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On a topological condition for strongly asymptotically stable differential inclusions

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Abstract— **In this paper, we will show that a system on a non-contractible manifold cannot be strongly asymptotically stabilized in Filippov's sense, even if discontinuous feedback is used. The fact is well-known for C^1 feedback case, and we extend it to the discontinuous feedback case. To consider the stabilization problem on a non-contractible manifold, the assumption convexity or upper semicontinuity is restrictive. We will propose a new type of differential inclusion without upper semicontinuity by defining a function indicating a rate of leaving from discontinuous set. By adopting the new differential inclusion, stabilization problems on non-contractible manifolds become possible for many cases.**

I. INTRODUCTION

The smooth system on a non-contractible manifold is not globally asymptotically stable. This fact means the control system on a non-contractible manifold is not C^1-stabilizable. Many researchers have noticed this fact. For example, Sontag wrote this theorem and its proof in his book[1]. Byrnes and Isidori[2] also mentioned this fact briefly. The proof of the theorem is performed by using a continuity of the one-parameter group of transformation $\varphi_t(x)$. We can guess that Hopf might know this fact, because in the proof of Poincaré-Hopf index theory a similar procedure as written above has appeared.

On the other hand, it has been expected that this theorem stands for discontinuous system cases also, because there exist Filippov solutions[3] staying on the set of discontinuous points. However, the one-parameter group of transformation describing the flow does not exist for this case, so a similar method of the smooth case is not applicable. In this paper, we will prove the theorem for a discontinuous case by constructing a Lyapunov function by the method of Clarke et al.[4]. We will show that the sublevelsets of the Lyapunov function are homeomorphic to a closed ball, which leads the main theorem.

The main theorem show that the assumption convexity or upper semicontinuity of the differential inclusion is restrictive for the stabilization problem on a non-contractible manifold. We will show a new type of differential inclusion without upper semicontinuity. By defining a function indicating a rate of leaving from the set of discontinuous points, we can decide the existence of a solution outgoing the discontinuous points. Under the new definition of differential inclusion, there exists a solution of the initial value problem for the positive direction of time.

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II. DISCONTINUOUS SYSTEM AND DIFFERENTIAL INCLUSIONS

In this paper, we consider a system

$$\dot{x} = f(x), \qquad x \in M, \tag{1}$$

where M is an n-dimensional C^∞-differential manifold that is regular and second countable. From Urysohn's metrization theorem, the manifold M is metrizable, i.e. there exists a distance function $d(x_1, x_2)$ such that $d(x_1, x_2) \geq 0$, $d(x_1, x_2) = d(x_2, x_1)$, $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$, and $d(x_1, x_2) + d(x_2, x_2) \geq d(x_1, x_3)$. We do not assume that the vector field $f(x)$ is continuous. The existence and uniqueness of the solution of (1) are not guaranteed without continuity of $f(x)$. Usually, a differential inclusion

$$\dot{x} \in F(x), \qquad x \in M \tag{2}$$

is considered instead of the differential equation (1) in such cases. A function $x(t)$ ($[t_1, t_2) \to M$) is a solution of (2), if (2) is satisfied for almost everywhere in $[t_1, t_2)$. In this paper, the followings are assumed for the multifunction F:

(H1) $F(x)$ is a nonempty compact convex subset in $\Re^n$ for every $x \in M$.

(H2) The multifunction $F(x)$ is upper semicontinuous, i.e., given $x \in M$, for $^\forall\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, x') < \delta \Rightarrow F(x') \subseteq F(x) + \epsilon B, \tag{3}$$

where B denotes the open unit ball.

It is well known that under the assumptions **(H1)** and **(H2)** a solution of (2) exists locally[5], that is, for every $x_0 \in M$ there exists a solution of (2) such that $x(0) = x_0$ in $[0, T)$ for some $T > 0$. Note that uniqueness of the solution does not guaranteed even if **(H1)** and **(H2)** hold.

