Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Direct Adaptive Control of Multi-Input Plants with Magnitude Saturation Constraints

Mac Schwager and Anuradha M. Annaswamy

Abstract—**A direct-adaptive controller is proposed for multi-input plants with input magnitude saturation. It is proved that for a bounded region of initial states and parameter errors, bounded trajectories are ensured for an unstable plant. For open-loop stable plants, global stability is guaranteed regardless of initial conditions. The proposed controller is demonstrated in numerical simulation with an unstable second order, two input plant.**

I. Introduction

Control of systems with constrained inputs is a theoretically challenging problem and one of immense practical importance. Actuators subject to saturation and bandwidth constraints are ubiquitous in control applications. Actuator constraints can degrade system performance and potentially lead to instability if they are not accounted for in control system design. In particular, adaptive control techniques are especially susceptible to destabilization from saturating actuators since saturation errors can lead to parameter estimation errors, in turn leading to instability.

In [1]-[6], adaptive control approaches for plants in the presence of saturation have been developed. In [1], Karason and Annaswamy introduce a saturation compensation method for direct-adaptive control, and show that for a single input plant with output feedback, bounded trajectories can be guaranteed for a range of initial conditions regardless of the stability properties of the open-loop plant. Reference [2] presents a modification of [1] that allows for stable adaptation without hard actuator saturation. In [3], a viable control strategy is proposed without formal proof of stability or boundedness. In [4], the authors review the current state of the art of adaptive control with input constraints. They note that most current algorithms are applicable to indirect-adaptive control, and many rest on the assumption that the plant is open-loop stable. No multi-input saturation compensation strategies are mentioned. An indirect adaptive control strategy is developed in [5] for open-loop stable plants with possible zeros in the RHP. In [6], current saturation compensation methods are used for the problem of reconfigurable flight control.

All of these papers have focused primarily on single input, single output (SISO) systems. To the authors' knowledge, there currently exist no formal attempts to demonstrate stability or boundedness for multi-input, multi-output (MIMO), direct-adaptive systems subject to saturation constraints. In fact, it can be argued that susceptibility to saturation errors is a major factor inhibiting the implementation of direct-adaptive control in a wide range of MIMO applications.

In this paper, we develop an extension of the result in [1] to multivariable plants. The technique proposed in [1] consists of modifying the error signal used for training adaptive gains so as to remove the effects of saturation from the error, and is often given the name Training Signal Hedging (TSH) as in [2,10]. The proof methodology for the multi-input case is similar to the single input case, though there are features that require significant modification. As in [1], we provide a region of initial conditions for which bounded trajectories are guaranteed for a multi-input adaptive system with magnitude-constrained inputs. This region is shown to extend to the entire state space if the plant is open-loop stable.

This paper is organized as follows: In section II we pose the multi-input problem, state the stability result in a theorem, and provide a proof. Section III contains simulation results, and a summary is presented in section IV.

Manuscript received March 7, 2005. This work was supported by the Boeing Strategic University Initiative.

M. Schwager (schwager@mit.edu) and A. M. Annaswamy (aanna@mit.edu) are with the Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA, 02139, USA.

II. Formulation for a Multi-Input Plant

Consider a plant model of the form

$$\dot{x} = A_p x + B_p \Lambda E_s(u) + B_p \Lambda f \,, \qquad (1)$$

where $x \in \Re^n$ is the fully measurable state vector, $u \in \Re^m$ is the input vector. Also, $A_p \in \Re^{n\times n}$ is *unknown*, $B_p \in \Re^{n\times m}$ is *known*, $\Lambda \in \Re^{m\times m}$ is *unknown* and diagonal with positive entries, and $f \in \Re^m$ is *unknown*. The requirement that B_p is known with Λ diagonal and unknown can be replaced by a completely unknown B_p, but the complication of the resulting adaptive controller obscures the main stability result. The function $E_s(.)$ is an elliptical multi-dimensional saturation function defined by

$$E_s(u) = \begin{cases} u & if \quad \|u\| \le g(u) \\ \bar{u} & if \quad \|u\| > g(u) \end{cases}, \qquad (2)$$

where $g(u)$ is given by

$$g(u) = \left(\sum_{i=1}^{m} \left[\frac{\hat{e}_i}{u_{\max i}} \right]^2 \right)^{-1/2},$$

$\hat{e} = \frac{u}{\|u\|}$ denotes the unit vector in the direction of u, $u_{\max i}$ is the saturation limit of the i^{th} actuator, and $\bar{u}$ is given by

$$\bar{u} = \hat{e} g(u) . \tag{3}$$

The vector norm ||.|| is the Euclidean norm. The actuators saturate symmetrically in this formulation, though asymmetric saturation limits can be treated straightforwardly with the same approach. Two aspects of this saturation function should be noted. First, the function $g(u)$ returns the magnitude of the projection of u onto the boundary of the m-dimensional ellipsoid defined by $\|u\| = g(u)$, and hence $E_s(.)$ is denoted as an elliptical saturation function. Second, from (3) it is clear that the output of $E_s(.)$ is direction preserving (see Figure 1).

