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Well-posedness of Nonconvex Integral Functionals

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Abstract— We find a sufficient condition guaranteeing well-posedness in a strong sense of the minimization of a multiple integral on the Sobolev space $W^{1,1}(\Omega; R^m)$ with boundary datum equal to zero. We remark that this condition does not involve global convexity of the integrand and therefore it allows us to find well-posedness properties of two classes of nonconvex problems recently studied: functionals depending only on the gradient and radially symmetric functionals.

I. INTRODUCTION

Consider the class of integral functionals defined by

$$J(u) = \int_\Omega L(x, u(x), Du(x))\, dx \qquad (1)$$

on some space of Sobolev functions with boundary datum equal to zero.

In [1] Ioffe and Zaslavski proved a variational principle thanks to which it is possible to show that the set of normal integrands giving raise to a well-posed minimization problem in a very strong sense is generic with respect to a suitable topology. This implies that the classical hypotheses needed in order to apply the direct method of the calculus of variations, namely convexity of the integrand in the gradient variable and a growth assumption, are not necessary to have well-posedness of the minimization problem and therefore also existence and uniqueness of the minimizer.

For this reason it is useful to find explicit conditions generically satisfied by integral functionals guaranteeing well-posedness of a given problem.

We deal with two types of well-posedness: Tykhonov well-posedness which is probably the best-known and the weakest well-posedness concept (see [2]) and well-posedness under perturbations, introduced by Zolezzi in [3] which is instead the strongest known concept of well-posedness.

This paper is organized as follows: in Section II we recall the basic definitions of well-posedness and of bounded Hausdorff convergence, which was introduced in [4] and [5]. Moreover we illustrate a result of general nature, proved in [7], in order to deal with nonconvex problems defined as in (1). Theorem 1 is an extension to the nonconvex case of a theorem of Beer-Lucchetti in [6] about the connections between Tykhonov well-posedness and well-posedness under perturbations with respect to the bounded Hausdorff convergence of a given function. The convexity assumption is replaced by a suitable a priori localization condition for the approximate minimizers of elements of any bounded Hausdorff converging sequence of functions (see [7]).

According to this result, in order to get a characterization of well-posedness under perturbations of a functional it is enough to get appropriate characterizations of Tykhonov well-posedness.

In Section IV we find a sufficient condition of local character for Tikhonov wellposedness of an integral functional defined on the Sobolev space $W_0^{1,1}(\Omega; R^m)$ by extending some results of [8], [9] and [10]. The key assumption is strict convexity of the integrand at certain points and the geometric approach of the proof allows us to consider integrands which may take the value $+\infty$ and this gives the possibility to apply our results also to problems with constraints.

Finally, in Section IV, we apply our results to study wellposedness of two classes of integral functionals. The first is the case of integrands depending only on the gradient, and with linear boundary datum: for this class we extend a previous result of [11] and we get stability of the solution by perturbations of the boundary data. The second is the case of radially symmetric functionals, treated for instance in [12] and [13]. For this class we have that the hypotheses which ensure existence and uniqueness of the solution guarantee also wellposedness (see [14]).

There are not many well-posedness results in the classical calculus of variations, however results on this subject are proved in [15], [11], [16], [17], [2], [3] and [18], with respect to different types of perturbations.

In [15] a Tikhonov wellposed one-dimensional integral functional is considered. In particular it is proved that under suitable hypotheses this is enough to get the strong convergence in some Sobolev space of the asymptotically minimizing sequences corresponding to perturbations of the integrand with respect to the variational convergence not involving the derivative.

In [3] sufficient and necessary conditions are obtained for wellposedness by perturbations of the boundary data for integral functionals depending only on the gradient. The integrand is assumed to be continuous in the one-dimensional case and to have moreover polynomial growth in the multi-dimensional one.

II. PRELIMINARIES AND GENERAL RESULTS

As we said in the Introduction there is a very strong link between the concepts of Tykhonov well-posedness

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and well-posedness under perturbations. We start giving the definition of the variational convergence with respect to which the perturbations are taken: the bounded Hausdorff one.

Given a Banach space X, for any set $A \subset X$ and $\rho \geq 0$, we set $A_\rho := A \cap B_\rho(0)$. For $A, B \subset X$, $e(A,B) := \sup_{x \in A} d(x,B)$, is the excess of A on B.