As a manner to generate the multifunction $F(\cdot)$ from the vector field $f(\cdot)$, Filippov's differential inclusion[3]

$$\dot{x} \in F(x) = \bigcap_{\delta > 0} \bigcap_{\mathrm{meas}(\mathcal{N}) = 0} \overline{\mathrm{co}} f(x + \delta B \backslash \mathcal{N}) \tag{4}$$

is often used. If the vector field $f(\cdot)$ is bounded on bounded sets and is measurable, the assumption **(H1)** and **(H2)** are satisfied, and a Filippov solution of (1) exists locally.

We assume that $0 \in F(0)$, where the origin $x = 0$ is a point on M. This means that the origin is a stationary point of (2). In this paper, stability of a system means a global stability of the origin. To precisely state the stability of the system, norm of the state $|x|$ must be defined as

$$|x| = d(x, 0). \tag{5}$$

We define strong asymptotical stability of the differential inclusion (2) as follows:

Definition 1 (Clarke et al.): The differential inclusion (2) is strongly asymptotically stable iff no solution has finite-escape time and the following hold:

(a) *Uniform Attraction.* For any $r > 0$, $R > 0$, there exists $T = T(r, R)$ such that for any solution $x(\cdot)$ of (2) with $|x(0)| \leq R$, one has

$$|x(t)| \leq r, \qquad {}^{\forall}t \geq T. \tag{6}$$

(b) *Uniform Boundedness.* There is a continuous nonincreasing function m: $(0, \infty) \to (0, \infty)$ such that for any solution $x(\cdot)$ of (2) with $|x(0)| \leq R$ one has

$$|x(t)| \leq m(R), \qquad {}^{\forall}t \geq 0. \tag{7}$$

(c) *Lyapunov Stability.*

$$\lim_{R \downarrow 0} m(R) = 0. \tag{8}$$

□

This definition implies the classical Lyapunov stability and the attractiveness $x(t) \to 0$ $(t \to \infty)$. For the strong asymptotical stable system, all solution of the system should tend to the origin as $t \to \infty$.

The purpose of this paper is to clarify the topological properties of M for a strongly asymptotically stable differential inclusion.

III. MAIN THEOREM

For the smooth vector field $f(\cdot)$, the following theorem is known:

Theorem 2: Suppose that $f(\cdot)$ is a smooth vector field. Then, the domain of attraction of the origin is contractible. □

A topological space X such that the identity map on X is homotopic to a constant map is called contractible, i.e. a contractible space can deform continuously to one point. The precise proof of Theorem 2 is written in Sontag[1], for example. Roughly saying, the one-parameter group of transformation $\varphi_t(x)$, describing the flow of (1), maps a point of the domain of attraction to neighborhood of the origin along the flow, which means the domain of attraction can be deformed into a point. On the other hand, for a discontinuous vector field, a one-parameter group of transformation may not exist, because the solution of the differential inclusion is not unique, and the multivalued semiflow $x(t) = \varphi_t(x(0))$ may not have some continuous properties.

Theorem 2 immediately derives the following theorem:

Theorem 3: Suppose that $f(\cdot)$ is a smooth vector field, and that the origin of (1) is globally asymptotically stable. Then, the manifold M is contractible. □

A contractible open manifold does not mean that the manifold is homeomorphic to Euculidian space $\Re^n$, though a Euclidian space is always contractible. One of the counterexample is 'Whitehead link[6].' The theorem can be restated for control systems as the following corollary:

Corollary 4 (Sontag[1]): Consider the system

$$\dot{x} = f(x, u), \quad x \in M \tag{9}$$

with an input u, where $f(\cdot)$ is smooth. If the manifold M is not contractible, the system is not C^1 globally asymptotically stabilizable. □

The above corollary can be extended to locally Lipschtz case.

In this paper, we obtain the differential-inclusion version of the theorem 3.