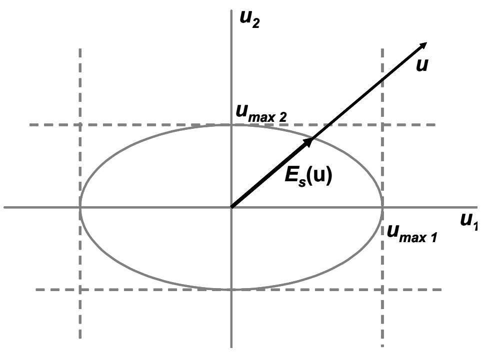

Fig. 1. A schematic of the elliptical saturation function $E_s(.)$ for $m = 2$ is shown in this figure. Notice that it is direction preserving.

The elliptical saturation function was used in this formulation for analytical expediency. A more realistic rectangular saturation function will be discussed later.

A reference model whose state represents the desired state of the plant is chosen as

$$\dot{x}_m = A_m x_m + B_m r , \tag{4}$$

where $x_m \in \Re^n$, and $r \in \Re^l$ is a bounded reference input vector so that $\|r\| < r_{\max}$. Note that r and u are not necessarily of the same dimension, therefore this architecture can be used for adaptive control allocation, as described in [9]. We also require that $A_m \in \Re^{n \times n}$ is Hurwitz, which is a standard requirement in adaptive control. The goal is to choose u so that $e = x - x_m$ is as small as possible, and all signals in the closed-loop system remain bounded.

A standard feedback/feedforward control structure is chosen as

$$u = K_x x + K_r r + k_f . \tag{5}$$

We assume that there exist ideal gain matrices $K_x^* \in \Re^{m \times n}$ and $K_r^* \in \Re^{m \times l}$, and an ideal constant vector $k_f^* \in \Re^m$ that result in perfect model following, so that $A_p + B_p \Lambda K_x^* = A_m$, $B_p \Lambda K_r^* = B_m$, and $B_p \Lambda (k_f^* + f) = 0$. Define the parameter errors to be

$$\tilde{K}_x = K_x - K_x^*,\ \tilde{K}_r = K_r - K_r^*, \text{ and } \tilde{k}_f = k_f - k_f^* . \tag{6}$$

Then subtracting (4) from (1) and substituting with (5) and (6), the closed-loop error dynamics can be written

$$\dot{e} = A_m e + B_p \Lambda (\tilde{K}_x x + \tilde{K}_r r + \tilde{k}_f + \Delta u) , \tag{7}$$

where $\Delta u = u - E_s(u)$ represents the control deficiency signal, as in [1]. To eliminate the adverse effects of the disturbance Δu, we generate a signal e_Δ as

$$\dot{e}_\Delta = A_m e_\Delta + B_p diag(\hat{\lambda}) \Delta u ,$$

where $\hat{\lambda} \in \Re^m$ is a vector, the terms of which are the estimates of the diagonal terms of the unknown matrix Λ. The effects due to saturation can be removed from the error in (7) by defining the augmented error as $e_u = e - e_\Delta$, which can be written

$$\dot{e}_u = A_m e_u + B_p \Lambda (\tilde{K}_x x + \tilde{K}_r r + \tilde{k}_f) + B_p diag(\Delta u) \tilde{\lambda}, \tag{8}$$

where $diag(\tilde{\lambda}) = \Lambda - diag(\hat{\lambda})$, and exploiting the fact that $diag(\tilde{\lambda}) \Delta u = diag(\Delta u) \tilde{\lambda}$.

