delay. They obtain a TVKF by solving a Kalman filtering problem on an enlarged state space.

The estimation scheme is set up in Section II, followed by a discussion of communication schedulers and their stochastic models. The main result, which includes the construction of communication schedules and their stochastic stability properties, appears in Section III. Section IV proves the theorems, and Section V provides a simulation example.

II. AN ESTIMATION SCHEME IN NCS

The estimation scheme consists of a smart sensor, a remote estimator and a network in between, as shown in Fig. 3. The smart sensor has a Kalman filter and a communication scheduler. The Kalman filter state $\tilde{\mathbf{x}}(t)$ is computed locally in a continuous fashion (or with a short sampling period) and sent to the remote estimator at times determined by the communication scheduler. The decision on when to send data is based on how well the remote estimate $\hat{\mathbf{x}}$ matches the current Kalman filter state $\tilde{\mathbf{x}}$. To implement this idea, the smart sensor keeps an identical copy of the remote estimate $\hat{\mathbf{x}}$. To reduce notations, the copy of $\hat{\mathbf{x}}$ on the smart sensor is also denoted by $\hat{\mathbf{x}}$.

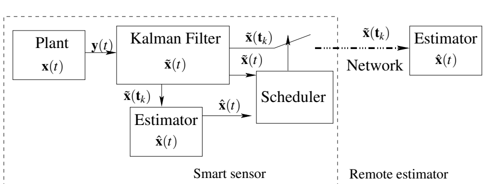

Fig. 3. A detailed structure of the estimation scheme.

The Kalman filter for plant (1) is

$$\dot{\tilde{\mathbf{x}}} = A\tilde{\mathbf{x}} + L(\mathbf{y} - C\tilde{\mathbf{x}}) \tag{3}$$

where $L \in \mathbb{R}^{n\times l}$ makes $A - LC$ Hurwitz. The estimator, which has two identical copies on both the smart sensor and the remote site, propagates the estimate in open-loop most of the time, and resets to the Kalman filter state $\tilde{\mathbf{x}}$ when data is received

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= A\hat{\mathbf{x}} \\ \hat{\mathbf{x}}(\mathbf{t}_k) &= \tilde{\mathbf{x}}(\mathbf{t}_k^-) - \mathbf{z}_k \end{aligned} \tag{4}$$

where $\mathbf{t}_k$, $k \geq 0$, indicates the instants of data arrivals, and the term $\mathbf{z}_k$ models the quantization errors. It is assumed that $\mathbf{z}_k$ has probability distribution μ_k such that all finite moments are bounded, *i.e.*,

$$\mathrm{E}\left[\|\mathbf{z}_k\|^{2m}\right] < \Delta_z(m) < \infty, \quad m \geq 1.$$

We assume *i.i.d.* quantization errors with probability distribution function $\mu(z)$ on a finite support, and use $\mathbf{z}$ in place of $\mathbf{z}_k$ when the context allows. Two assumptions are implicit in the equations above:

Assumption 1. *Delay is negligible.*

Assumption 2. *In case of lossy networks, an instantaneous acknowledgment of the packet arrival is available to the smart senor.*

Assumption 1 can be relaxed if data packets are *time-stamped* so that state propagation is possible. This is pursued in [10]. Assumption 2 is needed to synchronize the remote estimator and the estimator on the smart sensor.

The communication instants are decided by the communication scheduler, for which several patterns will be considered. Before talking about communication patterns, we derive the error dynamics, assuming the communication instants are given.

A. Error dynamics

The estimation error $\hat{\mathbf{e}}$, the propagation error $\tilde{\mathbf{e}}$, and the Kalman filtering error ξ are defined as follows,

$$\hat{\mathbf{e}} := \mathbf{x} - \hat{\mathbf{x}}, \quad \tilde{\mathbf{e}} := \tilde{\mathbf{x}} - \hat{\mathbf{x}}, \quad \xi := \mathbf{x} - \tilde{\mathbf{x}} \tag{5}$$

of which only $\tilde{\mathbf{e}}$ is known to the smart sensor.