Theorem 5 (Main theorem): Suppose that the differential inclusion (2) satisfies the assumptions **(H1)** and **(H2)**, and that (2) is strongly asymptotically stable. Then, the manifold M is contractible. □

This theorem shows that it is impossible to stabilize the system (9) in Filippov's sense when M is not contractible, even if discontinuous feedback is used.

Example 6: Consider the system:

$$\dot{\theta} = u, \tag{10}$$

where u is an input and θ is a state on a circle S_1. To stabilize the system locally, one can use a smooth feedback $u = -\sin\theta$. Under this smooth feedback, its origin is locally asymptotically stable, but the system is not globally asymptotically stable because of the existence of an unstable equilibrium at $\theta = \pm\pi$. To avoid this difficulty, a discontinuous feedback, for example

$$u = -\theta, \qquad \theta \in (-\pi, \pi], \tag{11}$$

is useful. The feedback (11) is discontinuous at $\theta = \pm\pi$. Such a discontinuous feedback works well practically. However, the closed system is not strongly asymptotically stable in Filippov's sense, because Filippov's definition admits a parasitic solution staying at $\theta = \pm\pi$. □

In the next section, we will prove the main theorem 5.

IV. PROOF OF MAIN THEOREM

In the discontinuous cases, the flow $\varphi_t(x)$ does not exist in general. Hence, to prove the main theorem (Theorem 5), the topological structure must be generated by another method without the flow. In this paper, we use a Lyapunov function of the differential inclusion.

Definition 7: A pair of continuous functions (V, W) on M where V is C^∞-function on M and W is C^∞-function on $M \backslash \{0\}$ is a C^∞-smooth strong Lyapunov pair for the differential inclusion (2), if the following conditions are satisfied[4]:

(L1) *Positive Definiteness.* $V(x) > 0$ and $W(x) > 0$ for all $x \neq 0$. In addition, $V(0) = 0$.

(L2) *Properness.* The sublevelsets

$$\{x \in M \,:\, V(x) \leq a\} \tag{12}$$

are bounded for every $a \geq 0$.

(L3) *Strong Infinitesimal Decrease.* For nonzero x,

$$\max_{v \in F(x)} \langle dV(x), v \rangle \le -W(x) \tag{13}$$

is satisfied, where $dV(x) = \partial V / \partial x$. □

When M is a Euclidian space, a converse Lyapunov theorem has been given by Clarke at el.[4]:

Theorem 8 (Clarke at el.[4]): Suppose that M is a Euclidian space, and let the multifunction $F(\cdot)$ satisfy hypotheses **(H1)** and **(H2)**. Then the differential inclusion (2) is strongly asymptotically stable iff there exists a C^∞-smooth strong Lyapunov pair (V, W). □

This Theorem shows that a *smooth* Lyapunov function exists even if the system is discontinuous.

By tracing the proof by Clarke at el., one can find that the above theorem holds without the assumption that M is a Euclidian space.

Theorem 8': Let the multifunction $F(\cdot)$ satisfy hypotheses **(H1)** and **(H2)**. Then the differential inclusion (2) is strongly asymptotically stable iff there exists a C^∞-smooth strong Lyapunov pair (V, W). □

We will show that the sublevelsets $\{x : V(x) \le a\}$ $(a > 0)$ are homeomorphic to $\overline{B_n}$, where $\overline{B_n}$ is the n-dimensional closed unit ball. First of all, it will be proven that a sublevelset $\{x : V(x) \le \epsilon\}$ is homeomorphic to $\overline{B_n}$, where the positive constant ϵ is small enough. If the Lyapunov function is a Morse function, there exists a local coordinate $(x_1, \ldots, x_n)^T$ near the origin such that

$$V(x) = x_1^2 + \cdots + x_n^2, \tag{14}$$

where $|x|$ is small enough. In this case, we obtain

$$\{x : V(x) \le \epsilon\} = \{x : x_1^2 + \cdots + x_n^2 \le \epsilon\} \tag{15}$$

for a small ϵ, which means the sublevelset is homeomorphic to $\overline{B_n}$. However, in general, the Lyapunov function $V(x)$ may not be a Morse function, and its Hessian matrix may degenerate at the origin.