Since (8) is in a standard form relevant to several adaptive systems, we choose the adaptive laws for adjusting the parameters K_x, K_r, k_f, and $\hat{\lambda}$ as

$$\dot{K}_x = -\Gamma_x B_p^T P e_u x^T,\ \dot{K}_r = -\Gamma_r B_p^T P e_u r^T ,$$

$$\dot{k}_f = -\Gamma_f B_p^T P e_u, \text{ and } \dot{\hat{\lambda}} = \Gamma_\lambda diag(\Delta u) B_p^T P e_u , \tag{9}$$

where $A_m^T P + P A_m = -Q$, and Q, Γ_x, Γ_r, Γ_f, and Γ_λ are positive definite. Define a Lyapunov function candidate V as

$$\begin{aligned} V &= e_u{}^T P e_u + Tr(\tilde{K}_x^T \Gamma_x^{-1} \Lambda \tilde{K}_x) + \\ &Tr(\tilde{K}_r^T \Gamma_r{}^{-1} \Lambda \tilde{K}_r) + \tilde{k}_f{}^T \Gamma_f{}^{-1} \Lambda \tilde{k}_f + \tilde{\lambda}^T \Gamma_\lambda^{-1} \tilde{\lambda}. \end{aligned} \tag{10}$$

Since all Γ and Λ are positive definite, V is positive definite in $e_u, \tilde{K}_x, \tilde{K}_r, \tilde{f}$ and $\tilde{\lambda}$. Taking the time derivative along the system trajectories leads to $\dot{V} = -e_u^T Q e_u \le 0$, which implies that the signals $e_u, \tilde{K}_x, \tilde{K}_r, \tilde{k}_f$ and $\tilde{\lambda}$ are bounded. Define K_{max} so that

$$K_{\max} = \max\left(\sup\|\tilde{K}_x\|, \sup\|\tilde{K}_r\|, \sup\|\tilde{k}_f\|\right).$$

For efficiency of notation we define the following:

$$q_{\min} = \min eig(Q),$$

$$p_{\min} = \min eig(P),$$

$$p_{\max} = \max eig(P),$$

$$\rho = \sqrt{\frac{p_{\max}}{p_{\min}}},$$

$$\bar{u}_{\min} = \min_i \left(u_{\max i}\right),$$

$$\bar{u}_{\max} = \max_i \left(u_{\max i}\right),$$

$$P_B = \left\|P B_p \Lambda\right\|,$$

$$\gamma_{\max} = \max(eig(\Gamma_x), eig(\Gamma_r), eig(\Gamma_f), eig(\Gamma_\lambda)),$$

$$\lambda_{\min} = \min(eig(\Lambda)),$$

All vector norms are Euclidean norms and the matrix norm P_B is the induced matrix norm, which has the property $\|P B_p \Lambda x\| \le P_B \|x\|$. Also, define

$$\beta = \frac{P_B K_{\max}}{\left\|K_x^*\right\| + K_{\max}} \quad (11)$$

$$a_0 = \frac{\bar{u}_{\min} K_{\max}}{\left\|K_x^*\right\| + K_{\max}} - 2\|f\| \quad (12)$$

$$x_{\min} = \frac{3P_B K_{\max}(r_{\max}+1) + 3P_B\left\|K_r^*\right\| r_{\max} + 2P_B \bar{u}_{\max} + P_B\left\|k_f^*\right\|}{q_{\min} - 3P_B K_{\max}}, \quad (13)$$

$$x_{\max} = \frac{P_B a_0}{\left|q_{\min} - 2P_B\left\|K_x^*\right\|\right|}, \text{ and} \quad (14)$$

$$\bar{K}_{\max} = \frac{q_{\min} - \frac{\rho}{a_0}\left(3\left\|K_r^*\right\| r_{\max} + 2\bar{u}_{\max} + \left\|k_f^*\right\|\right)\left|q_{\min} - 2P_B\left\|K_x^*\right\|\right|}{3P_B + 3\frac{\rho}{a_0}(r_{\max}+1)\left|q_{\min} - 2P_B\left\|K_x^*\right\|\right|}. \quad (15)$$

Assumption A1. $\bar{u}_{\min}$ *is such that* $a_0 > 0$.

Assumption A1 implies that there is a constraint on the size of the unknown disturbance $\|f\|$ given by

$$\|f\| < \frac{\bar{u}_{\min} K_{\max}}{2(\left\|K_x^*\right\| + K_{\max})}.$$

Theorem 1. *Under assumption A1, for the system in (1) with the controller in (5) and the adaptive laws in (9), x(t) has bounded trajectories for all t* $\geq t_0$ *if*

i. $\left\|x(t_0)\right\| < x_{\max}\dfrac{1}{\rho}$, *and*

ii. $\sqrt{V(t_0)} < \bar{K}_{\max}\sqrt{\dfrac{\lambda_{\min}}{\gamma_{\max}}}$.

