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Asymptotic Properties of Extended Kalman Filters for a Class of Nonlinear Systems

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***Abstract*— We study the closed-loop behavior of the extended Kalman filter for a class of deterministic nonlinear systems that are transformable to the special normal form with linear internal dynamics. We argue that the closed-loop system is asymptotically stable and the estimation error exponentially converges to zero. We compare the performance of the extended Kalman filter to a high-gain observer through the use of numerical examples.**

I. INTRODUCTION

The extended Kalman filter has seen wide use in the areas of control and signal processing as a state estimator for nonlinear stochastic systems. For an introduction see [7]. In the noise free case, the EKF can be parameterized to function as a deterministic observer for nonlinear systems. A method for constructing deterministic observers as asymptotic limits of filters was studied in [2]. Additional work on the convergence properties of extended Kalman filters used as observers has been conducted in [5],[14],[15],[16]. Many of these results are local and often the convergence properties are shown under assumptions on the behavior of the state under control. Furthermore, analysis of the closed-loop system under EKF feedback has been limited. In [5] it was recognized that, for a particular parameterization, the EKF is a time-varying high-gain observer that asymptotically approaches a fixed gain observer as the gain is pushed higher. Furthermore, it was shown that the EKF is a global exponential observer for a class of nonlinear systems where the nonlinearities appear in a lower triangular form. This argument was based on a global Lipschitz property for the system nonlinearities. Here we attempt to study the closed-loop behavior of a class of systems under EKF feedback. Section II presents the main result. We relax the global Lipschitz condition and consider a class of systems transformable to the special normal form with linear internal dynamics. Based on a parameterization of the Riccati equation, the closed-loop system under EKF feedback is placed in the standard singularly perturbed form. We note that by relaxing the global Lipschitz condition, difficulties may arise as a result of the peaking phenomenon. In addition to globally bounding the control, the time-varying terms of the Riccati equation must be globally bounded in order to have a well defined solution. We argue that the origin of the closed-loop system is asymptotically stable under EKF output feedback. In section III we compare through simulation the use of the extended Kalman filter versus a fixed-gain high-gain observer.

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II. MAIN RESULT

Consider the system

$$\dot{z} = A_{11}z + B_1 x_1 \quad (1)$$

$$\dot{x} = Ax + B_2\phi(x,z,u) \quad (2)$$

$$y = Cx \quad (3)$$

where $x \in \mathbb{R}^r$ and $z \in \mathbb{R}^q$ are the states, u is the input, and y is the output. The function ϕ is assumed to be continuously differentiable and satisfies $\phi(0,0,0) = 0$. The $q \times q$ matrix A_{11} is Hurwitz. The $r \times r$ matrix A, the $r \times 1$ matrix B_2, and the $1 \times r$ matrix C are given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots 0 \end{bmatrix}$$

The internal dynamics (1) are driven by the output $y = x_1$. Given this structure, the system (1)-(2) is said to be in the special normal form [9]. Let $\chi = [z \ \ x]^T$ and rewrite (1)-(2) as

$$\dot{\chi} = f(\chi, u) \quad (4)$$

The extended Kalman filter for this system is given by

$$\dot{\hat{\chi}} = f(\hat{\chi}, u) + P(t)C_e^T R^{-1}(y - C_e\hat{\chi}) \quad (5)$$

$$\dot{P} = A_e P + P A_e^T + Q - P C_e^T R^{-1} C_e P \quad (6)$$

where R, Q, and $P(0)$ are positive definite symmetric matrices and

$$C_e = [0_{1\times q} \ \ C] \quad (7)$$

The matrix A_e takes the form

$$A_e = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

in which

$$A_{12} = [B_1 \quad 0 \quad \cdots \quad 0]_{q \times r}$$

$$A_{21} = B_2 \frac{\partial \phi}{\partial z}(\hat{z}, \hat{x}, u), \quad A_{22} = A + A_0$$

where

$$A_0 = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ d\phi_1 & d\phi_2 & \cdots & d\phi_r \end{bmatrix}, \quad d\phi_i = \frac{\partial \phi}{\partial x_i}(\hat{z}, \hat{x}, u) \tag{8}$$

We consider the following full state feedback controller

$$u = \gamma(z, x) \tag{9}$$

The closed-loop system under state feedback is given by

$$\dot{z} = A_{11} z + B_1 x_1 \tag{10}$$

$$\dot{x} = A x + B_2 \phi(x, z, \gamma(x, z)) \tag{11}$$

We state our assumptions.