Lemma 9: The sublevelset $\{x : V(x) \le \epsilon\}$ is homeomorphic to $\overline{B_n}$ for a small $\epsilon > 0$, even if the Hessian matrix of the Lyapunov function $V(x)$ degenerates at the origin. □

Proof: Because M is a differential manifold, there exists a local map $z = \Phi(x)$ that maps the neighborhood U of the origin to Y, where Y is an open subset in $\Re^n$. Without loss of generality, we assume $\Phi(0) = 0$. Let $\bar{V}(z)$ be $V(\Phi^{-1}(z))$. Obviously, $\bar{V}(0) = 0$, $\bar{V}(z) > 0$ $(z \in Y, z \ne 0)$. As $\bar{V}(z)$ is a positive definite function on a Euclidian space locally, there exists a small ϵ (> 0) such that $\{z : \bar{V}(z) \le \epsilon\}$ is homeomorphic to $\overline{B_n}$ and is included in Y. Therefore, $\{x : V(x) \le \epsilon\}$ is homeomorphic to $\overline{B_n}$ also for the same ϵ. See Milnor[7] also. ■

To show that all sublevelsets of the Lyapunov function are homeomorphic to $\overline{B_n}$, we use the following theorem:

Theorem 10: Let $\alpha(x)$ be a smooth function on a manifold M, and suppose $[a, b] \subset \alpha(M)$. Moreover, assume that there is no critical point of $\alpha(\cdot)$ in $\alpha^{-1}([a, b])$. Then, $\alpha^{-1}((-\infty, a])$ is homeomorphic to $\alpha^{-1}((-\infty, b])$. Furthermore, $\alpha(\cdot)$ is a deformation retract from $\alpha^{-1}(a)$ to $\alpha^{-1}(b)$. (See Fig. 1.) □

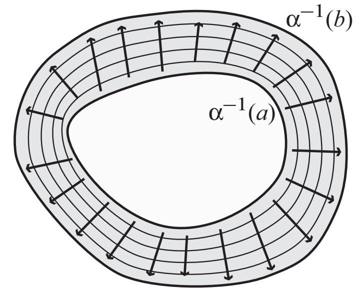


Fig. 1. Illustration of Theorem 10

Proof: See Milnor[8]. ■

We must show that there is no critical point of $V(x)$ except the origin.

Lemma 11: Let (V, W) be a C^∞-smooth strong Lyapunov pair for the strongly asymptotically stable differential inclusion (2). Then there is no critical point of $V(x)$ except the origin. □

Proof: Assume that there is a critical point x_0 of $V(x)$ such that $x_0 \ne 0$, i.e. $dV(x_0) = 0$. Clearly

$$\langle dV(x_0), v \rangle = 0 \tag{16}$$

for any v. However, from the definition of the Lyapunov pair **(L3)**,

$$\max_{v \in F(x_0)} \langle dV(x_0), v \rangle \le -W(x_0) < 0. \tag{17}$$

It contradicts (16), so there is no critical point of $V(x)$ except the origin. ■

By using Lemma 9, Theorem 10, Lemma 11, and the properness of $V(x)$, we can show that all sublevelsets $\{x : V(x) \le a\}$ $(a > 0)$ are compact and homeomorphic to a closed ball $\overline{B_n}$. Moreover, from Theorem 10, Lyapunov function $V(x)$ is a deformation retract from $\{x : V(x) = a\}$ for all $a > 0$ to the origin. This fact shows that the manifold M is contractible. Hence, the main theorem 5 has been proven.