Further,

$$\|x(t)\| < x_{\max} \quad \forall t \geq t_0,$$

and the output error e is of the order

$$\|e\| = O\left[\sup_{\tau \leq t}\|\Delta u(\tau)\|\right].$$

Proof. We shall choose a positive definite function, $W(x)$, as

$$W = x^T P x \quad (16)$$

and define a level set, B, of W as

$$B : \left\{x \mid W(x) = p_{\min} x_{\max}^2\right\}, \quad (17)$$

where x_{max} is as defined in (14). Now define the annulus region A as

$$A : \left\{x \mid x_{\min} < \|x\| \leq x_{\max}\right\}, \quad (18)$$

where x_{min} is defined in (13). The proof proceeds in 2 steps. In step 1 we show that condition (*ii*) from theorem 1 implies that $B \subset A$. In step 2 we show that $\dot{W} < 0 \;\; \forall x \in A$. Condition (*i*) from theorem 1 implies that

$$W(x(t_0)) < W(B).$$

Therefore the results of step 1, and step 2, imply that

$$W(x(t)) < W(x(t_0)) \;\; \forall t \geq t_0.$$

Theorem 1 follows directly.

Step 1:
In this step we show that $B \subset A$. First, from condition (*ii*) from theorem 1 it follows that $K_{\max} < \bar{K}_{\max}$. Substituting the expression for $\bar{K}_{\max}$, leads to

$$\rho\frac{3K_{\max}(r_{\max}+1) + 3\left\|K_r^*\right\| r_{\max} + 2\bar{u}_{\max} + \left\|k_f^*\right\|}{q_{\min} - 3P_B K_{\max}} < \frac{a_0}{\left|q_{\min} - 2P_B\left\|K_x^*\right\|\right|}. \quad (19)$$

The inequality in (19) with the definitions of x_{min} and x_{max} from (13) and (14) respectively, directly implies

$$\rho x_{\min} < x_{\max}. \quad (20)$$

From (16), $W(x)$ can be bounded from below by $p_{\min}\|x\|^2 \leq W(x)$, which implies from (17) that

$$\|x\| \leq x_{\max} \quad \forall x \in B.$$

Likewise, from (16), $W(x)$ can be bounded from above by $W(x) \leq p_{\max}\|x\|^2$, which implies from (20) and (17) that

$$x_{\min} < \frac{1}{\rho}x_{\max} \leq \|x\| \quad \forall x \in B.$$

From the definition of A in (18) we conclude therefore that $B \subset A$.

Step 2:
We now show that $\dot{W} < 0 \;\; \forall x \in A$.

Case A: $\|\Delta u\| = 0$

From (1), (2), and (6), we get

$$\dot{x} = A_m x + B_p \Lambda(\tilde{K}_x x + K_r r + \tilde{k}_f),$$

which leads to

$$\dot{W} = x^T(Q + 2PB_p\Lambda\tilde{K}_x)x + 2PB_p\Lambda(\tilde{K}_r r + K_r^* r + \tilde{k}_f)x.$$

Bounding quantities on the right hand side using previous definitions gives

$$\dot{W} < (2P_B K_{\max} - q_{\min})\|x\|^2 + 2P_B K_{\max}(r_{\max}+1)\|x\| + 2P_B\left\|K_r^*\right\| r_{\max}\|x\|.$$

From condition (*ii*) of theorem 1 and from the definition of K_{max} it follows that

$$K_{\max} < \bar{K}_{\max} < \frac{q_{\min}}{3P_B}. \quad (21)$$

Therefore

$$\dot{W} < 0 \text{ if } \|x\| > \frac{2P_B K_{\max}(r_{\max}+1) + 2P_B\left\|K_r^*\right\| r_{\max}}{\left(q_{\min} - 2P_B K_{\max}\right)}.$$

The choice of x_{min} in (13) implies that

$$x_{\min} > \frac{2P_B K_{\max}(r_{\max}+1) + 2P_B\left\|K_r^*\right\| r_{\max}}{\left(q_{\min} - 2P_B K_{\max}\right)}.$$