The main objective of the estimation structure is to keep the $2m^{th}$ moment of $\hat{\mathbf{e}}$ small, $m \geq 1$ while reducing the *communication rate*. The communication rate R is the long-term average message rate

$$R := \limsup_{T\to\infty} \mathrm{E}^{\mathbf{w},\mathbf{v}}\left[\frac{\text{number of messages in } [0,T]}{T}\right],$$

where $\mathrm{E}^{\mathbf{w},\mathbf{v}}[\cdot]$ stands for the expectation with respect to $\mathbf{w}$ and $\mathbf{v}$. This communication rate definition is in contrast to the logarithm based concepts such as *entropy rate* or *channel capacity* [11].

Given $m \geq 1$, the process $\hat{\mathbf{e}}$ is *stable in the* $2m^{th}$ *moment* if for $\Delta_0(m) > 0$ such that $\mathrm{E}\left[\|\hat{\mathbf{e}}(0)\|^{2m}\right] < \Delta_0(m)$, $\exists\, \Delta(m) < \infty$ such that

$$\mathrm{E}\left[\|\hat{\mathbf{e}}(t)\|^{2m}\right] < \Delta(m), \quad \forall t > 0.$$

From (5), $\hat{\mathbf{e}}(t)$ is stable if both $\mathrm{E}\left[\|\tilde{\mathbf{e}}(t)\|^{2m}\right]$ and $\mathrm{E}\left[\|\xi(t)\|^{2m}\right]$ are bounded. The error systems of $\tilde{\mathbf{e}}(t)$ and

$\xi(t)$ are from (1), (3)-(4),

$$\dot{\tilde{\mathbf{e}}} = A\tilde{\mathbf{e}} + LC\xi + L\mathbf{v} \quad \tilde{\mathbf{e}}(\mathbf{t}_k) = \mathbf{z}_k \tag{6a}$$

$$\dot{\xi} = (A - LC)\xi - L\mathbf{v} + \mathbf{w} \tag{6b}$$

the first of which is a *jump diffusion process* [12]. The stability of $A - LC$ guarantees that ξ has all finite-order moments bounded, *i.e.*,

$$\mathrm{E}\left[\|\xi(t)\|^{2m}\right] \le \Delta_{\xi}(m). \tag{7}$$

Remark 1. *The process* $\tilde{\mathbf{e}}(t)$ *in* (6a) *is right continuous and has a left limit. It is known as* cadlag *in stochastic literature [13, pp 3]. The left limit is denoted as* $\tilde{\mathbf{e}}(t^-)$*, or* $\tilde{\mathbf{e}}^-$ *when* t *is implicit.*

We regard $\tilde{\mathbf{e}}(t)$ in (6a) as a process driven by $\xi(t)$ as given in (6b). Alternatively, stack the states, $\mathbf{e} := \begin{bmatrix} \tilde{\mathbf{e}} \\ \xi \end{bmatrix}$,

$$\dot{\mathbf{e}} = \bar{A}\mathbf{e} + \bar{\mathbf{w}}, \quad \tilde{\mathbf{e}}(\mathbf{t}_k) = \mathbf{z}_k \tag{8}$$

where

$$\bar{A} := \begin{bmatrix} A & LC \\ 0 & A - LC \end{bmatrix}$$

and $\bar{\mathbf{w}} \in \mathbb{R}^{2n}$ has a covariance matrix,

$$\Sigma = \begin{bmatrix} L\Sigma_v L' & L\Sigma_v L' \\ L\Sigma_v L' & L\Sigma_v L' + \Sigma_w \end{bmatrix} \ge 0. \tag{9}$$

B. Data communication Patterns

The communication scheduler on the smart sensor and the communication network are modeled in this subsection. The emphasis is on the former. The network is lossy, dropping packets with probability p in an *i.i.d.* fashion.