Assumption 1:

1) The origin $(x = 0, z = 0)$ of (10)-(11) is globally asymptotically stable.
2) The function γ is locally Lipschitz in its arguments and globally bounded in x. Furthermore, $\gamma(0, 0) = 0$.

In addition we assume that the closed-loop system satisfies the following ISS property

Assumption 2: The system

$$\dot{z} = A_{11} z + B_1 x_1 \tag{12}$$

$$\dot{x} = A x + B_2 \phi(x, z, \gamma(x, z + v)) \tag{13}$$

with v viewed as the input, is input-to-state stable (ISS).

Assumption 3: The functions

$$\frac{\partial \phi}{\partial z}(\hat{z}, \hat{x}, u) \quad and \quad \frac{\partial \phi}{\partial x_i}(\hat{z}, \hat{x}, u)$$

for $i = 1, \cdots, r$ are globally bounded in $\hat{z}$ and $\hat{x}$.[1]

Assumption 3 ensures that the matrices A_{21}, A_{22} of the Riccati equation are bounded. We parameterize Q in the following way

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2 & \frac{1}{\varepsilon^2} D^{-1} Q_3 D^{-1} \end{bmatrix} \tag{14}$$

where Q_1 and Q_3 are chosen to be positive definite symmetric, $D = diag[1, \varepsilon, \cdots, \varepsilon^{r-1}]$, and $\varepsilon > 0$. We take R to be the identity matrix. The above parameterization produces a two-time scale behavior in the solution to the Riccati equation (6). We partition and scale P according to

$$P = \begin{bmatrix} P_1 & P_2 D^{-1} \\ D^{-1} P_2^T & \frac{1}{\varepsilon} D^{-1} P_3 D^{-1} \end{bmatrix}$$

[1] Global boundedness can always be achieved by saturating $\hat{x}$ and $\hat{z}$ outside a compact region of interest.

Then, the observer can be written as

$$\dot{\hat{z}} = A_{11} \hat{z} + B_1 \hat{x}_1 + P_2 D^{-1} C^T (y - C\hat{x}) \tag{15}$$

$$\dot{\hat{x}} = A\hat{x} + B_2 \phi(\hat{x}, \hat{z}, u) + \frac{1}{\varepsilon} D^{-1} P_3 D^{-1} C^T (y - C\hat{x}) \tag{16}$$

The gain $\frac{1}{\varepsilon} D^{-1} P_3 D^{-1} C^T$ has the structure of a high-gain observer ([1], [5]). This was exploited in [5], using a parameterization similar to the above, to show global exponential stability of the extended Kalman filter. For the fast estimation error we use the standard rescaling

$$\xi_i = \frac{x_i - \hat{x}_i}{\varepsilon^{r-i}} \tag{17}$$

for $i = 1, \cdots, r$. Thus, $x - \hat{x} = D_2 \xi$, where $D_2 = diag[\varepsilon^{r-1}, \varepsilon^{r-2}, \cdots, 1]$. Define the estimation error for the internal states by $\eta = z - \hat{z}$. The closed-loop system under output feedback can now be written in the standard singularly perturbed form

$$\dot{z} = A_{11} z + B_1 x_1 \tag{18}$$

$$\dot{x} = A x + B_2 \phi(x, z, \gamma(x - D_2 \xi, z - \eta)) \tag{19}$$

$$\dot{\eta} = A_{11} \eta + \varepsilon^{r-1} (B_1 - P_2 C^T) \xi_1 \tag{20}$$

$$\varepsilon \dot{\xi} = (A - P_3 C^T C) \xi + \varepsilon B_2 \delta(x, z, \eta, \xi) \tag{21}$$

$$\begin{aligned} \dot{P}_1 = {} & A_{11} P_1 + A_{12} P_2^T + P_1 A_{11}^T + P_2 A_{12}^T \\ & + Q_1 - P_2 C^T C P_2^T \end{aligned} \tag{22}$$

