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On Synchronization of Kuramoto Oscillators

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Abstract— **Synchronization is a key concept to the understanding of self-organization phenomena occurring in coupled oscillators of the dissipative type. In this paper we study one of the most representative models of coupled phase oscillators, the Kuramoto model. The traditional Kuramoto model (all-to-all connectivity) is said to synchronize if the angular frequencies of all oscillators converge to the mean frequency of the group and the oscillators get phase locked. Recently, Jadbabaie et. al. calculated a lower bound on the coupling gain which is necessary for the onset of synchronization in the traditional Kuramoto model. It was also shown that there exists a large enough coupling gain so that the phase differences are locally asymptotically stable. Furthermore, the authors demonstrated that the convergence is exponential when all oscillators have the same natural frequency. In this paper we assume that the natural frequencies of all oscillators are arbitrarily chosen from the set of reals. We develop a tighter lower bound on the coupling gain, as compared to the one proposed by Jadbabaie et. al., which is necessary for the onset of synchronization in the traditional Kuramoto model. Our main result says that it is possible to find a coupling gain such that the angular frequencies of all oscillators locally exponentially synchronize to the mean frequency of the group. To the best of our knowledge, this is the first result which demonstrates that in the traditional Kuramoto model, with all-to-all coupling and different natural frequencies, the oscillators locally exponentially synchronize. Simulations are also presented to validate the proposed results.**

I. Introduction

Collective synchronization phenomena have been observed in biological, chemical, physical and social systems for centuries. The concept of synchronization implies that multiple periodic processes with different natural frequencies come to acquire a common natural frequency as a result of their mutual or one-sided interaction. This phenomenon is observed when system of oscillators lock on to a common frequency despite differences in the natural frequency of the individual oscillators. Biological examples include groups of synchronously flashing fireflies [1], crickets that chirp in unison [13] etc. Examples in physics include the superconducting Josephson junction [16]. The importance of synchronization in nature may be realized from the fact that what looks like a single periodic process on a macroscopic scale often turns out to be collective oscillation resulting from the mutual synchronization among large number of constituent oscillators. The human heartbeat may be taken as an example of this phenomenon. As the constituent oscillators in nature rarely posses identical frequencies, mutual synchronization appears to be a unique mechanism for producing and maintaining macroscopic rhythmicity.

Collective synchronization was first studied by Weiner [15], who speculated that it is involved in the generation of alpha rhythms in the brain. It was then taken up by Winfree [17] who used it to study circadian rhythms in living organisms. He contended that theoretical understanding of the origin of collective rhythmicity would be best studied by studying its onset, that is, by treating it as a kind phase transition or bifurcation. His attempt to study mutual synchronization in multi-oscillator systems was based on a phase description [18]. Winfree's model was significantly extended by Kuramoto in [5], [6] where he developed results what is now popularly known as the Kuramoto model. Kuramoto's work, and the later attempts to answer the questions that were raised by his formulations, have been elegantly summarized in [11].

Recently, control theoretic methods have been used in [14], [10], [4], [2] to address the synchronization phenomenon. Phase models of coupled oscillators were used to derive control laws for stabilizing collective motion of a group of self-propelled particles in [10]. In [7] consensus problems were discussed for a network of dynamic agents with fixed and switching topologies. Stability analysis was carried out for a unidirectional ring of oscillators with Kuramoto type dynamics in [9]. In [8] it was shown that only phase locking solutions corresponding to $(-\frac{\pi}{2}, \frac{\pi}{2})$ can be locally asymptotically stable, and a condition for guaranteeing local asymptotic stability was also derived. However, the condition was dependent on the parameter r_∞, along with the coupling gain and natural frequencies of the oscillators. Recently in [3], control and graph theoretic methods were used to analyze Kuramoto oscillators for an arbitrary bidirectional graph topology. The authors derived the value of the coupling gain K_L for the onset of synchronization in the traditional Kuramoto Model (all-to-all connectivity). It was also shown that there exists a large enough coupling gain K gain so that the phase differences are locally $(-\frac{\pi}{2}, \frac{\pi}{2})$ asymptotically stable. Furthermore, the authors also demonstrated that the convergence is exponential, when all oscillators have the same natural frequency.

In this paper we study the case of a large but finite (N) number of Kuramoto oscillators where every oscillator is connected to every other oscillator (the original Kuramoto model). We assume in this note that the natural frequencies ω_i of all oscillators are arbitrarily chosen from the set of reals and we do not impose any particular distribution on them. We construct a tighter lower bound on the coupling

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gain, as compared to the one developed by Jadbabaie et. al., which is necessary for the onset of synchronization in the traditional Kuramoto model. Our main result says that it is possible to find a coupling gain $K = K_{inv}$ such that all trajectories which start within the set defined by

$$\mathcal{D} = \{\theta_i, \theta_j \in R \mid |\theta_i - \theta_j| \le \frac{\pi}{2} - 2\epsilon\}$$

where $\epsilon < \frac{\pi}{4}$ is an arbitrary positive number, exponentially synchronize.