We have made several assumptions for the manifold M, so the following stronger result than the smooth case (Theorem 2) can also derived:

Theorem 12: If the differential inclusion (2) satisfies the assumptions **(H1)** and **(H2)**, and that (2) is strongly asymptotically stable, the manifold M is homeomorphic to Euclidean space. □

Proof: Let U_a denote open set $\{x : V(x) < a\}$. Since the sublevelsets $\{x : V(x) \le a\}$ are compact and homeomorphic to a closed ball, all sets U_a $(a > 0)$ are homeomorphic to a open ball B_n. Thus, we can obtain a sequence of open sets that are homeomorphic to a open ball B_n as follows:

$$U_1 \subset U_2 \subset U_3 \subset \cdots, \tag{18}$$

where U_1, U_2,... are σ-covering of the metrizable manifold M. By using the result of Brown[9], it can be shown that M is homeomorphic to a Euclidean space. ■

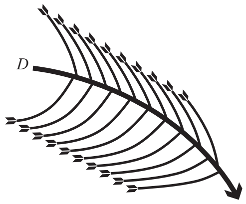


Fig. 2. Trajectories sliding on D (the case of $R(x) \leq 0$).

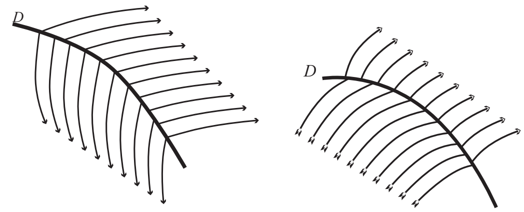
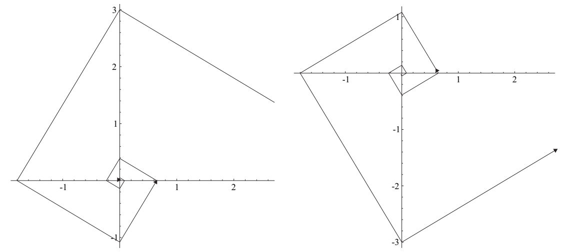


Fig. 3. Trajectories outging from the set of discontinuous points.



(a) **(b)**

Fig. 4. Zeno trajectories: **(a)** stable case and **(b)** unstable case.

V. DIFFERENTIAL INCLUSIONS WITHOUT UPPER SEMICONTINUITY/CONVEXITY

The main theorem suggests that when the manifold M is not contractible, the system (9) is not strongly asymptotically stabilizable in Filippov's sense, even if discontinuous feedback is used. However, in Example 6, the closed loop system

$$\dot{\theta} = -\theta, \quad \theta \in (-\pi, \pi] \tag{19}$$

has a unique solution in the classical sense to positive direction of time, and the solution tends to the origin as $t \to \infty$. In this case, the solution staying at $\theta = \pm\pi$ does not exist practically.

Filippov's multifunction $F(x)$ has a convex set for each x, which generates a practically nonexistent solution. When the manifold M is not contractible, the assumption of convexity or upper semicontinuity is too restrictive to consider the stabilizing control. To avoid such a problem, the concept of Euler solution[10] is often used, which is a definition of the solution of a discontinuous differential equation without differential inclusion. However, when a parasitic solution exists, one can choose the value of $f(x)$ such that the Euler solution remains on the set of the discontinuous point, even if a solution of the differential inclusion goes out of the set. Actually, the choice of $f(x)$ avoiding the parasitic solution is not obtained explicitly. Moreover, in the continuous and non-Lipschtz case, a Caratheodory's solution may not be an Euler solution. In addition to these, it is complicate to check stability using Euler solution, while the stability of the differential inclusion can be verified by Lyapunov function. Therefore, in this paper we persist in using differential inclusion. For disallowing parasitic solutions, we must remove vectors keeping the state on the set of discontinuous points, from the multifunction. Such a modification of the differential inclusion violates the assumption **(H1)** or **(H2)**. However, when all solutions in Filippov's sense stay on the set of discontinuous points (See Fig. 2), removing the vector deprives the diffrential inclusion of the existence of the solution. So, we must distinguish the cases of Fig. 2 from the cases of Fig. 3.