$$\begin{aligned} \varepsilon \dot{P}_2 = {} & \varepsilon A_{11} P_2 + A_{12} P_3 + \varepsilon P_1 A_{21}^T D + \varepsilon Q_2 D \\ & + P_2 (A + \varepsilon A_{0\varepsilon})^T - P_2 C^T C P_3 \end{aligned} \tag{23}$$

$$\begin{aligned} \varepsilon \dot{P}_3 = {} & \varepsilon^2 D A_{21} P_2 + (A + \varepsilon A_{0\varepsilon}) P_3 + Q_3 \\ & + P_3 (A + \varepsilon A_{0\varepsilon})^T + \varepsilon^2 P_2^T A_{21}^T D - P_3 C^T C P_3 \end{aligned} \tag{24}$$

where $\delta = \phi(x, z, u) - \phi(\hat{x}, \hat{z}, u)$ and

$$A_{0\varepsilon} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ \varepsilon^{r-1} d\phi_1 & \varepsilon^{r-2} d\phi_2 & \cdots & d\phi_r \end{bmatrix}$$

Note that $A_{12} D^{-1} = A_{12}$. Equations (18)-(20) and (22) characterize the "slow" dynamics and (21), (23), and (24) the "fast" ones. We have the following result.

Theorem 1: Consider the closed-loop system (18)-(24) under output feedback. Let Assumptions (1)-(3) hold and let $\mathcal{M}$ and $\mathcal{N}$ be any compact subsets of $\mathbb{R}^{q \times r \times q}$ and $\mathbb{R}^r$ respectively. Then, for trajectories $(z, x, \eta) \times \hat{x}$ starting in $\mathcal{M} \times \mathcal{N}$ the following holds:

- There exists ε^* such that, for all $0 < \varepsilon \leq \varepsilon^*$ the origin $(z = \eta = 0, x = \xi = 0)$ of the closed-loop system is asymptotically stable and $\mathcal{M} \times \mathcal{N}$ is a subset of its region of attraction.
- The origin of the estimation error equations (20)-(21) is exponentially stable.

Proof: The quasi steady-state equations of the fast dynamics are

$$0 = (A - \hat{P}_3 C^T C) \hat{\xi} \tag{25}$$

$$0 = A_{12} \hat{P}_3 + \hat{P}_2 A^T - \hat{P}_2 C^T C \hat{P}_3 \tag{26}$$

$$0 = A\hat{P}_3 + \hat{P}_3 A^T + Q_3 - \hat{P}_3 C^T C \hat{P}_3 \quad (27)$$

By standard results on algebraic Riccati equations [13], with Q_3 positive definite and (C, A) observable, (27) has a unique solution $P_3^+ = P_3^{+T} > 0$ such that $(A - P_3^+ C^T C)$ is a Hurwitz matrix. Hence, we have from (25) and (26) that $\hat{\xi} = 0$ and

$$P_2^+ = -A_{12} P_3^+ (A - P_3^+ C^T C)^{-T} \quad (28)$$

It is easy to show that

$$-A_{12} P_3^+ (A - P_3^+ C^T C)^{-T} = A_{12} \quad (29)$$

Therefore, $P_2^+ = A_{12}$. This results in the reduced system

$$\begin{aligned} \dot{z} &= A_{11} z + B_1 x_1 & (30)\\ \dot{x} &= Ax + B_2 \phi(x, z, \gamma(x, z - \eta)) & (31)\\ \dot{\eta} &= A_{11} \eta & (32)\\ \dot{P}_1 &= A_{11} P_1 + P_1 A_{11}^T + Q_1 + B_1 B_1^T & (33) \end{aligned}$$

where $Q_1 + B_1 B_1^T$ is positive definite and symmetric. Therefore, $P_1(t)$, with $P_1(0) = P_1^T(0) > 0$ is a bounded positive definite symmetric solution to (33) for all $t \geq 0$ and approaches $P_1^+ = P_1^{+T} > 0$ as $t \to \infty$.