The smart sensor uses certain mechanisms to schedule when to send data. There are many ways to specify the *intermittent time*, *i.e.*, the time interval between one data sending and the next. The simplest one is periodic. It can also follow any other pre-specified time sequences. The intermittent time can also be random, of which a simple case is exponential distribution with average T. The associated integer process is known as the Poisson process, with *Poisson rate* $\lambda = \frac{1}{T}$. Viewed from another point, *data sending is triggered whenever the Poisson process has a jump.*

The intermittent time can also be determined by controlled random processes so as to use system dynamics information. Motivated by the Poisson process mechanism, an integer-valued random measure $\mathbf{N}(t)$ is constructed, assigning a rate that depends on the system dynamics. $\mathbf{N}(t)$ is right-continuous, non-decreasing, and at any time t, its increment is either zero or one, i.e., $\mathbf{N}(t) - \lim_{s\to t^-}\mathbf{N}(s) \in \{0,1\}$. For stricter definitions, the readers are referred to [13, pp 65-71]. Because the state $\hat{\mathbf{x}}(t)$ of the estimator evolves as a right-continuous process, as in (4), at time t, the smart sensor does not know $\hat{\mathbf{x}}(t)$. Denote the information available to the smart sensor at time t as $\mathscr{I}(t^-)$, which is the maximum information set for communication decision making. Let $\Lambda(\mathscr{I}(t^-))$ be the intensity of the jumps, i.e., in any infinitesimal time interval $(t, t+dt]$,

$$\Pr\left[\mathbf{N}(t+dt) - \lim_{s\to t^-}\mathbf{N}(s) = 1\right] = \Lambda\left(\mathscr{I}(t^-)\right)dt$$

Note that the Poisson process is a special case with a constant intensity $\Lambda(\mathscr{I}(t^-)) = \lambda$. In this paper, to minimize the error variance, the intensity is chosen as a function of error $\tilde{\mathbf{e}}$ as in (5), i.e., $\Lambda(\mathscr{I}(t^-)) = \lambda(\tilde{\mathbf{e}}(t^-))$, which can be proved not to lose optimality [4].

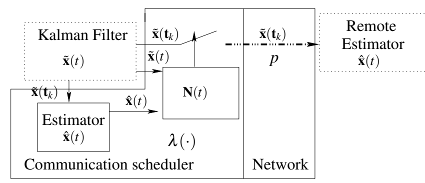

Fig. 4. A communication pattern: a communication scheduler on the smart sensor and a lossy network.

The communication pattern models the communication scheduling and the network uncertainties, as in Fig. 4. Specifically, we consider two patterns:

1) The communication scheduler is driven by a Poisson process with a constant Poisson rate λ, and the packets get lost with probability p, $0 \le p < 1$. Since packet loss is independent of the Poisson process, data arrives at the remote estimator according to another Poisson process with rate $(1-p)\lambda$.
2) The communication scheduler is driven by an integer-valued process with jump intensity $\lambda(\tilde{\mathbf{e}}^-)$, and the packets get lost with probability p. In this case, the effective intensity becomes $(1-p)\lambda(\tilde{\mathbf{e}}^-)$.

For analysis purposes, the data loss probability p does not add complexity. From now on, λ or $\lambda(\tilde{\mathbf{e}}^-)$ refers to the intensity that data are *received* on the remote estimator.

III. Main results

This section gives a tight bound on the Poisson rate for stochastic moment stability in uncontrolled communication, and sufficient stability conditions for controlled communications. The results are interpreted in the discrete-time domain, and compared to the corresponding theorem from [7]. It is assumed that the matrix A is not Hurwitz. Otherwise, a trivial estimator does the job with zero communication rate.

A. Uncontrolled communication

Consider a communication scheduler that is driven by a Poisson process with a rate $\gamma > 0$. Define

$$\gamma_{2m} := 2m \max\{\Re[\mathrm{Eig}(A)]\}.$$

Theorem 1. *Let the estimation error* $\hat{\mathbf{e}}$ *be defined as in* (1), (3)-(5)*, in which the time sequence* $\mathbf{t}_k$*,* $k \ge 0$*, is generated by a Poisson process with a nonnegative rate* γ*. For any*

$m \geq 1$, *if* $\gamma > \gamma_{2m}$, $\mathrm{E}[(\hat{\mathbf{e}}(t)'\hat{\mathbf{e}}(t))^m]$ *is bounded,* $\forall\ t \geq 0$, *and if* $\gamma < \gamma_{2m}$, $\lim_{t\to\infty} \mathrm{E}[(\hat{\mathbf{e}}(t)'\hat{\mathbf{e}}(t))^m] \to \infty$.