In this section, we propose a method to construct a differential inclusion without upper semicontinuity or convexity.

The following example shows the difficulty:

Example 13: Consider the following two systems

$$\Sigma_{Z1}: \begin{cases} \dot{x}_1 = -\mathrm{sgn}(x_1) - 4\mathrm{sgn}(x_2) \\ \dot{x}_2 = 4\mathrm{sgn}(x_1) - \mathrm{sgn}(x_2), \end{cases} \tag{20}$$

$$\Sigma_{Z2}: \begin{cases} \dot{x}_1 = \mathrm{sgn}(x_1) - 4\mathrm{sgn}(x_2) \\ \dot{x}_2 = 4\mathrm{sgn}(x_1) + \mathrm{sgn}(x_2), \end{cases} \tag{21}$$

where Σ_{Z1} is stable, and Σ_{Z2} is unstable (See Fig. 4). In Σ_{Z1}, the state converges to the origin in finite time. For the positive time direction, the Filippov solution of Σ_{Z1} starting from the origin is unique and stays on the origin. Hence, the differential inclusion for Σ_{Z1} must include zero at the origin to assure the existence of the solution for the positive time direction. On the other hand, in the system Σ_{Z2}, for any $z \in \Re^2$ there exists a solution reaching z from the origin in finite time. One may want to remove the solution staying at the origin, and may remove zero from the differential inclusion at the origin. This modification of the differential inclusion does not violate the existence of the solution for the positive time direction. However, it is not easy to distinct these two cases from the local information of vector fields. For example, in the area $\{x : x_1 > 0, x_2 > 0\}$, $\dot{x}_1 < 0$ and $\dot{x}_2 > 0$ in both cases. In this example, checking the local stability is necessary for the determination of the differential inclusion, which requires us to trace all trajectories. □

Let $\dot{x} \in F_f(x)$ be the Filippov's differential inclusion derived from the differential equation (1). To eliminate kinks of $f(x)$, we use $F_f(x)$ as the basis of the desired differential inclusion rather than $f(x)$. Let $\Phi(x_0)$ denotes the set of the Filippov solution starting from x_0. We define a set of discontinuous points of (1) as follows:

Definition 14: Let S be a set of points such that $F_f(x)$ contains only one value. The set D is defined as a complementary set of S. □

If D has a constant dimension near a point $x \in D$, x is called as a regular point of D.

To construct the differential inclusion eliminating parasitic solutions, tracing trajectories from x is necessary. Let $\Phi_R(x_0)$ be the set of the solutions starting from x_0 such that leave from D, i.e.:

$$\begin{aligned}\Phi_R(x_0) = \{&x(\cdot) \,:\, x(\cdot) \in \Phi(x_0), \\ &x(t) \in S, \text{ for almost all } t \in [0, \epsilon], {}^{\exists}\epsilon > 0\}.\end{aligned} \tag{22}$$

If $\Phi_R(x_0)$ is an empty set, each solution remains on D locally. We propose a new non-convex differential inclusion by using $\Phi_R(x_0)$. The multivalued function

$$F_t(x_0) = \begin{cases} \bigcap_{\epsilon>0} \overline{\left\{ \dfrac{x(t) - x_0}{t} \,:\, t \in (0, \epsilon], x(\cdot) \in \Phi_R(x_0) \right\}} \\ \qquad\qquad\qquad\qquad (\Phi_R(x_0) \neq \emptyset) \\ F_f(x_0) \qquad\qquad\quad (\Phi_R(x_0) = \emptyset) \end{cases} \tag{23}$$

derives a differential inclusion $\dot{x} \in F_t(x)$ that may not satisfy the assumption **(H1)**. By definition, $x(t) \in \Phi_R(x_0)$ ($x_0 \in D$) is a solution of the new differential inclusion. Hence, the existence of the solution is guaranteed for the differential inclusion $\dot{x} \in F_t(x)$.