Hence,

$$\dot{W} < 0 \qquad \forall x \in A \text{ in case A.} \quad (22)$$

Case B: $\|\Delta u\| \neq 0$

Equations (1), (2), (5), and (6) give

$$\dot{x} = A_m x - B_p \Lambda K_x^* x + B_p \Lambda \bar{u} + B_p \Lambda f . \quad (23)$$

Take the time derivative of (16) along the trajectories of (23) to get

$$\dot{W} = -x^T Q x - 2x^T P B_p \Lambda K_x^* x + 2x^T P B_p \Lambda (\bar{u} + f). \quad (24)$$

Sub-case (i): $2x^T P B_p \Lambda \bar{u} < -\bar{u}_{\min} \beta \|x\|$

Using (24) and the condition for sub-case (*i*), and previously defined upper bounds, we can bound $\dot{W}$ by

$$\dot{W} < \|x\|^2 \left| 2P_B \|K_x^*\| - q_{\min} \right| + \|x\| 2P_B \|f\| - \|x\| \bar{u}_{\min} \beta.$$

This implies that if

$$\|x\| \le \frac{P_B a_0}{\left| q_{\min} - 2P_B \|K_x^*\| \right|} \text{ then } \dot{W} < 0 . \quad (25)$$

From the definition of x_{max} in (14), we have therefore that

$$\dot{W} < 0 \quad \forall x \in A \text{ in case B, sub-case } (i). \quad (26)$$

Sub-case (ii): $2x^T P B_p \Lambda \bar{u} \ge -\bar{u}_{\min} \beta \|x\|$

From (3), the condition for sub-case (*ii*) implies that

$$2x^T P B_p \Lambda \frac{u}{\|u\|} \|\bar{u}\| + \|x\| \bar{u}_{\min} \beta \ge 0 .$$

Substituting for u from (5), and for K_x from (6) gives

$$2x^T P B_p \Lambda \left(\tilde{K}_x x + K_r r + k_f \right) + \|x\| \bar{u}_{\min} \beta \frac{\|u\|}{\|\bar{u}\|} \ge - 2x^T P B_p \Lambda K_x^* x.$$

Then adding terms to create $\dot{W}$ on the right hand side gives

$$- x^T Q x + 2x^T P B_p \Lambda \left(\tilde{K}_x x + K_r r + k_f \right) + \|x\| \bar{u}_{\min} \beta \frac{\|u\|}{\|\bar{u}\|} + 2x^T P B_p \Lambda \bar{u} + 2x^T P B_p f \ge \dot{W}.$$

Simplifying with (6),

$$- x^T Q x + 2x^T P B_p \Lambda \left(\tilde{K}_x x + K_r r + \tilde{k}_f \right) + \|x\| \bar{u}_{\min} \beta \frac{\|u\|}{\|\bar{u}\|} + 2x^T P B_p \Lambda \bar{u} \ge \dot{W},$$

and using (15) and previous definitions to bound unknown quantities

$$- q_{\min} \|x\|^2 + 2\|x\| P_B \left(K_{\max} \|x\| + K_{\max} r_{\max} + \|K_r^*\| r_{\max} + K_{\max} \right) + \|x\| \|u\| \beta + 2\|x\| P_B \bar{u}_{\max} > \dot{W}. \quad (27)$$

We note from (5) that

$$\|u\| \le \left(\|K_x^*\| + K_{\max} \right) \|x\| + \left(\|K_r^*\| + K_{\max} \right) r_{\max} + \left(\|k_x^*\| + K_{\max} \right), \quad (28)$$

and from (11) that

$$0 \le \beta \le P_B . \quad (29)$$

Using (28) and (29) in (27) gives

$$\|x\|^2 \left(3P_B K_{\max} - q_{\min} \right) + \|x\| \left(3P_B K_{\max} (r_{\max} + 1) + 3P_B \|K_r^*\| r_{\max} \right) + \|x\| \left(2P_B \bar{u}_{\max} + P_B \|k_f^*\| \right) > \dot{W}. \quad (30)$$

Then, from (21) and (30) we know that

$$\|x\| > \frac{3P_B K_{\max} (r_{\max} + 1) + 3P_B \|K_r^*\| r_{\max} + 2P_B \bar{u}_{\max} + P_B \|k_f^*\|}{q_{\min} - 3P_B K_{\max}}$$

implies that $\dot{W} < 0$. From the definition of x_{min} in (13), we have that

$$\dot{W} < 0 \quad \forall x \in A \text{ in case B, sub-case } (ii). \quad (31)$$

From (22), (26), and (31) it follows that

$$\dot{W} < 0 \qquad \forall x \in A .$$

□

Remark 1. In the case of magnitude saturation, global stability is impossible for an unstable plant. Initial conditions can always be found to cause the plant state to become unbounded regardless of the controller. Thus any stability result for unstable plants must be local in nature, as the one presented in Theorem 1.