It is seen that γ_{2m} is a *tight bound* for the $2m^{th}$ moment stability, *i.e.*, $\forall \varepsilon > 0$, $\exists \gamma > \gamma_{2m} - \varepsilon$ such that $\mathrm{E}[(\hat{\mathbf{e}}(t)'\hat{\mathbf{e}}(t))^m]$ is unbounded. Note that the bound γ_{2m} is only dependent on the least stable mode of the plant (1).

B. Controlled communication

Consider the communication scheduler that is driven by an integer-valued process $\mathbf{N}(t)$ with a jump intensity that depends on the measured error dynamics. One choice is

$$\lambda(\tilde{e}^-) = (\tilde{e}^{-\prime} P \tilde{e}^-)^k, \tag{10}$$

where $P > 0 \in \mathbb{R}^{n\times n}$ and $k \in \mathbb{R}^+$. Section V will show that for fixed communication rates, communication schedules of this type lead to smaller error variances compared to uncontrolled communication.

Theorem 2. *Let the estimation error* $\hat{\mathbf{e}}$ *be defined as in* (1)*,* (3)-(5)*, in which the time sequence* $\mathbf{t}_k$*,* $k \geq 0$*, is generated by an integer-valued process with jump intensity* $\lambda(\tilde{e}^-)$ *given by* (10)*. All finite moments of* $\hat{\mathbf{e}}(t)$ *are bounded,* $t \geq 0$*.*

C. Controlled communication: saturated intensity

When the intensity function (10) is "saturated", the moment stability is achieved as long as the saturation levels are given by Theorem 1.

Theorem 3. *Let the estimation error* $\hat{\mathbf{e}}$ *be defined as in* (1)*,* (3)-(5)*, in which the time sequence* $\mathbf{t}_k$*,* $k \geq 0$*, is generated by an integer-valued process with jump intensity*

$$\lambda(\tilde{e}^-) = \min\{\gamma, (\tilde{e}^{-\prime} P \tilde{e}^-)^k\}$$

where $P \in \mathbb{R}^{n\times n} > 0$*,* $\gamma > \gamma_{2m}$*,* $m > 0$*, and* $k > 0$*. Then* $\mathrm{E}[(\hat{\mathbf{e}}(t)'\hat{\mathbf{e}}(t))^m]$ *is bounded,* $t \geq 0$*.*

D. Periodic communication over lossy networks

If data is sent to the network periodically with period T_s and gets lost with probability p, Theorem 1 leads to the following corollary.

Corollary 1. *When the data loss probability* p *satisfies*

$$p < \exp(-2mT_s \max\{\Re[\mathrm{Eig}(A)]\})$$

for $m \geq 1$*, the* $2m^{th}$ *moment of the estimation error is bounded.*

Proof: [Corollary 1] In applying Theorem 1, subtleties arise from the fact that the Poisson process does not necessarily jump at the sampling instants, and there may be more than one jump in one sampling period. We start from the process $\mathbf{e}(t)$ in (8) with constant Poisson rate γ, and define a continuous-time jump diffusion process $\mathbf{e}_p = \begin{bmatrix} \tilde{\mathbf{e}}_p \\ \xi_p \end{bmatrix}$, $\forall k \geq 0$

$$\dot{\mathbf{e}}_p = \bar{A}\mathbf{e}_p + \bar{\mathbf{w}} \tag{11}$$
$$\tilde{\mathbf{e}}_p(kT_s) = \mathbf{z}_k \quad \text{if } \tilde{\mathbf{e}}(t) \text{ has jumps in } \big((k-1)T_s, kT_s\big],$$

i.e., $\mathbf{e}_p$ is driven by the same disturbance and noise $\bar{\mathbf{w}}$ as $\mathbf{e}$ is, but has a jump on any instant kT_s if and only if $\mathbf{e}$ has jump(s) in the previous interval $\big((k-1)T_s, kT_s\big]$. By construction, the process $\mathbf{e}_p(t)$ is the error system for a periodic sending scheme with period T_s and data loss rate $p = \exp(-\gamma T_s)$, which is the probability that the process $\mathbf{e}(t)$ has no jumps on the interval $\big((k-1)T_s, kT_s\big]$.