We proceed by studying the Riccati equation (22)-(24) alone. To do so, we will treat A_{21} and A_{22} as bounded time-varying matrices and use singular perturbation theory [12] to argue that the solutions of (22)-(24) are bounded. We begin by viewing the following equations as a nominal model ($\varepsilon = 0$ on the right hand side) of (22)-(24)

$$\dot{\bar{P}}_1 = A_{11}\bar{P}_1 + \bar{P}_1 A_{11}^T + A_{12}\bar{P}_2^T + \bar{P}_2 A_{12}^T + Q_1 - \bar{P}_2 C^T C \bar{P}_2^T \quad (34)$$

$$\varepsilon \dot{\bar{P}}_2 = \bar{P}_2 (A - \bar{P}_3 C^T C)^T + A_{12} \bar{P}_3 \quad (35)$$

$$\varepsilon \dot{\bar{P}}_3 = A\bar{P}_3 + \bar{P}_3 A^T + Q_3 - \bar{P}_3 C^T C \bar{P}_3 \quad (36)$$

Since Q_3 is positive definite, we can show [3] that

$$\|\bar{P}_3(t) - P_3^+\| \leq g_3 e^{(-\sigma_3 t/\varepsilon)} \quad (37)$$

for some positive constants g_3 and σ_3. Next, rewrite (35) as

$$\varepsilon \dot{\bar{P}}_2 = \bar{P}_2 (A - P_3^+ C^T C - (\bar{P}_3 - P_3^+) C^T C)^T + A_{12} \bar{P}_3 \quad (38)$$

we have

$$\|\bar{P}_2(t) - P_2^+\| \leq g_2 e^{(-\sigma_2 t/\varepsilon)} \quad (39)$$

for some positive constants g_2 and σ_2. From (34) we have that the driving terms are bounded and exponentially approach $Q_1 + B_1 B_1^T$. We can show that

$$\|\bar{P}_1(t) - P_1^+\| \leq g_1 e^{(-\sigma_1 t/\varepsilon)} \quad (40)$$

for some positive constants g_1 and σ_1. Hence, we have that each $\bar{P}_i$ is bounded for all $t \geq 0$.

Now let $\tilde{P}_i(t) := P_1(t) - \bar{P}_i$ for $i = 1, 2, 3$. We point out that $\tilde{P}_i(0) = 0$ and that the equations (22)-(24) are an ε perturbation of (34)-(36). Using this information we can show that each P_i is bounded uniformly in t and ε for all $t \geq 0$.[2] Next we argue that P_3 in (24) is positive definite. Let $\tau = t/\varepsilon$ and rewrite (24) as

$$\frac{dP_3}{d\tau} = A_2 P_3 + P_3 A_2^T + Q_3 + P_3 C^T C P_3 + \varepsilon \psi(P_i) \quad (41)$$

where ψ contains depends continuously on P_2 and P_3, $Q_3 + P_3 C^T C P_3$ is positive definite, and

$$A_2 = A - P_3^+ C^T C - (P_3 - P_3^+) C^T C$$

We note that A_2 is bounded by some constant L for all $t \geq 0$. It can be shown that the corresponding state transition matrix satisfies the lower bound

$$\|\Phi(\tau, t_0) x\| \geq \|x\| e^{-2L(\tau - t_0)}$$

Following analysis similar to Theorem 4.12 of [10] we can show that P_3 is positive definite uniformly in t and ε. From these arguments we arrive at

$$\kappa_1 I \leq P_3(t) \leq \kappa_2 I \quad (42)$$

It can be seen that

$$S_3 = P_3^{-1}$$

satisfies

$$\begin{aligned} \varepsilon \dot{S}_3 = & -(A + \varepsilon A_{0\varepsilon})^T S_3 - S_3 (A + \varepsilon A_{0\varepsilon}) + C^T C \quad (43)\\ & - S_3 Q_3 S_3 - \varepsilon^2 S_3 D A_{21} P_2 S_3 - \varepsilon^2 S_3 P_2^T A_{21}^T S_3 \end{aligned}$$

And by the argument above S_3 will have a bounded, positive definite, symmetric solution for all $t \geq 0$. Thus, we have

$$\kappa_3 I \leq S_3(t) \leq \kappa_4 I \quad (44)$$

for all $t \geq 0$ where the κ_i's are positive constants, independent of ε.