Remark 2. In the case of an open loop stable plant, bounded trajectories are guaranteed for all initial conditions. If A_p is stable, (1) is BIBO stable, and (2) implies that $E_s(u)$ is bounded, therefore x is bounded.

Remark 3. Determining $\bar{u}$ in (2) can be computationally burdensome, especially for $m > 2$. A simpler and more intuitively compelling saturation function is given by $R_s(u)$, where the elements of R_s are defined by the ordinary saturation function

$$R_{si} = sat(u_i) = \begin{cases} u_i & if \quad |u_i| \le u_{\max i} \\ u_{\max i} \operatorname{sgn}(u_i) & if \quad |u_i| > u_{\max i} \end{cases}, \quad (32)$$
$$\text{for } i = 1, \dots, m .$$

This saturation function can be expressed as the sum of a direction preserving component and an error component, so that

$$R_s = \begin{cases} u & if \quad \|u\| \le h(u) \\ u_d + \tilde{u} & if \quad \|u\| > h(u) \end{cases},$$

where

$$u_d = \hat{e} h(u) .$$

As before, $\hat{e}$ is the unit vector in the direction of u, and $h(u)$ returns the magnitude of the projection of u onto the hyper-rectangle defined by the saturation limits $u_{\max i}$, where $i = 1, \ldots, m$. In this formulation, u_d is in the same direction as u and $\tilde{u}$ is an error vector. Figure 2 illustrates the nature of $R_s(.)$ for the case when $m = 2$.

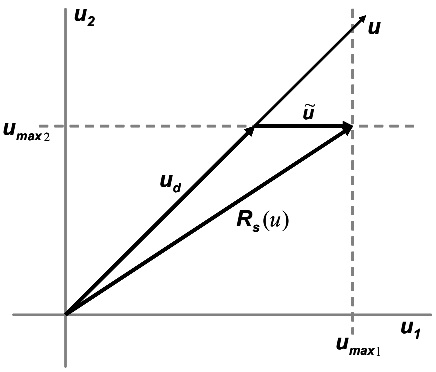

Fig. 2. A schematic of the rectangular saturation function for $m = 2$ is shown above. Notice that it is not necessarily direction preserving, that is, $\tilde{u} \neq 0$ in general.

It can be shown that $\tilde{u}$ is a bounded vector, whose bound depends upon the bound of the state $x_{\max}$. This in turn causes the extension of the above proof to the case of rectangular saturation to become more complex, and is beyond the scope of this paper. However, this case is explored in the simulation results in the next section, which show that for a rectangular saturation function, bounded trajectories can be achieved for a larger set of initial conditions than for the elliptical saturation function.

III. Simulations

Numerical simulations were carried out to demonstrate the usefulness of the proposed saturation compensation technique. An unstable, second order, two input plant in the form of (1) whose dynamics are given by

$$A_p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and}\ f = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{33}$$

was chosen for simulation. The reference model was selected in the form of (4) with dynamics given by

$$A_m = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix},\ B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The input, u, to the plant in (33) was constrained according to the rectangular saturation function $R_s(u)$ given in (32) where

$$u_{\max 1} = u_{\max 2} = 2.5.$$

All initial conditions were set to zero except for the initial value for $\hat{\lambda}(t)$, which was set to

$$\hat{\lambda}(0) = [1 \quad 1]^T.$$

A constant input r was given to the reference model at $t = 0$ where

$$r = [-1 \quad 1]^T.$$

As stated previously, the task is for all signals in the adaptive system to remain bounded while the error, e, remains as small as possible.

Simulation results are shown in Figs. 3-5. Figure 3 shows the L_2 norm of the tracking error for the adaptive system with and without multi-input TSH. It can be clearly seen that with TSH the tracking error converges to zero asymptotically, whereas without TSH the tracking error becomes unbounded.

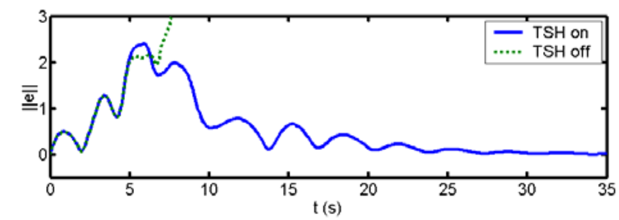

Fig. 3. The L_2 norm of the tracking error is shown in this figure for the adaptive system with and without muli-input TSH. With TSH the error converges asymptotically to zero. Without TSH the error becomes unbounded.