If the process $\mathbf{e}(t)$ in (8) has bounded $2m^{th}$ moment, $t \geq 0$, so does $\mathbf{e}_p(t)$ in (11). A proof can be adapted from the proof of Theorem 3 in [10]. It is omitted here due to space limitations.

According to Theorem 1, if $\gamma > 2m\max\{\Re[\mathrm{Eig}(A)]\}$, the $2m^{th}$ moment of $\mathbf{e}$ in (8) is bounded. Hence the $2m^{th}$ moment of $\mathbf{e}_p$ in (11) is also bounded. Therefore, it is sufficient to have

$$p = \exp(-\gamma T_s)) < \exp(-2mT_s \max\{\Re[\mathrm{Eig}(A)]\}),$$

which completes the proof. $\square$

E. Discussions

This section compares the stability conditions between the discrete-time estimation schemes in which the Kalman filter state is sent (Fig. 1) and schemes in which the raw measurement is sent (Fig. 2).

To this effect, Corollary 1 is interpreted in the discrete-time domain. Let the discrete-time LTI plant (2) be a zero-order-hold discretization of (1) with sampling time T_s, i.e., $A_d = \exp(AT_s)$ and $C_d = C$. For simplicity, suppose there is no quantization error, *i.e.*, $\mathbf{z}_k = 0$ in (11). When the sensor sends the Kalman filter state (Fig 1) with a data loss probability $p = \exp(-\gamma T_s)$, the discrete-time estimator has the same dynamics as the process $\mathbf{e}_p$ defined in (11), sampled at times kT_s, $k \geq 1$. From Corollary 1, the $2m^{th}$ moment of the discrete-time estimation error is bounded provided that the data loss probability p satisfies

$$p < \frac{1}{(\max\{|\mathrm{Eig}(A_d)|\})^{2m}}. \tag{12}$$

The authors of [7] consider an estimation scheme in which raw measurements are sent and show the existence of a critical value p_c for the drop rate, beyond which a transition to an unbounded state error covariance occurs. This critical value is bounded by $\underline{p} \leq p_c \leq \bar{p}$, where $\underline{p}$ and $\bar{p}$ are solved by LMIs and the latter has a closed form,

$$\bar{p} = \frac{1}{(\max\{|\mathrm{Eig}(A_d)|\})^2}.$$

The lower bound $\underline{p}$ depends on A_d and C_d. In general, $p_c < \bar{p}$, $p_c = \bar{p}$ only in special cases.

This shows that the remote estimation structure in Fig. 1 tolerates a data loss rate, which is in general higher than the one for Fig. 2, to achieve the second moment stability. Moreover, the bound in (12) solely depends on the matrix A_d. Eq. (12) also provides stability conditions for higher order moments.

IV. PROOF OF THEOREMS

The theorems in Section III are proved here. The key is to establish the boundedness of $\tilde{\mathbf{e}}(t)$ as in (6a). We omit the proof of Theorem 1 [14].

A. Stochastic generators

Given a twice continuously differentiable function V defined on $\mathbb{R}^n$ and a jump diffusion process $\mathbf{e}$, the *infinitesimal generator* $\mathscr{L}$ *of* $\mathbf{e}$ is defined by

$$(\mathscr{L}V)(e) = \lim_{\tau \to t} \frac{\mathrm{E}[V(\mathbf{e}(\tau))|\mathbf{e}(t)=e] - V(e)}{\tau - t}, \quad \forall e \in \mathbb{R}^n,\ \tau > t \geq 0. \tag{13}$$

The following lemma can be obtained from [15], [12].