II. Summary of Kuramoto's Results

In this section we describe the original Kuramoto model and summarize the main findings. The Kuramoto model consists of a population of N oscillators who dynamics are governed by the following equations

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} sin(\theta_j - \theta_i), \quad i = 1, 2, ..., N \quad (1)$$

where $\theta_i \in S^1$ is the phase of the $i-th$ oscillator, $\omega_i \in R$ is its natural frequency and $K > 0$ is the coupling gain. The natural frequencies are distributed with probability density $g(\omega)$, where $g(\omega)$ is assumed to be unimodal and symmetric about the mean frequency Ω i.e., $g(\Omega + \omega) = g(\Omega - \omega)$. By making a suitable choice of a rotating frame, $\theta_i \to \theta_i + \Omega t$, where Ω is the first moment (mean) of $g(\omega)$, the dynamics (1) get transformed to an equivalent system of phase oscillators whose natural frequencies have a zero mean. Therefore we have that $g(\omega) = g(-\omega)$ for all ω. To get a nice intuition about the problem, the oscillators may also be thought of as points moving on a unit circle. The problem is then to characterize the coupling gain K so that the oscillators synchronize.

The oscillators are said to synchronize if

$$\boxed{\dot{\theta}_i - \dot{\theta}_j \to 0 \quad as\ t \to \infty \quad \forall i, j = 1, \ldots, N}$$

or in other words the phase differences given by $\theta_i - \theta_j \quad \forall i, j = 1, 2, ..., N$ become constant asymptotically. Imagining these oscillators on circle as points, the points then move with the same angular frequency and hence, angular distance (phase difference) between the points remain constant with time. Define the order parameter r as

$$re^{i\Psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

The order parameter r(t) with $0 \le r(t) \le 1$ is a measure of phase coherence of the oscillator population. If the oscillators synchronize, then the parameter converges to a constant $r_\infty \le 1$, but if the oscillators add incoherently then the order parameter r remains close to zero. Using the order parameter, the model (1) can be rewritten as [11]

$$\dot{\theta}_i = \omega_i + \frac{K}{N} r sin(\theta_i - \Psi), \quad i = 1, 2, ..., N$$

In the continuum limit case where $N \to \infty$, Kuramoto showed that there exists a value of the coupling gain K such that for all $K < K_c$, the oscillators are incoherent (or remain unsynchronized), but for $K > K_c$ the incoherent state becomes unstable, the oscillators start synchronizing and eventually $r(t)$ settles at some $r_\infty(K) < 1$. Kuramoto calculated closed form solutions for the gain K_c (the critical gain for the onset of synchronization), and $r_\infty(K)$. Furthermore it has been shown via simulations that for $K > K_c$, the population of oscillators divides into two groups. The oscillators whose natural frequencies is close to the mean frequency, lock on to form a synchronized cluster and start rotating with the mean frequency Ω, while those whose natural frequencies are far way from the mean of the group, drift relative to the synchronized cluster oscillators.

However there are some important questions still associated with the Kuramoto model and they form the motivation for this paper. We list here one of the open problems which we address in this paper. It has been shown via numerical simulations that for when $K > K_c$, the parameter $r(t)$ grows exponentially and saturates at some $r_\infty < 1$. Till date there is no analysis which shows that the oscillators in the original Kuramoto model (where the oscillators have different natural frequencies) synchronize exponentially (even locally) and quoting Strogatz [11] "Nobody has even touched the problems of global stability and convergence".

The paper is organized as follows. In the next section we calculate the critical gain K_c which is necessary for the onset of synchronization in the whole population of oscillators. In Section (IV) we develop a lower bound on the coupling gain $K = K_{inv}$ which is sufficient for oscillator synchronization within an arbitrary compact set of $(-\frac{\pi}{2}, \frac{\pi}{2})$, and then in Section (V) it is demonstrated that the oscillators locally exponentially synchronize. The results are validated by simulations in (VI) and summarized in (VII).

III. Onset of Synchronization

As we are interested in the evolution of the phase differences, the phase difference dynamics can be written down using (1) as

$$\begin{aligned}\dot{\theta}_i - \dot{\theta}_j = \omega_i - \omega_j + \frac{K}{N}\Big\{ -2sin(\theta_i - \theta_j) \\ + \sum_{k=1\ k\ne i,j}^{N}\big(sin(\theta_k - \theta_i) + sin(\theta_j - \theta_k)\big)\Big\} \qquad (2)\end{aligned}$$

If the oscillators are to synchronize i.e. $\dot{\theta}_i - \dot{\theta}_j \to 0 \quad as\ t \to \infty \quad \forall i, j = 1, \ldots, N$, the R.H.S of (2) must go to zero. In this section, we calculate a lower bound on the coupling gain K so that there is a possibility of the R.H.S of (2) to go to zero. In other words, we calculate a necessary condition for the onset of synchronization in (1). The oscillators can only synchronize if equation (1) has at least one fixed point $\forall i, j = 1, \ldots, N \quad i \ne j$. The fixed point equation can be written down as

$$\begin{aligned}\omega_j - \omega_i = \frac{K}{N}\Big\{2sin(\theta_j - \theta_i) + \sum_{k=1\ k\ne i,j}^{N}\big(sin(\theta_k - \theta_i) \\ + sin(\theta_j - \theta_k)\big)\Big\} \qquad (3)\end{aligned}$$