Next we argue for the boundedness and ultimate boundedness of $(z, x, \hat{z}, \hat{x})$. Denote the right hand side of (18)-(20) as

$$\begin{aligned} \dot{\chi} &= F(\chi, \eta, \xi) & (45)\\ \dot{\eta} &= A_{11}\eta + \varepsilon^{r-1}(B_1 - P_2 C^T)\xi_1 & (46) \end{aligned}$$

With $\xi = 0$ we have from Assumptions 1 and 2 that (45)-(46) has a globally asymptotically stable equilibrium at the origin. Thus, there exists a positive definite radially unbounded function $V_1(\chi, \eta)$ and a positive definite function $U(\chi, \eta)$ such that

$$\frac{\partial V_1}{\partial \chi} F(\chi, \eta, 0) + \frac{\partial V_1}{\partial \eta} A_{11} \eta \leq -U(\chi, \eta) \quad (47)$$

for all χ and η. Let $\mathcal{M}$ be any compact subset of $\mathbb{R}^{q \times r \times q}$. Choose c such that $\mathcal{M} \subset \Omega_c = \{V_1(\chi, \eta) \leq c\} \subset \mathbb{R}^{q \times r \times q}$. Due to the global boundedness of f and δ in $\hat{x}$, for all $(\chi, \eta) \in \Omega_c$ and $\xi \in \mathbb{R}^r$, we have

$$\|F(\chi, \eta, \xi)\| \leq k_1 \quad , \quad \|\delta(x, z, \eta, \xi)\| \leq k_2 \quad (48)$$

[2] A detailed argument of this statement will be made in the full paper.

where k_1 and k_2 are positive constants independent of ε. Letting $W(\xi) = \xi^T S_3 \xi$ it can be shown that

$$\dot{V}_1 \leq -U(\chi, \eta) + k_4 \|\xi\| \tag{49}$$

$$\dot{W} \leq -\frac{k_3}{\varepsilon}\|\xi\|^2 + 2\kappa_4\|\xi\| k_2 \tag{50}$$

for all $(\chi, \eta, \xi) \in \Omega_c \times \{W(\xi) \leq \rho\varepsilon^2\}$, for some positive constants k_3,k_4, $\rho > 0$ independent of ε. Following analysis similar to [1] we have that for ε sufficiently small, (χ, η, ξ) enters the invariant set $\Omega_c \times \{W(\xi) \leq \rho\varepsilon^2\}$ during a finite time period $[0, T(\varepsilon)]$ where $T(\varepsilon) \to 0$ as $\varepsilon \to 0$. During $[0, T(\varepsilon)]$, ξ will be bounded by an $O(1/\varepsilon^{r-1})$ value. Also, from (48), $T(\varepsilon)$ can be made small enough that (χ, η) will remain in Ω_c for all $t \in [0, T(\varepsilon)]$. Consequently, all closed-loop trajectories are bounded for all $t \geq 0$. Furthermore, it can be shown that given any $\mu > 0$ there exists $\varepsilon_1^* > 0$ and $T_1(\mu)$ such that for all $0 < \varepsilon < \varepsilon_1^*$, $\|(\chi, \eta)\| + \|\xi\| \leq \mu$ for all $t \geq T_1$.

From the ultimate boundedness of $(z, x, \hat{z}, \hat{x})$ we can work locally to argue asymptotic stability of the closed-loop system. First, define $\zeta = [\eta \quad \xi]^T$. We have that $\|\delta(x, z, \eta, \xi)\| \leq L_1\|\zeta\|$ for all $(\chi, \eta) \in B(0, \mu) \times \{\|\xi\| \leq \mu\}$ where μ is the ultimate bound from above. Let

$$V_2 = \zeta^T \begin{bmatrix} P_L & 0 \\ 0 & S_3 \end{bmatrix} \zeta \tag{51}$$

be a Lyapunov function candidate for ζ where P_L satisfies $P_L A_{11} + A_{11}^T P_L = -I$. Due to the boundedness of A_{12} and A_{21} in $\hat{x}$ and $\hat{z}$, the boundedness of P_2, S_3, and the fact that $\varepsilon A_{0\varepsilon}$ is bounded and $O(\varepsilon)$, it can be shown that there exits ε_2^* sufficiently small such that