Lemma 1. *The generator for the jump diffusion process* (8) *is given by*

$$\mathscr{L}V(e) = \frac{\partial V(e)}{\partial e} \cdot Ae + \frac{1}{2}\operatorname{trace}\left(\Sigma \frac{\partial^2 V(e)}{\partial e^2}\right) + \lambda(e^-)\Big(\int V(z,\xi)\, d\mu(z) - V(e^-)\Big),$$

where $\frac{\partial V(e)}{\partial e}$ *and* $\frac{\partial^2 V(e)}{\partial e^2}$ *denote the gradient and Hessian matrix of* V*, respectively,* Σ *is defined in* (9)*, and* ξ *is as in* (6b)*.*

B. Proof of Theorem 2

Lemma 2. *[14] Given a random variable* $\mathbf{x}$ *that is nonnegative with probability one, any constant* $\delta > 0$ *and* $k > \ell > 0$*,* $\mathrm{E}[\mathbf{x}^k] \geq \delta^{\ell}\,\mathrm{E}[\mathbf{x}^{k-\ell}] - \delta^k$*.*

Proof: [Theorem 2] Since the jump intensity $\lambda(\cdot)$ in (10) is only dependent on $\tilde{\mathbf{e}}(t^-)$ and it is known that $\xi(t)$ is stochastically bounded, to simplify the proof, take $V := (\tilde{e}'P\tilde{e})^m$, $P > 0 \in \mathbb{R}^{n\times n}$. Following similar derivation, the generator for the jump diffusion process (6a) is given by

$$\mathscr{L}V(\tilde{e}) = \frac{\partial V(\tilde{e})}{\partial \tilde{e}} \cdot \big(A\tilde{e} + LC\xi\big) + \frac{1}{2}\operatorname{trace}\left(\Sigma_{Lv} \frac{\partial^2 V(\tilde{e})}{\partial \tilde{e}^2}\right) + \lambda(\tilde{e}^-)\Big(\int V(z)\, d\mu(z) - V(\tilde{e}^-)\Big),$$

where $\frac{\partial V(\tilde{e})}{\partial \tilde{e}}$ and $\frac{\partial^2 V(\tilde{e})}{\partial \tilde{e}^2}$ denote the gradient and Hessian matrix of V, respectively, $\Sigma_{Lv} := L\Sigma_v L'$, and ξ is considered external as given by (6b).

For a positive definite matrix P, $\exists c_1, c_2 > 0$ such that

$$PA + A'P \leq c_1 P$$
$$P\Sigma_{Lv}P \leq c_2 P$$

and $\exists c_3 > 0$ such that

$$\tilde{e}'(PLC + C'L'P)\xi \leq c_3(\tilde{e}'P\tilde{e})^{\frac{1}{2}}(\xi'\xi)^{\frac{1}{2}}$$

$\forall$ $\tilde{e}$ and $\xi \in \mathbb{R}^n$. Now consider the generator,

$$\begin{aligned}
&\mathscr{L}V(\tilde{e}) \\
&= m(\tilde{e}'P\tilde{e})^{m-1}\,\tilde{e}'(PA + A'P)\tilde{e} \\
&\quad + m(\tilde{e}'P\tilde{e})^{m-1}\,\tilde{e}'(PLC + C'L'P)\xi \\
&\quad + (\tilde{e}^{-\prime}P\tilde{e}^-)^k\,\mathrm{E}^{\mathbf{z}}\left[V(\mathbf{z})\right] + 2m(m-1)(\tilde{e}'P\tilde{e})^{m-2}\,\tilde{e}'P\Sigma_{Lv}P\tilde{e} \\
&\quad + m(\tilde{e}'P\tilde{e})^{m-1}\operatorname{trace}(\Sigma_{Lv}P) - (\tilde{e}^{-\prime}P\tilde{e}^-)^{m+k} \\
&\leq c_1 mV(\tilde{e}) + c_3 m(\tilde{e}'P\tilde{e})^{m-\frac{1}{2}}(\xi'\xi)^{\frac{1}{2}} + (\tilde{e}^{-\prime}P\tilde{e}^-)^k\,\mathrm{E}^{\mathbf{z}}\left[V(\mathbf{z})\right] \\
&\quad + m\big(2c_2(m-1) + \operatorname{trace}(\Sigma_{Lv}P)\big)(\tilde{e}'P\tilde{e})^{m-1} - (\tilde{e}^{-\prime}P\tilde{e}^-)^{m+k}.
\end{aligned}$$