Assuming without loss of generality that $\omega_j > \omega_i$, to calculate a lower bound on the coupling gain K satisfying (3), we need to maximize the expression

$$E = 2sin(\theta_j - \theta_i) + \sum_{k=1\ k\neq i,j}^{N} sin(\theta_k - \theta_i) + sin(\theta_j - \theta_k) \quad (4)$$

Using elementary calculus, the first order necessary conditions for maximizing (4) are given by

$$\frac{\partial E}{\partial \theta_i} = -2cos(\theta_j - \theta_i) - \sum_{k=1\ k\neq i,j}^{N} cos(\theta_k - \theta_i) = 0 \quad (5)$$

$$\frac{\partial E}{\partial \theta_j} = 2cos(\theta_j - \theta_i) + \sum_{k=1\ k\neq i,j}^{N} cos(\theta_j - \theta_k) = 0 \quad (6)$$

$$\frac{\partial E}{\partial \theta_k} = cos(\theta_k - \theta_i) - cos(\theta_j - \theta_k) = 0 \quad (7)$$

Using (7), we get that either

$$\theta_k = \frac{\theta_i + \theta_j}{2} \quad or \quad \theta_i = \theta_j$$

It is easily seen that $\theta_i = \theta_j = 0$ implies that $E = 0$, as we are looking for a maximum we investigate the other solution. Substituting θ_k as $\frac{\theta_i+\theta_j}{2}$ in (5) we get the condition

$$2cos(\theta_j - \theta_i) + \sum_{k=1\ k\neq i,j}^{N} cos\Big(\frac{\theta_j - \theta_i}{2}\Big) = 0$$

$$\Rightarrow 2cos(\theta_j - \theta_i) + (N-2)cos\Big(\frac{\theta_j - \theta_i}{2}\Big) = 0$$

$$\Rightarrow 4cos^2\Big(\frac{\theta_j - \theta_i}{2}\Big) - 2 + (N-2)cos\Big(\frac{\theta_j - \theta_i}{2}\Big) = 0$$

It is to be noted that the same equation will be obtained by substituting θ_k as $\frac{\theta_i+\theta_j}{2}$ in (6). Solving the above quadratic equation we get

$$cos\Big(\frac{\theta_j - \theta_i}{2}\Big) = \frac{-(N-2) \pm \sqrt{(N-2)^2 + 32}}{8}$$

As $cos(x) \leq 1\ \forall x \in R$, a well defined solution for all N is given by

$$cos\Big(\frac{\theta_j - \theta_i}{2}\Big) = \frac{-(N-2) + \sqrt{(N-2)^2 + 32}}{8} \quad (8)$$

Denote the optimal value of $\theta_j - \theta_i$ maximizing (4) by $(\theta_j - \theta_i)_{opt}$. It can also be verified that the second order necessary condition for optimality (maximum in this case) given by

$$\frac{\partial E}{\partial \theta_m \partial \theta_n} \leq 0 \quad \forall m, n = 1, \ldots, N$$

is also satisfied by $(\theta_j - \theta_i)_{opt}$, and hence the optimal (maximum) value of E is given as

$$E_{max} = 2sin(\theta_j - \theta_i)_{opt} + 2(N-2)sin\Big(\frac{(\theta_j - \theta_i)_{opt}}{2}\Big)$$

Thus the critical gain coupling gain required for onset of synchronization in (2) is given by

$$K_c = \frac{(\omega_j - \omega_i)N}{E_{max}}$$

If the natural frequencies belong to a compact set, then the critical gain coupling gain required for onset of synchronization in (1) is given as

$$K_c = \frac{(\omega_{max} - \omega_{min})N}{E_{max}} \quad (9)$$

where $\omega_{max} > 0, \omega_{min}$ is the maximum and minimum frequencies in the set of natural frequencies. The phrase *critical gain coupling gain required for onset of synchronization* does not imply that at K_c the oscillators synchronize. The critical gain K_c is the gain below which the oscillators cannot synchronize.

It is interesting to compare the condition (9) with that for the critical gain obtained in [3]. The value for the critical coupling in [3] is given as

$$K_L = \frac{(\omega_{max} - \omega_{min})N}{2(N-1)} \quad (10)$$

Therefore, comparing (9) with (10) we find that in (10) $E_{max} = 2(N-1)$ implicitly. We contend that $E_{max} = 2(N-1)$ is a value not achievable by the function E. This is so because in [3] the authors assumed that at E_{max}, $|\theta_m - \theta_n| = \frac{\pi}{2} \quad \forall \quad m, n = 1, \ldots, N$. This clearly is not possible as the phase differences $\theta_m - \theta_n\ \forall\ m, n = 1, \ldots, N$ are not independent. Thus the onset of synchronization is not possible for all coupling gains K satisfying $K_L \leq K < K_c$. Only for $K \geq K_c$, the dynamical system given by (1) may synchronize.