The definition of the differential inclusion based on the trajectories in Filippov's sense, i.e. one must obtain all Filippov's solution locally to determine the new multivalued function $F_t(x)$. Therefore, we propose another differential inclusion without solving Filippov's differential inclusion. The differential inclusion cannot remove parasite solutions perfectly, e.g. the case of Example 13, but can be constructed explicitly.

The rate leaving from D at $x \in D$ with the vector $F_f(x + u)$ is defined as

$$\gamma(x, u) = \langle \beta(x, u), F_f(x + u) \rangle, \quad x \in D, \tag{24}$$

where

$$\beta(x, u) = \begin{cases} \dfrac{1}{|u|} \left(u + x - \operatorname*{argmin}_{s \in D} |u + x - s| \right), & u \neq 0 \\ 0, & u = 0. \end{cases} \tag{25}$$

Note that $\beta(x, u) = 0$ if $x + u \in D$. Fig. 5 illustrates the definition of $\beta(\cdot)$ and $\gamma(\cdot)$.

At first glance, it seems that the existence of the solution leaving from D can be determined by checking the sign of

$$\zeta(x) = \lim_{\epsilon \downarrow 0} \max_{u \in \epsilon B} \gamma(x, u), \quad x \in D, \tag{26}$$

but it is incorrect. Fig. 6 shows the counter example. There exists no solution leaving from x_0, while $\zeta(x_0) > 0$. In this case, the trajectory from a point on D near x_0 enters D again in a very short time. To avoid such a case, we must improve this method.

Let U be an arbitrary neighborhood of $x \in D$. Because M is a manifold, there exists a local coordinate for a small U. Assume that there exists $L > 0$ such that

$$\left\| \frac{\partial F_f(x)}{\partial x} \right\| < L, \quad x \in U \cap S, \tag{27}$$

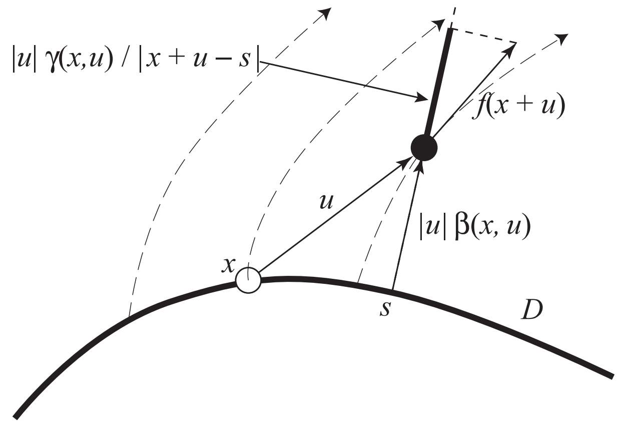


Fig. 5. Geometric understanding of $\beta(\cdot)$ and $R(\cdot)$.

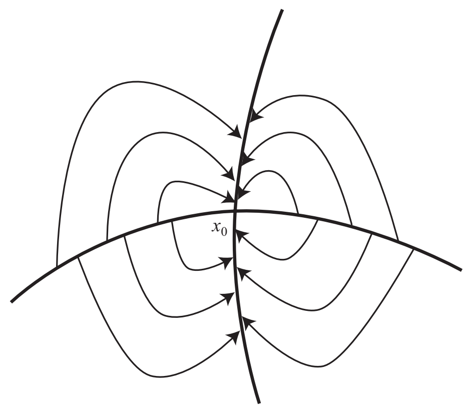


Fig. 6. Vector field such that $\zeta(x_0) > 0$ and no solution leaving from x_0 exists.

where $\| \cdot \|$ denotes the spectral radius of a matrix. This condition guarantees the direction of F_f does not change quickly near x.