Given two subsets of a normed space, the ρ-Hausdorff distance between A and B is the following quantity:

$$\mathrm{haus}_\rho(A,B) := \max\{e(A_\rho, B), e(B_\rho, A)\}.$$

Identifying each extended real valued function with its epigraph we are able to give the following definition.

Definition 1: For $\rho \geq 0$, the ρ-Hausdorff-distance between two extended real valued functions f and g defined on X is

$$\mathrm{haus}_\rho(f,g) := \mathrm{haus}_\rho(\mathrm{epi}f, \mathrm{epi}g),$$

where the unit ball of $X \times R$ is the set $B_1(0) := \{(x,\alpha) : \|x\| \leq 1, |\alpha| \leq 1\}$.

Definition 2: Let $f, f_h : X \to [-\infty, +\infty]$, $h = 1, 2, \ldots$, be lower semicontinuous functions. We say that f_h converge to f with respect to the bounded Hausdorff convergence, and we write $f = bH - \lim f_h$, if and only if there exists $\rho_0 > 0$ such that for every $\rho > \rho_0$

$$\mathrm{haus}_\rho(f, f_h) \to 0 \text{ as } h \to +\infty.$$

Finally we give the definitions of the two concepts of well-posedness involved in this paper ([2], [3]).

Definition 3: We say that a function $F : X \to R \cup \{+\infty\}$ is Tikhonov wellposed if it satisfies the following conditions:

a) there exists a unique global minimizer u_0 of F;
b) if u_h is any minimizing sequence, i.e. a sequence such that $F(u_h) \to F(u_0)$, then $u_h \to u_0$.

Now we consider a convergence space $\mathcal{A}$ and a fixed point $f \in \mathcal{A}$. We are given the proper extended real-valued functions

$$F : X \to (-\infty, +\infty], \;\; I : \mathcal{A} \times X \to (-\infty, +\infty]$$

such that

$$F(u) = I(f,u), \; u \in X.$$

The corresponding value function is given by

$$V(g) = \inf\{I(g,u) \,|\, u \in X\}, \; g \in \mathcal{A}.$$

Definition 4: ([19]) The problem of minimizing F on X is called wellposed under perturbations (with respect to the embedding defined by I) iff

1) $V(g) > -\infty$ for all $g \in \mathcal{A}$,
2) there exists a unique global minimizer u_0 for F,
3) for every sequences $f_h \to f$ in $\mathcal{A}$ and $u_h \in X$ such that

$$I(f_h, u_h) - V(f_h) \to 0 \,\text{as}\, h \to +\infty \qquad (2)$$

we have $u_h \to u_0$ in X.

Sequences satisfying condition (2) are called asymptotically minimizing sequences.

Let $L(X) = \{G : X \to R \cup \{+\infty\}$, bounded from below, proper and l.s.c.$\}$ and $\psi : X \to R \cup \{+\infty\}$ such that $\lim_{\|u\| \to +\infty} \psi(u) = +\infty$. Set

$$\mathcal{A} = \{G \in L(X) \,|\, G(u) \geq \psi(u) \text{ for each } u\}, \qquad (3)$$

endowed with the bH-convergence, and define

$$I : \mathcal{A} \times X \to (-\infty, +\infty]$$
$$I(G,u) = G(u). \qquad (4)$$

The following result is a rewriting in different terms of Theorem 3.5 of [7].

Theorem 1: Let X be a Banach space, $\mathcal{A}$ and I defined by (3) and (4). Assume that $F \in \mathcal{A}$ is Tikhonov wellposed. Then F is wellposed under perturbations with respect to the embedding defined by I.

As we mentioned in the Introduction this result is the backbone for the applications in the next sections.

III. WELL-POSEDNESS OF INTEGRAL FUNCTIONALS

Consider Ω an open and bounded subset of R^n, $L : \Omega \times R^{nm} \to [0, +\infty]$ a normal integrand and define the associated integral functional

$$J(u) = \int_\Omega L(x, Du(x))\, dx \qquad (5)$$

on the space $W_0^{1,1}(\Omega; R^m)$. The key result of this section is the following theorem (see [14]).

Theorem 2: Let L and J be defined as in Corollary 2. Suppose that J is coercive having a unique minimum point $u_0 \in W_0^{1,1}(\Omega; R^m)$.