IV. Synchronization of Kuramoto Oscillators

In the previous section we developed the lower bound on the critical gain K denoted by K_c which is necessary for the onset of synchronization in the traditional Kuramoto model. In this section we develop a lower bound on the coupling gain K which is sufficient for synchronization of the oscillators within an arbitrary compact set of $(-\frac{\pi}{2}, \frac{\pi}{2})$. The assumption in the analysis that follows is that the initial phase of all oscillators lie within the set described by

$$\mathcal{D} = \{\theta_i, \theta_j \in R \mid |\theta_i - \theta_j| \leq \frac{\pi}{2} - 2\epsilon\}$$

where $\epsilon < \frac{\pi}{4}$ is an arbitrary positive number. We will develop a lower bound on the coupling gain K denoted by K_{inv} which makes this set positively invariant for all oscillators, i.e. $\theta_i - \theta_j \in \mathcal{D}$ at t=0 $\Rightarrow \theta_i - \theta_j \in \mathcal{D}\ \forall\ t > 0$. Then having phase-locked the oscillators in $\mathcal{D}$, we will show that the oscillators synchronize.

The phase difference dynamics as described by (2) can be rewritten as

$$\dot{\theta}_i - \dot{\theta}_j = K\Big\{\frac{\omega_i - \omega_j}{K} - sin(\theta_i - \theta_j) + \frac{1}{N}\Big(\sum_{k=1}^{N} sin(\theta_i - \theta_j) + sin(\theta_k - \theta_i) + sin(\theta_j - \theta_k)\Big)\Big\} \quad (11)$$

Consider the term

$$\frac{1}{N}\big(sin(\theta_i - \theta_j) + sin(\theta_k - \theta_i) + sin(\theta_j - \theta_k)\big)$$

This can be rewritten using some trigonometric rearrangements as

$$\frac{1}{N}sin(\theta_i-\theta_j)\Big(1-\frac{cos(\theta_k-\frac{(\theta_i+\theta_j)}{2})}{cos(\frac{(\theta_i-\theta_j)}{2})}\Big)$$
$$=\frac{1}{N}sin(\theta_i-\theta_j)C_k$$

where $C_k = \Big(1-\frac{cos(\theta_k-\frac{(\theta_i+\theta_j)}{2})}{cos(\frac{(\theta_i-\theta_j)}{2})}\Big)$. It is easy to see that $\forall(\theta_i-\theta_j)\in\mathcal{D}$, $0\leq C_k<1$. Using this, (11) can be rewritten as

$$\dot\theta_i-\dot\theta_j = K\big\{\frac{\omega_i-\omega_j}{K}-sin(\theta_i-\theta_j)$$
$$+\frac{1}{N}\sum_{k=1}^{N}C_k sin(\theta_i-\theta_j)\big\}$$
$$=K\big\{\frac{\omega_i-\omega_j}{K}-sin(\theta_i-\theta_j)\big(1-\frac{1}{N}\sum_{k=1}^{N}C_k\big)\big\} \quad (12)$$

We are now in a position to state the first result of this section.

Theorem 4.1: Consider the system dynamics as described by (12). Let all initial phase differences at t=0 be contained in the compact set $\mathcal{D}=\{\theta_i,\theta_j \mid |\theta_i-\theta_j|\leq\frac{\pi}{2}-2\epsilon \ \ \forall i,j=1,\ldots,N\}$. Then there exists a coupling gain $K_{inv}>0$ such that $(\theta_i-\theta_j)\in\mathcal{D} \ \ \forall t>0$.

Proof: Let the positive definite Lyapunov function for the dynamic system governed by (12) be given as

$$V=\frac{1}{2K}\Big(\theta_i-\theta_j\Big)^2$$

The derivative of the Lyapunov function along trajectories of the system (12) is given as

$$\dot V=\frac{1}{K}(\theta_i-\theta_j)\big(\dot\theta_i-\dot\theta_j\big)$$
$$=(\theta_i-\theta_j)\Big(\frac{\omega_i-\omega_j}{K}-sin(\theta_i-\theta_j)\big(1-\frac{1}{N}\sum_{k=1}^{N}C_k\big)\Big)$$
$$\leq|\theta_i-\theta_j||\frac{\omega_i-\omega_j}{K}|-(\theta_i-\theta_j)sin(\theta_i-\theta_j)\big(1-\sum_{k=1}^{N}\frac{C_k}{N}\big)$$
$$\leq|\theta_i-\theta_j||\frac{\omega_i-\omega_j}{K}|-(\theta_i-\theta_j)sin(\theta_i-\theta_j)\big(1-\frac{N-2}{N}\big)$$

where it has been used in the last equation that $C_k<1$ and that $C_k=0$ for $k=i,j$. Thus the derivative can be written as

$$\dot V\leq|\theta_i-\theta_j||\frac{\omega_i-\omega_j}{K}|-(\theta_i-\theta_j)sin(\theta_i-\theta_j)\frac{2}{N}$$