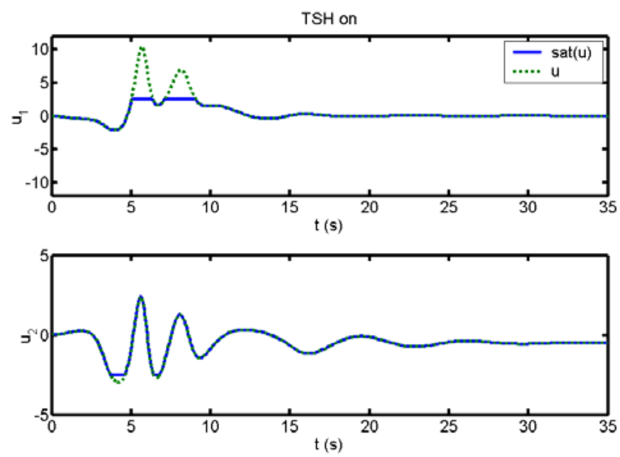

Fig. 4. The control input activity is shown in this figure for the adaptive system with multi-input TSH. Saturation occurs initially for both inputs, however the control signals remain well behaved. After $t = 10s$ the control signal remains within the saturation limits.

The control input activity for the adaptive system with and without TSH is shown in Figs. 4 and 5. With TSH (Fig. 4), the controls saturate, but remain well behaved. After $t = 10s$ the control signals remain within the saturation limits. Without TSH (Fig. 5), both input signals saturate and diverge.

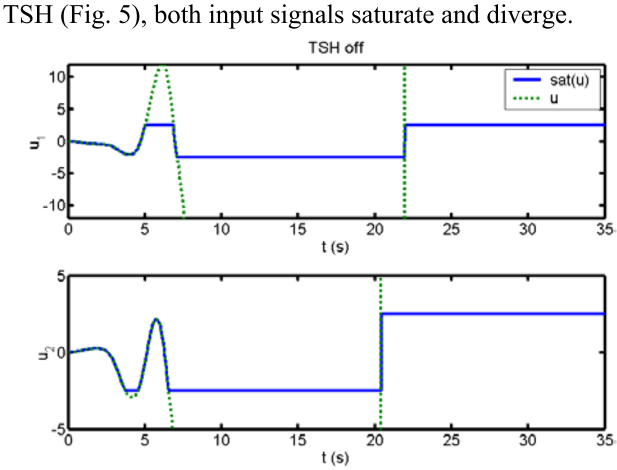

Fig. 5. The control input activity is shown in this figure for the adaptive system without multi-input TSH. Both input signals saturate severely and become unbounded.

As discussed in Remark 3, it is interesting to see in simulation the difference between the more realistic rectangular saturation function, $R_s(u)$, and the more analytically attractive elliptical saturation function, $E_s(u)$. The same simulation scenario described above was performed with the two saturation functions. Figure 6 shows that for the system with R_s the error converges to zero asymptotically, whereas with E_s the error becomes unbounded, giving evidence that E_s leads to a smaller region of initial conditions for which bounded trajectories are obtained.

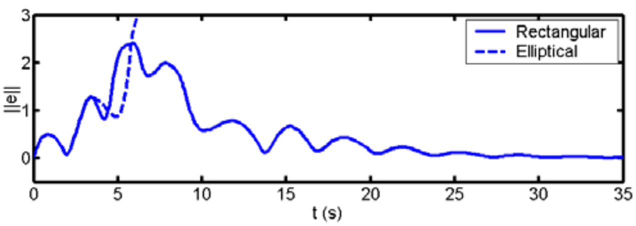

Fig. 6. The L_2 norm of the tracking error is shown in this figure for the adaptive system with rectangular and elliptical saturation functions. With the rectangular function the error converges asymptotically to zero. With the elliptical function the error becomes unbounded.