$$\dot{V}_2 \leq -k_5 V_2 \tag{52}$$

where k_5 is a positive constant. Hence, for all $(\chi(0), \eta(0), \xi(0))$ starting in $\mathcal{M} \times \mathcal{N}$ the estimation error, (η, ξ), converges exponentially. Finally, asymptotic stability of the closed-loop system follows from the following composite Lyapunov function

$$V(\chi, \eta, \xi) = \theta V_1(\chi, \eta) + (V_2(\zeta))^{1/2} \tag{53}$$

with $\theta > 0$. We have

$$\dot{V} \leq -\theta U(\chi, \eta) + \theta L_2 L_3\|\zeta\| - k_6\|\zeta\| \tag{54}$$

for some positive constant k_6, a Lipschitz constant L_2, and L_3, an upper bound on $[\partial V_1/\partial\chi \;\;,\;\; \partial V_1/\partial\eta]$ in Ω_c. Taking $\theta \leq k_6/2L_2L_3$ yields asymptotic stability. ■

III. Comparison

The stability results for locally Lipschitz nonlinear systems in the previous section came at the expense of sacrificing global results for semiglobal ones. An essential factor in this sacrifice is the effect of peaking on the closed-loop system. In high-gain observers, peaking is caused by the special structure of the observer gain

$$H^T = \begin{bmatrix} \frac{\alpha_1}{\varepsilon} & \frac{\alpha_2}{\varepsilon^2} & \cdots & \frac{\alpha_n}{\varepsilon^r} \end{bmatrix} \tag{55}$$

For high-gain observers, peaking can be overcome by globally bounding the control outside a compact region of interest [6]. This can be done by using a saturation function on the controller. For the case of the extended Kalman filter, globally bounding the control alone is not enough. Peaking in the estimates may induce numerical difficulties in the solution to the RDE as the following example shows.

Example 1: Consider the following system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_2^a + u \tag{56}$$

and the feedback linearizing controller

$$u = -x_2^a - x_1 - 3x_2 - 3x_3 \tag{57}$$

where a will be chosen later on. By saturating the control outside a compact region of interest the effect of peaking can be overcome and the closed-loop system under (fixed-gain) high-gain observer feedback can recover the response under state feedback as $\varepsilon \to 0$. Let $a = 3$. Using the extended Kalman filter parameterized as in the previous section we have that the matrix $A + \varepsilon A_{0\varepsilon}$ in (24) is

$$A + \varepsilon A_{0\varepsilon} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3\varepsilon\hat{x}_2^2 \end{bmatrix} \tag{58}$$

where in this example $a = 3$. During any occurrence of peaking, the estimate $\hat{x}_2$ will become $O(1/\varepsilon)$. Therefore, from (58) with saturation only on the control, the RDE will contain unbounded terms as $\varepsilon \to 0$. This system was simulated for $\varepsilon = 0.01$ with $x_1(0) = 0.9$, $x_2(0) = x_3(0) = 0$, $\hat{x}_1(0) = \hat{x}_2(0) = \hat{x}_3(0) = 0$, $P(0) =$ Identity, $Q_3 = diag[3, 3, 1]$, and with the control saturated outside (-20,20). Figure 1 illustrates the effect this has on the system response. The peaking in $\hat{x}_2$ induces a very large gain (from the solution to the RDE) and this gain in turn exacerbates the peaking in the estimate. Figure 1 shows that the saturation of the control prevents the system states from deviating too much from their initial values, but the estimate $\hat{x}_2$ and the gain $h_2(t)$ have become prohibitively large. These difficulties are overcome by saturating the estimates themselves outside a compact region of interest. This will globally bound the control and the time-varying terms in the RDE. This approach is shown in Figure 2 where we have saturated $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ outside (-2,2). Figure 2 shows that the estimate $\hat{x}_2$ saturates then quickly converges. Also, we see that the control remains bounded, the gain h_2 converges quickly to its steady-state value, and the output x_1 gracefully approaches the origin.

Like the HGO, the EKF achieves faster and more accurate reconstruction of the state x as $\varepsilon \to 0$. It can be shown from analysis similar to the above that the response for the EKF approaches the response for the HGO as $\varepsilon \to 0$. Here we illustrate this through a numerical example. Figure 3a plots the output $y = x_1$ of the closed-loop system for the EKF (solid) and the HGO (dashed) for $x_1(0) = 2$ and $\varepsilon = 0.1$ with all other parameters as above. Figure 3b shows that the two responses have converged for $\varepsilon = 0.001$.