It is to be noted that the function $sin(\theta_i-\theta_j)(\theta_i-\theta_j)$ is always nonnegative in the considered domain. Therefore, if $K>\frac{N|\omega_i-\omega_j|}{2cos(2\epsilon)}$, the derivative of the Lyapunov function is negative at $|\theta_i-\theta_j|=\frac{\pi}{2}-2\epsilon$ and thus the phase difference $\theta_i-\theta_j$ cannot leave the set $\mathcal{D}$. Finally, if $K=K_{inv}>\frac{N|\omega_{max}-\omega_{min}|}{2cos(2\epsilon)}$, all phase differences $\theta_i-\theta_j \ \ \forall i=1,2,...,N$ are positively invariant with respect to the compact set $\mathcal{D}$. ■

Having trapped the phase differences within the desired compact set $\mathcal{D}$ by appropriately choosing the coupling gain, we demonstrate that the oscillators synchronize.

Theorem 4.2: Consider the system dynamics as described by (12). Let all initial phase differences at t=0 be contained in the compact set $\mathcal{D}$. If the coupling gain K is chosen such that $K=K_{inv}$, then all the oscillators synchronize i.e. $\dot\theta_i-\dot\theta_j\to 0 \ \ as\ t\to\infty \ \ \forall i,j=1,\ldots,N$

Proof: Consider the positive function,

$$S=\frac{1}{2}\dot\theta^T\dot\theta$$

where $\dot\theta=\big[\dot\theta_1\ldots\dot\theta_N\big]^T$ Differentiating along trajectories of the system (1) we get

$$\dot S=\dot\theta_1\ddot\theta_1+\dot\theta_2\ddot\theta_2+\ldots+\dot\theta_n\ddot\theta_n$$
$$=\frac{\dot\theta_1}{\beta}\Big(cos(\theta_1-\theta_2)(\dot\theta_2-\dot\theta_1)+\ldots+cos(\theta_n-\theta_1)(\dot\theta_n-\dot\theta_1)\Big)$$
$$+\frac{\dot\theta_2}{\beta}\Big(cos(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2)+\ldots+cos(\theta_n-\theta_2)(\dot\theta_n-\dot\theta_2)\Big)$$
$$\vdots$$
$$+\frac{\dot\theta_n}{\beta}\Big(cos(\theta_2-\theta_n)(\dot\theta_2-\dot\theta_n)+\ldots+cos(\theta_1-\theta_n)(\dot\theta_1-\dot\theta_n)\Big)$$

where $\beta=\frac{N}{K}$. On rearranging terms and simplifying we have that,

$$\dot S=-\frac{K}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}cos(\theta_i-\theta_j)(\dot\theta_i-\dot\theta_j)^2 \quad (13)$$

Due to Theorem (4.1) we have that $(\theta_i-\theta_j)\in\mathcal{D}$, $\forall i,j$. This gives us that $cos(\theta_i-\theta_j)>0\ \forall i,j$ and hence $\dot S\leq 0$. Hence all angular frequencies (i.e. $\dot\theta_i\ \ \forall i$) are bounded. Consider the set $E=\{\theta_i-\theta_j,\dot\theta_i\in R\ \forall i,j \mid \dot S=0\}$. The set E is characterized by all trajectories such that $\dot\theta_i=\dot\theta_j,\ \forall i,j$. Let M be the largest invariant set contained in E. Using Lasalle's Invariance Principle, all trajectories starting in $\mathcal{D}$ converge to M as $t\to\infty$. Hence the oscillators synchronize asymptotically. ■

The above theorem tells us that all the oscillators start moving with the same angular frequency, but what is the consensus value of the group? Or in other words what is the common angular frequency to which all the oscillators converge? We provide an answer to this question in the next result.

Corollary 4.3: Consider the system represented by (1). If $K=K_{inv}$, then the oscillatory asymptotically converge to the mean natural frequency of all oscillators i.e., $\dot\theta_i=\dot\theta_j=\frac{\sum_{i=1}^{N}\omega_i}{N}=\Omega\ \ \forall i,j=1,\ldots,N$ as $t\to\infty$.