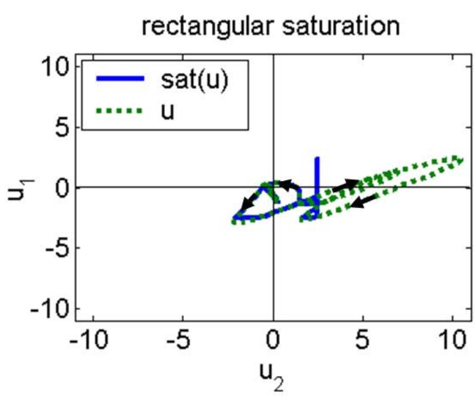

Fig. 7. The input space is shown for the rectangular saturation function. The outline of the saturation boundary is visible.

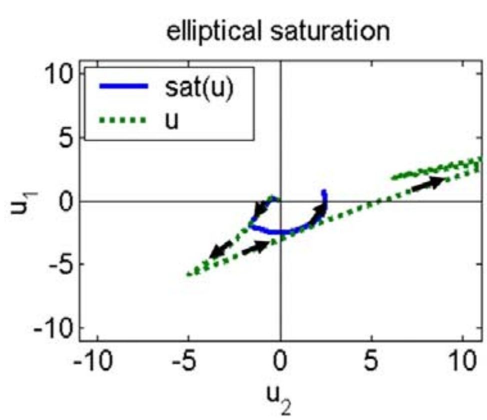

Fig. 8. The input space is shown for the elliptical saturation function. The outline of the saturation boundary is visible.

Figures 7 and 8 show phase trajectories of the input signals for R_s and E_s, respectively. Partial outlines of the saturation boundaries can be seen in both cases. The input signal for the rectangular function saturates intermittently, but settles within the saturation limits. The input signal with the elliptical function saturates and becomes unbounded, again giving evidence that E_s leads to a smaller region of initial conditions for which bounded trajectories are obtained.

IV. Summary

In this paper, we have developed an extension of the approach used in [1] to multivariable plants with magnitude-constrained inputs. It is shown that for initial conditions of the system state and the adaptive control parameters that lie inside a bounded region, bounded trajectories are guaranteed. This region is shown to extend to the entire state space if the plant is open-loop stable.

Acknowledgment

The authors would like to acknowledge Eugene Lavretsky for his insightful comments and suggestions in the proof of theorem 1.

References

[1] Karason, S. P., and A. M. Annaswamy, "Adaptive Control in the Presence of Input Constraints," *IEEE Transactions on Automatic Control*, Vol. 39, No. 11, pp. 2325-2330, November 1994.

[2] E. Lavretsky, N. Hovakimyan, "Positive μ-modification for Stable Adaptation in a Class of Nonlinear Systems with Actuator Constraints," *Proc. American Control Conf.*, Boston, MA, Jul. 2004, Vol. 3, pp. 2545-2550.

[3] Monopoli, R. V., "Model Reference Adaptive Control with an Augmented Error Signal," *IEEE Transactions on Automatic Control*, Vol. AC-19, No. 5, pp. 474- 484, October 1974.

[4] Cheng, J.-W. J., and Y.-M. Wang, "Adaptive Control for Systems with Input Constraints – A Survey," *Adaptive Control of Nonsmooth Dynamic Systems*, Edited by G. Tao and F. L. Lewis, Springer-Verlag, London, 2001, pp. 311-332.

[5] Suarez-Cerda, D. A., and R. Lozano, "Adaptive Control of Linear Systems with Poles in the Closed LHP with Constrained Inputs," *Adaptive Control of Nonsmooth Dynamic Systems*, Edited by G. Tao and F. L. Lewis, Springer-Verlag, London, 2001, pp. 349-359.

[6] Bodson, M., and W. A. Pohlchuck, "Command Limiting in Reconfigurable Flight Control," *Proc. of AIAA Guidance, Navigation, and Control Conf.*, New Orleans, LA, Aug. 1997, pp. 727-736.

[7] Narendra, K. S., and A. M. Annaswamy, *Stable Adaptive Systems*, Prentice-Hall, 1989.

[8] Slotine, J. J. E., and W. Li, *Applied Nonlinear Control*, Prentice Hall, Englewood Cliffs, NJ, 1991.

[9] Schwager, M., A. M. Annaswamy, and E. Lavretsky, "Adaptation-based Reconfiguration in the Presence of Actuator Failures and Saturation," *Proc. American Control Conf.*, Portland, OR, Jun. 2005, pp. 2640-2645.

[10] Johnson, E. N. and A. J. Calise, "Neural Network Adaptive Control of Systems with Input Saturation," *Proc. American Control Conference*, Arlington, VA, Jun. 2001, Vol. 5, pp. 3527-3532.