Consider the case in which $m > k$. Set $e = \mathbf{e}(t)$ in (13) and take the expectation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[V(\mathbf{e}(t))] = \mathrm{E}[(\mathscr{L}V)(\mathbf{e}(t))]. \tag{14}$$

The process $\tilde{\mathbf{e}}$ has only countably many jumps and at any time t, the probability of a jump is zero. We have that

$$\mathrm{E}[(\tilde{e}^{-\prime}P\tilde{e}^-)^k] = \mathrm{E}[(\tilde{e}'P\tilde{e})^k]$$
$$\mathrm{E}[(\tilde{e}^{-\prime}P\tilde{e}^-)^{m+k}] = \mathrm{E}[(\tilde{e}'P\tilde{e})^{m+k}].$$

Take the expectation of the generator w.r.t. $\tilde{\mathbf{e}}$ and ξ. $\forall$ $\delta_1 > 0$, $\delta_2 > 0$, and $\delta_3 > 0$

$$\begin{aligned}
\mathrm{E}\left[(\tilde{\mathbf{e}}'P\tilde{\mathbf{e}})^k\right] &\leq \frac{\mathrm{E}\left[V(\tilde{\mathbf{e}})\right]}{\delta_1^{m-k}} + \delta_1^k \\
\mathrm{E}\left[(\tilde{\mathbf{e}}'P\tilde{\mathbf{e}})^{m+k}\right] &\geq \delta_2^k\,\mathrm{E}\left[V(\tilde{\mathbf{e}})\right] - \delta_2^{m+k} \\
\mathrm{E}\left[(\tilde{\mathbf{e}}'P\tilde{\mathbf{e}})^{m-1}\right] &\leq \frac{1}{\delta_3}\mathrm{E}\left[V(\tilde{\mathbf{e}})\right] + \delta_3^{m-1}.
\end{aligned} \tag{15}$$

By Hölder's Inequality and Lemma 2, it is easy to verify that, $\forall$ $\delta_4 > 0$,

$$\begin{aligned}
&\mathrm{E}^{\tilde{\mathbf{e}},\xi}\left[(\tilde{e}'P\tilde{e})^{m-\frac{1}{2}}(\xi'\xi)^{\frac{1}{2}}\right] \\
&\leq \big(\Delta_\xi(m)\big)^{\frac{1}{2m}}\left(\frac{1}{\delta_4^{\frac{1}{2m}}}\mathrm{E}\left[V(\tilde{\mathbf{e}})\right] + \delta_4^{1-\frac{1}{2m}}\right)
\end{aligned} \tag{16}$$

where $\Delta_\xi(m)$ is as in (7). By the Comparison Lemma for ODEs, from (14), (15) and (16), it is established that $\mathrm{E}[V(\mathbf{e}(t))]$ is bounded by choosing large enough δ_2.

For $k \geq m$, use Lemma 2 again, $\forall$ $\delta_5 > 0$

$$\mathrm{E}\left[(\tilde{e}'P\tilde{e})^m\right] \leq \frac{\mathrm{E}\left[(\tilde{e}'P\tilde{e})^{k+1}\right]}{\delta_5^{k+1-m}} + \delta_5^m,$$

which leads to the boundedness of $\mathrm{E}\left[(\tilde{e}'P\tilde{e})^m\right]$ since the boundedness of $\mathrm{E}\left[(\tilde{e}'P\tilde{e})^{k+1}\right]$ is already established in the previous case. $\square$

C. Proof of Theorem 3

The rate can be written as $\lambda(\tilde{e}) = \gamma - b(e)$, where

$$b(\tilde{e}) = \begin{cases} \gamma - (\tilde{e}'P\tilde{e})^k & \text{if} \quad (\tilde{e}'P\tilde{e})^k < \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathrm{E}\left[b(\tilde{e})V(\tilde{e})\right]$ is uniformly bounded. Similar arguments go through as in the proof of Theorem 1 as long as γ satisfies the conditions in Theorem 1. The details are omitted [14].