Proof: It is easy to see from (1) that

$$\sum_{i=1}^{N}\dot\theta_i=\sum_{i=1}^{N}\omega_i \quad (14)$$

As $\dot\theta_i\to\dot\theta_j\ \ \forall i,j=1,\ldots,N$ as $t\to\infty$, we have that $\dot\theta_i\to\frac{\sum_{i=1}^{N}\omega_i}{N}\ \forall i=1,2,..,N$, and hence asymptotically all oscillators start moving with the mean natural frequency of the group. ■

V. Exponential Synchronization

In the previous section we demonstrated that with suitable choice of the coupling gain K, the oscillators synchronize. In this section we demonstrate that the oscillators converge exponentially to the mean natural frequency of the group. In the Kuramoto model, all oscillators are connected via $all-to-all$ topology, i.e. every oscillator (or node) is connected to every other oscillator (node). These nodes form a graph and our results in this section use some algebraic properties of the underlying graph. We provide a brief introduction of the graph theory tools used in this section. A graph theoretic approach to the Kuramoto Oscillator problem was used recently in [3] and we adapt their concise introduction in this section. The graph can be described by two matrices which encode the topology of the interconnection. The incidence matrix of an oriented graph $\mathcal{G}^{\alpha}$ with N vertices and e edges is the $N \times e$ matrix such that: B_{ij}=1 if the edge is incoming to vertex i, $B_{ij} = -1$ is the edge j is outcoming from the vertex i, and 0 otherwise. The symmetric $N \times N$ matrix defined as: $L = BB^T$ is called the Laplacian of $\mathcal{G}$ and is independent of the choice of orientation α. The Laplacian has several important properties: L is always positive semi-definite with a zero eigenvalue; the algebraic multiplicity is equal to the connected components in the graph; the N dimensional vector associated with the zero eigenvalue is the vector of ones 1_N. The spectrum of the Laplacian matrix of the graph captures many topological properties of the graph. It was shown by Fiedler that the first non-zero eigenvalue $\lambda_2(L)$ (also referred to as the algebraic connectivity and the Fielder eigenvalue) gives a measure of connectedness of the graph. If we associate a positive number W_i to each edge and we form the diagonal matrix $W_{e\times e} := diag(W_i)$, then the matrix $L_W(\mathcal{G}) = BWB^T$ is a weighted Laplacian which fulfills the aforementioned properties.

In the Kuramoto model, all nodes are connected to all other nodes and hence the dynamics which were previously described by (1) can be equivalently written down as

$$\dot{\theta} = \omega - \frac{K}{N} B sin(B^T\theta) \tag{15}$$

where B is the incidence matrix of the unweighted graph, θ and ω are $N \times 1$ vectors. It is also helpful to define the $e \times 1$ vector of the phase differences $\phi := B^T\theta$.

Let us revisit Theorem (4.2), where it was shown that the oscillators synchronize. The positive function S is given as

$$S = \frac{1}{2}\dot{\theta}^T\dot{\theta} \tag{16}$$

The derivative of this function along trajectories of (15) can be written as

$$\begin{aligned}\dot{S} &= -\frac{K}{N}\dot{\theta}^T B diag\big(cos(\phi)\big)B^T\dot{\theta} \\ &= -\frac{K}{N}\dot{\theta}^T L_K(\mathcal{G})\dot{\theta}\end{aligned} \tag{17}$$

The matrix $L_K(\mathcal{G}) = Bdiag\big(cos(\phi)\big)B^T \in N \times N$ is the weighted Laplacian and is described as follows

$$L_W(\mathcal{G})_{ii} = \sum_{k=1,k\neq i}^{N} cos(\theta_k - \theta_i) \quad \forall i = 1,\dots,N$$
$$L_W(\mathcal{G})_{ij} = -cos(\theta_i - \theta_j) \quad \forall i,j = 1,\dots,N \quad i \neq j$$

Clearly, if all phase differences $\phi \in \mathcal{D}$, then the weighted Laplacian matrix $L_K(\mathcal{G})$ is positive-semidefinite, and hence the result of Theorem (4.2) follows. In the next theorem we extend this result by developing an exponential bound on the synchronization rate of the oscillators.

Theorem 5.1: Consider the dynamics of the system as described by (15). If the phase differences given by $\phi \in \mathcal{D}$ at $t = 0$ and the coupling gain is selected such that $K = K_{inv}$, then the oscillators synchronize exponentially at a rate no worse that $\sqrt{Ksin(2\epsilon)}$.

Proof: It follows from (14) that

$$\Omega = \frac{\sum_{i=1}^{N}\dot{\theta}_i}{N} = \frac{\sum_{i=1}^{N}\omega_i}{N}$$

which implies that Ω is an invariant quantity. Following [7], the vector $\dot{\theta}$ can be written down as

$$\dot{\theta} = \Omega 1 + \delta \tag{18}$$

where 1 is the N dimensional vector of ones associated with the zero eigenvalue of the weighted Laplacian $L_W(\mathcal{G})$, $\delta \in R^n$ satisfies $\sum_{i=1}^{N}\delta = 0$ $(as \sum_{i=1}^{N}\dot{\theta}_i = N\Omega)$. The vector δ is orthogonal to 1 and was referred to as the group disagreement vector in [7]. Substituting (18) in (16), we have that

$$\frac{d(\delta^T\delta)}{dt} = -\frac{K}{N}\delta^T L_W(\mathcal{G})\delta \tag{19}$$