Moreover assume that $v \mapsto L(x,v)$ is strictly convex at the point $Du_0(x)$ for almost every $x \in \Omega$. Then J is Tikhonov wellposed in $W_0^{1,1}(\Omega; R^m)$.

Equivalently, if $u_h \in W_0^{1,1}(\Omega; R^m)$ is any minimizing sequence, then:

$$\|u_h - u_0\| \to 0 \;\text{ in } W_0^{1,1}(\Omega; R^m).$$

In order to illustrate Theorem 2 we state three lemmas which are the building blocks of its proof , after recalling the definition of strict convexity at a point. In the sequel we will use the following notations. Given a function $L : R^n \to [0, +\infty]$ we will denote by L^{**} its convex regularization and by L_r the function defined on R^n by setting $L_r = L$ on the closed ball $\overline{B}(u_0, r)$ with center u_0 and radius r and $L_r = +\infty$ otherwise.

Definition 5: Let U be a closed convex subset of R^n and $u_0 \in U$. We say that a function $L : R^n \to R \cup \{+\infty\}$ is convex at u_0 with respect to the set U if

$$\sum_{i=1}^m c_i L(v_i) \geq L(u_0) \qquad (6)$$

for every $m > 0$, for every $v_i \neq u_0$ and $c_i > 0$ such that $v_i \in U$, $\sum_{i=1}^m c_i = 1$ and $\sum_{i=1}^m c_i v_i = u_0$.

If inequality (6) is always strict, we say that the function L is strictly convex at u_0 with respect to U.

Clearly, there are many examples of convex functions at a point which are not globally convex; for instance every function is convex at its minimum point, if it has one.

Lemma 1 is one of the tools used to prove the main theorem, and gives a characterization of the points of strict convexity in terms of their geometric properties.

Lemma 1: Let $L : R^n \to [0, +\infty]$ be lower semicontinuous and proper. Then:

1) If $(u_0, L^{**}(u_0))$ is an extreme point of epiL^{**} then L is strictly convex at u_0.
2) If L is strictly convex at u_0, then $(u_0, L(u_0))$ is an extreme point of epi$(L_r)^{**}$ for all $r > 0$.

Another basic ingredient in the proof of Theorem 2 is the following semicontinuity result.

Lemma 2: Let $L : \Omega \times R^{nm} \to [0, +\infty]$ be a normal integrand and let J be defined as in (5). Consider $u_h, u_0 \in W^{1,1}(\Omega; R^m)$ and suppose that the function $v \mapsto L(x, v)$ is convex at the point $Du_0(x)$ for almost every $x \in \Omega$.

If $\|u_h - u_0\| \to 0$ in $L^1(\Omega; R^m)$ and $Du_h \rightharpoonup Du_0$ in $L^1(\Omega; R^l)$, then

$$\liminf J(u_h) \geq J(u_0).$$

The last lemma is a generalization to the nonconvex setting of Lemma 3 of [10].

Lemma 3: Let $L : \Omega \times R^{mn} \to [0, +\infty]$ be a normal integrand and $u_h, u_0 \in W^{1,1}(\Omega; R^m)$. Suppose that the function $v \mapsto L(x, v)$ is convex at the point $Du_0(x)$ for almost every $x \in \Omega$.

If $\|u_h - u_0\|_{L^1} \to 0$, $Du_h \rightharpoonup Du_0$ and $J(u_h) \to J(u_0)$ then $x \mapsto L(x, Du_h(x)) \rightharpoonup (x \mapsto L(x, Du_0(x)))$ in $L^1(\Omega; R)$.

As a consequence of Theorems 1 and 2, we get the following Corollary, which establishes a criterium of well-posedness under perturbations of nonconvex integral functionals.

Corollary 1: Let $L, L_h : \Omega \times R^{nm} \to [0, +\infty]$ be normal integrands and let J be defined as (5). Suppose that J is coercive having a unique minimum point $u_0 \in W_0^{1,1}(\Omega; R^m)$.

Moreover assume that $v \mapsto L(x, v)$ is strictly convex at the point $Du_0(x)$ for almost every $x \in \Omega$. Consider the sequence J_h defined by $J_h(u) := \int_\Omega L_h(x, Du(x))\, dx$ on $W_0^{1,1}(\Omega,\ R^m)$. Assume that

(i) $J_h \to J$ with respect to the bounded Hausdorff convergence on $W_0^{1,1}(\Omega; R^m)$;
(ii) J_h is equicoercive.