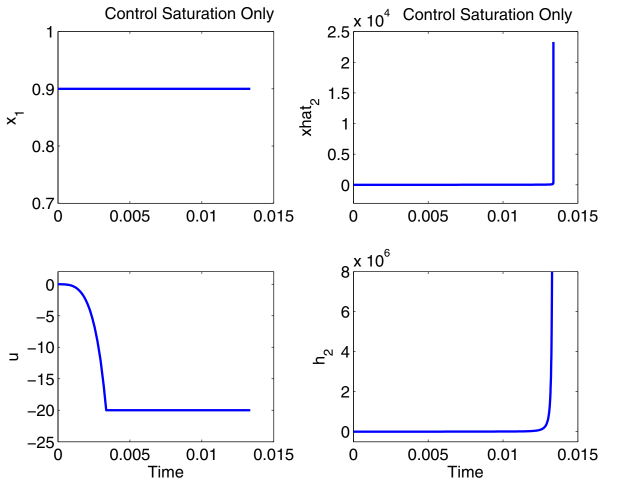


Fig. 1. Simulation for Example 1, showing the output x_1, the estimate $\hat{x}_2$, the control u, and the gain $h_2(t)$ for EKF feedback under control saturation only.

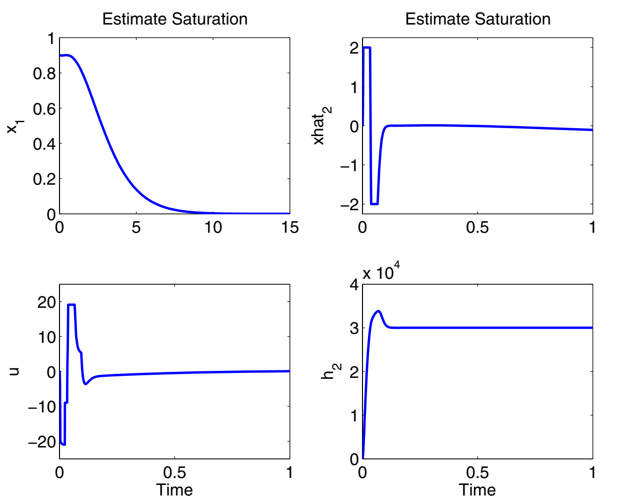
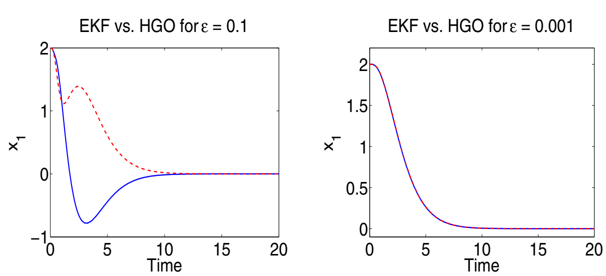


Fig. 2. Simulation for Example 1, showing the output x_1, the estimate $\hat{x}_2$, the control u, and the gain $h_2(t)$ for EKF feedback under estimate saturation.



(a) EKF(solid),HGO (dash)

(b) EKF(solid),HGO (dash)

Fig. 3. Simulation for Example 1, showing the output x_1, for (a) the EKF and HGO for $\varepsilon = 0.1$ and (b) the EKF and HGO for $\varepsilon = 0.001$.

Considering the foregoing observation we note that for relatively large values of ε, the time-varying terms in the Riccati equation will have more influence over the closed-loop response. Whether the added complexity of the time-varying gain gives an advantage over a time-invariant gain appears to be system dependent. We consider the system above (56) for two cases. First, let $a = 3$. This system was simulated for $\varepsilon = 0.3$, $x_1(0) = 2$. We choose an initial convariance $P(0)$ such that the EKF gain is equal to its steady-state value. The behavior of the estimate $\hat{x}_2$ will cause the gain to deviate from steady-state. For the HGO we used an observer with gain matrix

$$H^T = \begin{bmatrix} 3/\varepsilon & 3/\varepsilon^2 & 1/\varepsilon^3 \end{bmatrix}$$