where we have used the fact that Ω is an invariant quantity and that $1^T L_W(\mathcal{G}) = 0$ as 1 is an eigenvector associated with the zero eigenvalue of $L_W(\mathcal{G})$. It is easy to see from the above equation and the positive definiteness of the matrix $L_W(\mathcal{G})$ (in the projected space orthogonal to 1) that the disagreement vector δ exponentially converges to the origin. The exponential convergence of δ and (18) tells us that the oscillators start moving with the mean frequency of the group. As $\lambda_2(L_K(\mathcal{G}))$ is the Fiedler eigenvalue (smallest non-zero eigenvalue) of the weighted Laplacian $\lambda_2(L_K(\mathcal{G}))$, we have from (19) that

$$\begin{aligned}\frac{d(\delta^T\delta)}{dt} &\leq -\frac{K}{N}\delta^T\lambda_2(L_W(\mathcal{G}))\delta \\ &\leq -\frac{K}{N}\delta^T\lambda_2(Bdiag\big(cos(\phi)\big)B^T)\delta \\ &\leq -\frac{K}{N}\delta^T sin(2\epsilon)\lambda_2(BB^T)\delta \\ &\leq -Ksin(2\epsilon)\delta^T\delta\end{aligned}$$

as $\min\{\cos(\phi)\} : \forall\phi \in \mathcal{D} = \cos(\frac{\pi}{2} - 2\epsilon) = \sin(2\epsilon)$ and for an all-to-all connected topology $\lambda_2(BB^T) = N$. Thus the exponential convergence rate for synchronization is no worse that $\sqrt{Ksin(2\epsilon)}$.

■

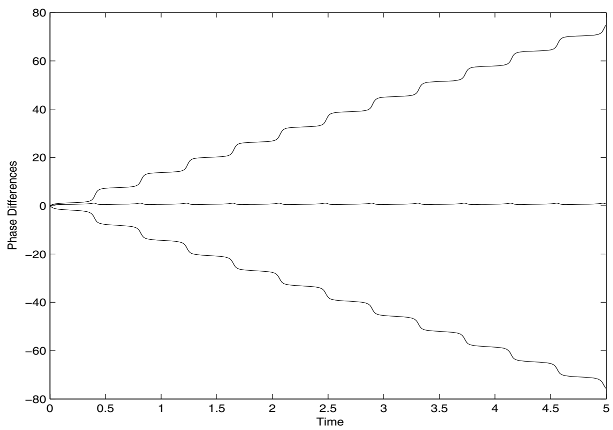

Fig. 1. The oscillators do not synchronize when $K_L < K < K_c$.

VI. Simulations

In this section we simulate the Kuramoto oscillator model with $N = 3$ oscillators. The oscillators are chosen such that their natural frequency are as follows, $\omega_1 = 10$, $\omega_2 = 30$, and $\omega_3 = 70$ (units are in rad/s). The mean frequency of the group is then given by $\Omega = 36.67$. The coupling gain $K = K_c$ which is necessary for the onset of synchronization is given by (9), and as $\omega_{max} = 70$, $\omega_{min} = 10$, we have that $K_c = 51.13$. Using (10), the necessary lower bound provided in [3] equals $K_L = 45$. To define the set in which we want to confine our phase differences, choose $\epsilon = 0.5$, and thus the desired compact set $\mathcal{D}$ is given as

$$\begin{aligned}\mathcal{D} &= \{\theta_i, \theta_j \in R \mid |\theta_i - \theta_j| \leq \frac{\pi}{2} - 2\epsilon\} \\ &= \{\theta_i, \theta_j \in R \mid |\theta_i - \theta_j| \leq 0.5708\}\end{aligned}$$

The desired coupling gain is given by the formula $K = K_{inv} > \frac{N|\omega_{max} - \omega_{min}|}{2cos(2\epsilon)}$ and thus substituting the relevant values, we select $K_{inv} = 167$. The simulations were performed for three values of the coupling gain $K_L < K = 50 < K_c$, $K_c < K = 53 < K_{inv}$ and $K = K_{inv}$. The initial phase differences at t=0 were selected so that they were in $\mathcal{D}$. In the first simulation, when the coupling gain K satisfies $K_L < K = 50 < K_c$, the phase differences diverge and the oscillators are unsynchronized as seen in Figure 1. In the next simulation scenario, the coupling gain is chosen to be $K_c < K = 53 < K_{inv} = 53$ and as seen in Figure 2, the oscillators synchronize. On the other hand we see that the set $\mathcal{D}$ is not able to attract the phase differences. Finally setting the coupling gain $K = K_{inv}$, we find find that the oscillators synchronize, and the phase differences are indeed invariant with respect to the compact set $\mathcal{D}$ as seen in Figure 3. Also the angular frequencies of all oscillators ($\dot{\theta}_i \;\; i = 1, 2, 3$) exponentially converge to the mean frequency Ω as seen in Figure 4. The next simulation we perform is with the initial phase differences outside the set $\mathcal{D}$ and with the coupling gain $K = K_{inv}$. It turns out that the phase differences still converge to the desired set $\mathcal{D}$ (Figure 5). This behavior seems interesting and shall be a topic of our future research.