V. Simulation

The second-order plant with two unstable poles in [7] is recaptured in continuous settings with $T_s = 0.1s$, and

$$A = \begin{bmatrix} 2.23 & 0 \\ 8.52 & 0.95 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

and the disturbance and noise covariance matrices are

$$\Sigma_w = \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix}, \quad \Sigma_v = 25.$$

The Kalman filter gain is $L = \begin{bmatrix} 31.88 \\ 8.36 \end{bmatrix}$. Figure 5 is the variance of 5000-sample Monte Carlo simulation for different constant Poisson rates γ, which are chosen around $\Gamma := 2\max\{\Re[\mathrm{Eig}(A)]\} \approx 4.46$. For $\gamma = 0.2\Gamma$, the error variance blows up, and for $\gamma = 1.8\Gamma$, it is well below 1000. For $\gamma = 1.1\Gamma$, the variance is still bounded, but not as clear-cut. The curve for $\gamma = 0.9\Gamma$ has a trend of going up. It is conjectured that limited samples in the simulation cause statistical variations.

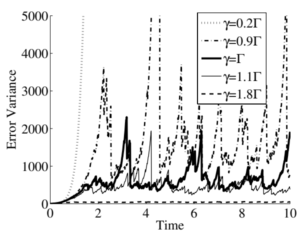

Fig. 5. Variances of 5000 samples by Monte Carlo simulation for different Poisson rates.

For communication schedules mediated by integer-valued processes, the intensity functions are $\lambda(\tilde{e}^-) = \tilde{e}^{-\prime} P \tilde{e}^-$, $P > 0$. Fig. 6 gives the variance and communication rate for different P. For $P = 0.2I_{2\times2}$, the corresponding rate is 5.25, which is roughly the same as that of constant $\gamma = 1.1\Gamma$. But the former has a variance around 5, as in Fig. 6, while the latter has 427, as in Fig. 5. The schedule associated with the integer-valued process reduces error variances given the same communication rates in this case.

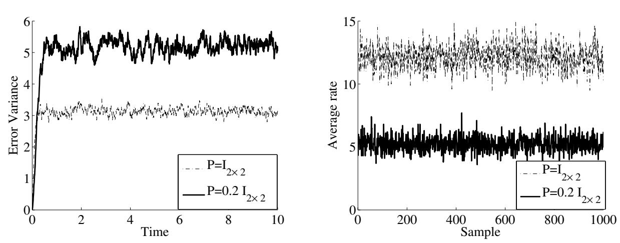

Fig. 6. Variances (left) and communication rates (right) of 5000 samples by Monte Carlo simulation for integer-valued processes with jump intensity $\tilde{e}^{-\prime} P \tilde{e}^-$.

VI. Conclusion and Future Work

An LTI estimation framework is proposed, in which Kalman filter estimates are sent to the remote estimator. Data communication patterns based on uncontrolled and controlled random processes are proposed and analyzed. The tight bound on the Poisson rate is given for stochastic moment stability in uncontrolled communications. Sufficient stability conditions for controlled communications are also proposed. Simulations show that the controlled communications improve the trade-off between the communication rate and the estimation variance.

In the Fig. 1 scheme, the remote estimator actually encodes more measurements, thanks to the Kalman filter, than that from Fig. 2. Simulations for the discrete-time estimators show that the former achieves lower error variances, but we need to calculate the expected covariance matrices for this case. It is left as future work to investigate stronger notions of performance, e.g., the system transient response, and to design the matrix P in (10) so as to improve the communication rate and estimation performance trade-off.

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