Let $\Lambda(x + u)$ is a ray such that

$$\Lambda(x + u) = \{x + u + kF_f(x + u) \,:\, k \in \Re^+\}. \tag{28}$$

We define a set

$$Q(x) = \{u \,:\, \Lambda(x + u) \cap D \cap U = \emptyset\}. \tag{29}$$

For $u \notin Q(x)$, let $r \in \Lambda(x + u) \cap D \cap U$ is a nearest point to $x + u$.

As the value of the multivalued function at $x \in D$, we will use a value of $F_f(\cdot)$ near x, if there exists a trajectory that leaves D and remains in S for a finite period of time. Otherwise, $F_f(x)$ itself will be adopted. In order to discriminate these two cases, the following is defined:

$$\begin{aligned} R = \Bigg\{ &x \in D \,:\, \left(\epsilon' B \cap Q(x) \neq \emptyset, \quad 0 < {}^{\forall}\epsilon' < {}^{\exists}\epsilon_{max}\right) \\ &\text{or } \Bigg[\left(\epsilon' B \cap Q(x) = \emptyset, \quad 0 < {}^{\forall}\epsilon' < {}^{\exists}\epsilon_{max}\right) \\ &\quad \text{and } \left(\lim_{\epsilon \downarrow 0} \sup_{u \in \epsilon B} |r - x| > 0 \right) \Bigg] \Bigg\}, \end{aligned} \tag{30}$$

i.e. if $x \in R$, there exists a trajectory leaving D and remaining in S for a finite period of time. We have used $\Lambda(x+u)$ instead of the exact trajectory, which stands on the expansion of the solution in S:

$$\begin{aligned} x(t) = x + u + F_f(x+u)t + \frac{1}{2!}\frac{\partial F_f}{\partial x}(x(t'))F_f(x(t'))t^2, \\ 0 \le t' \le t. \end{aligned} \tag{31}$$

The second term of the expansion can be evaluated as

$$\begin{aligned} &\left| \frac{1}{2!}\frac{\partial F_f}{\partial x}(x(t'))F_f(x(t')) \right| t^2 \\ &\quad \le \frac{L}{2}\left(\max_{x\in U} |F_f(x)| \right) t^2 = O(t^2), \end{aligned} \tag{32}$$

where the coefficient of t^2 is bounded owing to the assumption (27). Therefore, using the Euler approximation $x(t) = x + u + F_f(x+u)t$, we can check whether the trajectory from $x+u$ enters D for a small t or not.

Finally, we can get the following differential inclusion:

$$\dot{x} \in F_d(x) = \begin{cases} F_f(x), & x \notin R \\ \lim_{\epsilon \downarrow 0} F_f(x + u_0(x,\epsilon)), & x \in R, \end{cases} \tag{33}$$

where

$$u_0(x,\epsilon) = \begin{cases} \underset{u \in \epsilon B \cap Q(x)}{\operatorname{argmax}} \ \gamma(x,u), & \epsilon B \cap Q(x) \neq \emptyset \\ \underset{u \in \epsilon B}{\operatorname{argmax}} \, |r - x|, & \text{otherwise.} \end{cases} \tag{34}$$

As the new definition of the multifunction $F_d(\cdot)$ does not satisfy the assumption **(H2)**, the existence of the solution of the new differential inclusion does not guaranteed. Actually, the backward (in time) solution may not exist for the new definition. However, a solution for the initial value problem always exists for the positive direction of time, because on the trajectory sliding on D the multifunction has a same set as the Filippov's multifunction. The new differential inclusion can remove parasitic solutions in many cases, but it cannot purge all parasitic solutions perfectly, e.g. the case of Example 13.

VI. Conclusions

In this paper, we have shown that a system on a non-contractible manifold cannot be strongly asymptotically stabilized in Filippov's sense, even if discontinuous feedback is used. To consider the stabilization problem on a non-contractible manifold, the assumption **(H1)** or **(H2)** is restrictive, so we have proposed new types of differential inclusions without upper semicontinuity or convexity.

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