VII. Conclusions

In this paper we studied the phenomenon of synchronization in the Kuramoto model with an arbitrary but finite number of oscillators. A necessary condition in the form of a lower bound on the coupling gain $K = K_c$ was established for the onset of synchronization in the Kuramoto model. A lower bound on the coupling gain $K = K_{inv}$ was developed which is sufficient for oscillator synchronization within an arbitrary compact set of $(-\frac{\pi}{2}, \frac{\pi}{2})$, provided the oscillators phases are contained in that compact set at t=0. Finally it was shown that the oscillators synchronize exponentially. In [3], exponential convergence was only demonstrated for the case **when the natural frequencies ω_i are same for all oscillators**. In this paper we have extended this for the case when the natural frequencies may be different for all oscillators. Simulations were also presented to justify the proposed results. Future work involves extending this work to arbitrary switching topologies and for networks with time delays.

VIII. Acknowledgements

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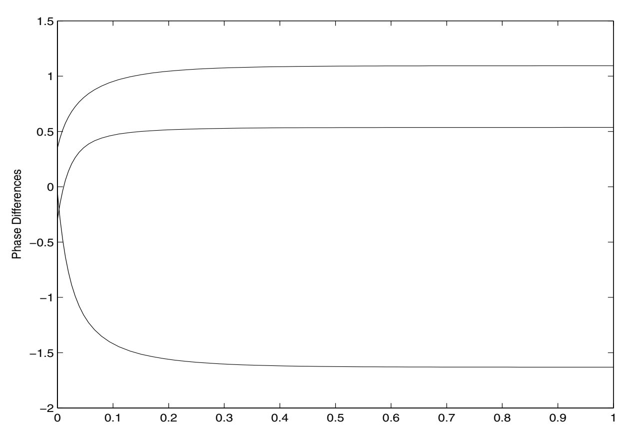

Fig. 2. The oscillators synchronize when the coupling gain $K_c < K < K_{inv}$.

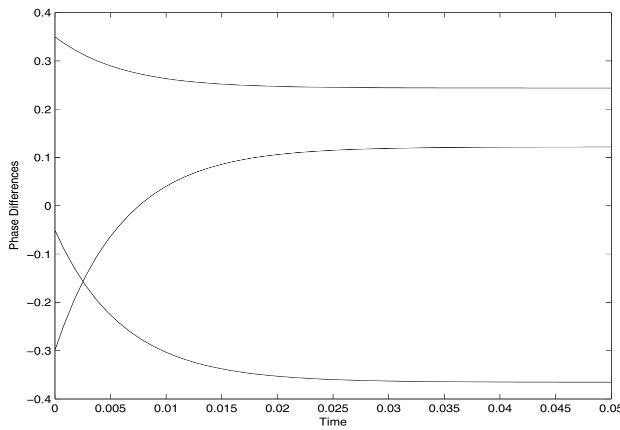

Fig. 3. The oscillators synchronize and are invariant with respect to $\mathcal{D}$.

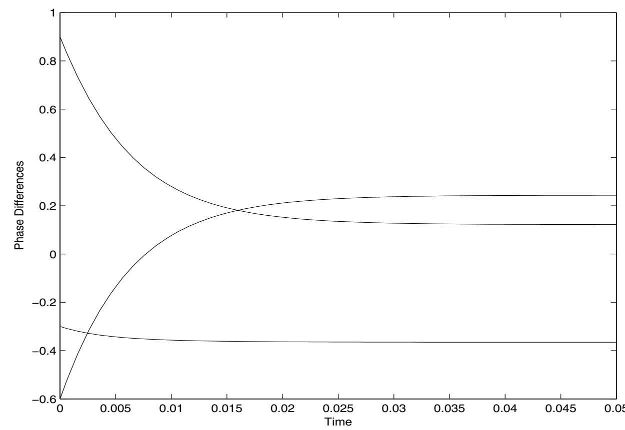

Fig. 5. The set $\mathcal{D}$ is able to attract the phase difference even when they start outside $\mathcal{D}$.

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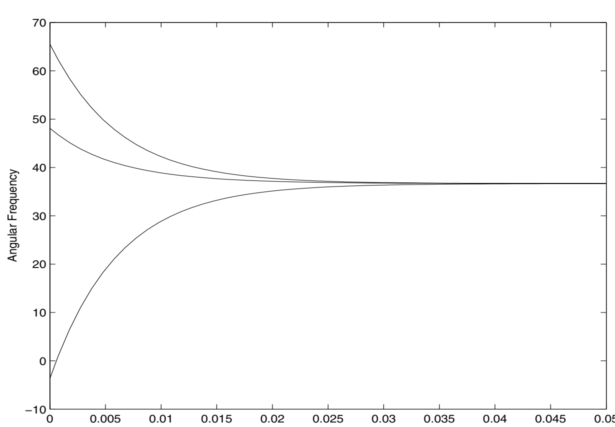

Fig. 4. The angular frequencies $\dot{\theta}_i \; i = 1, 2, 3$ of the oscillators exponentially converge to the mean frequency Ω.