which is the steady-state value of the EKF gain. Figure 4 shows the response of the output x_1 and the control signal u for the closed-loop system under EKF feedback (left) and HGO (right). The time-varying gain was able to stabilize the system where as the response under the HGO observer went unstable. We compare these observations with the second case where we now take $a = 2$ in (56) and (57). Conversely, the fixed-gain high-gain observer was able to achieve stability and the time-varying observer went unstable. This is a result of the sensitivity of the Riccati equation to the transient response of the estimate $\hat{x}_2$ as illustrated in Figure 5. We emphasize that both the EKF and the HGO can stabilize each system $(a = 2, 3)$ by making ε small enough (e.g. $\varepsilon = 0.01$). But, these examples indicate that advantages to using a time-varying high-gain observer versus a fixed gain high-gain observer at steady state are, at least, system dependent.

Finally we remark that for the EKF, the initial condition of the Riccati equation can be chosen to eliminate peaking during the initial transient. Initialization strategies to overcome peaking have been explored for observers with time-varying gains in [4],[8], and for sampled data output feedback control in [11]. However, it has been pointed out [4],[11] that these designs may suffer from peaking through impulsive-like disturbances that occur after the initial transient. Therefore, the peaking phenomenon is relevant irrespective of the initial gain choice.

IV. Conclusions

We have examined the closed-loop behavior of a class of deterministic nonlinear systems with locally Lipschitz nonlinearity under extended Kalman filter feedback. We have shown that the origin of the closed-loop system is asymptotically stable and the origin of the estimation error is exponentially stable. We have seen that in addition to globally bounding the control, the time-varying functions in the Riccati equation must be globally bounded for the Riccati equation to have a well defined solution. Through simulation we have compared the closed-loop performance of the time-varying EKF versus the time-invariant HGO. Efforts to expand this result beyond the special normal form are complicated by a difficulty in rescaling the partitioned

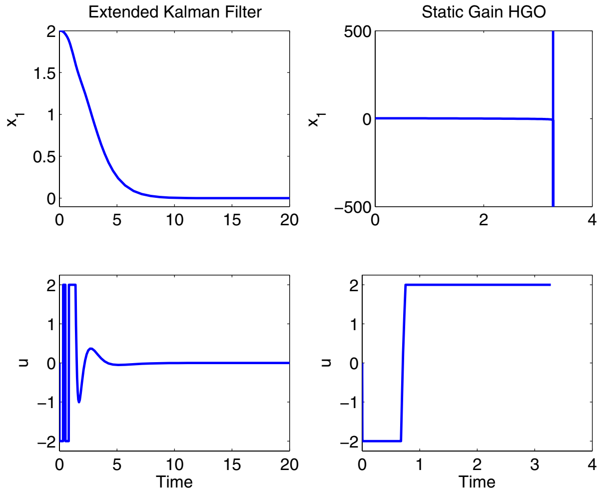


Fig. 4. Simulation for Example 1, showing the output x_1, and the control u for EKF feedback and HGO feedback for $a = 3$.

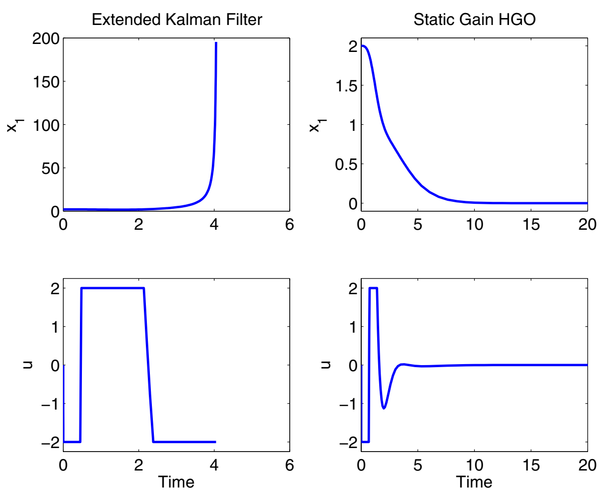


Fig. 5. Simulation for Example 1, showing the output x_1, and the control u for EKF feedback and HGO feedback for $a = 2$

Riccati equation such that it will be in the standard singularly perturbed form.

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