Then:

$$\|u_h - u_0\| \to 0 \ \text{ in } W_0^{1,1}(\Omega; R^m)$$

for every asymptotically minimizing sequence u_h.

In other words the problem of minimizing J is well-posed under perturbations with respect to the embedding defined by (4).

IV. APPLICATIONS AND EXAMPLES

In the papers [20], [21], [11], [22] the following problem of minimizing a functional of the gradient under linear boundary conditions is studied:

$$(P_a) \quad \text{Minimize} \int_\Omega L(Du(x))dx$$

subject to $u \in u_a + W_0^{1,1}(\Omega)$. Here $\Omega \subset R^n$ is open, bounded and with piecewise C^1 boundary and $u_a(x) := \langle a, x\rangle$, where $a \in R^n$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in R^n. The function $L : R^n \to R$ is supposed to be lower semicontinuous and bounded from below. Moreover the function L satisfies the growth condition:

$$(G) \quad L(y) \geq \Phi(|y|) \ \text{ for any } y \in R^n,$$

where $\Phi : [0, +\infty) \to R$ is such that $\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty$.

In particular the paper [11] studies the continuous dependence of the solutions on the boundary data. For this purpose we need to introduce the problem

$$(P_a^{**}) \quad \text{Minimize} \int_\Omega L^{**}(Du(x))\, dx$$

subject to $u \in u_a + W_0^{1,1}(\Omega)$. As well-known (see Chapter X of [23]), it turns out that

$$\inf P_a = L^{**}(a)|\Omega| \tag{7}$$

and that the function u_a is a minimizer.

Using the results of Section III we can improve the results contained in [11].

Theorem 3: Let $L : R^n \to R$ be lower semicontinuous, bounded from below and satisfying the growth condition (G). Suppose that $(a, L^{**}(a))$ is an extreme point of epiL^{**}.

Then any asymptotically minimizing sequence of P_{a_k} converges strongly to u_a in $W^{1,1}(\Omega)$.

This Theorem moreover shows how the concept of well-posedness under perturbations also extends the classical concept of Hadamard well-posedness.

The following Theorem shows that also the class of radially symmetric functionals is well-posed.

Consider the variational problem

$$\text{Minimize} \int_{B(0,R)} [L(|Du(x)|) + h(u(x))]\, dx \tag{8}$$

subject to $u \in W_0^{1,1}(B(0, R))$. where $B(0, R)$ is the ball of R^n of radius R centered at the origin; the function $L : [0, +\infty[\to [0, +\infty]$ is lower semicontinuous and $h : R \to [0, +\infty[$ is a convex function.

Problems of this kind arise in various fields as nonlinear elasticity, fluid dynamics and optimal design. We refer to [12] and [13] for results about existence and uniqueness of minimizers, and we remark that wellposedness was not considered there.

Applying the results of [12] and Theorem 2 we obtain the following theorem.

Theorem 4: Assume that:

(h1) $L : [0, +\infty[\to [0, +\infty]$ is lower semicontinuous and has superlinear growth;

(h2) $h : R \to [0, +\infty[$ is convex and monotonic;

and that either h or L^{**} is strictly monotonic . Then problem ((8)) is wellposed in the sense of Tikhonov on the space $W_0^{1,1}(B(0, R))$.

We conclude with an example in order to show some useful features of the definition of well-posedness under perturbations.

Example 1: Let us consider the functionals

$$J(u) = \int_0^1 (u(x)^2 + u'(x)^4)\, dx$$

and

$$J_h(u) = \int_0^1 (u(x)^2 + (u'(x)^2 - \frac{1}{h})^2)\, dx,$$

with $u \in W_0^{1,4}(0, 1)$.

It is easy to show that the perturbed functionals J_h do not have a minimizer for any h. This means that even if J is strictly convex and Tykhonov well-posed it is possible to perturb it with perturbations as small as wished and destroy the existence of a minimizer. Anyway, since J is Tykhonov well-posed and the sequence J_h is convergent with respect to the bounded Hausdorff topology, every asymptotically minimizing sequence strongly converges. This shows that the concept of well-posedness under perturbations allows us to avoid the requirement of the existence of the minimizer of the perturbed problems.

References

[1] A. D. Ioffe and A. J. Zaslavski, “Variational principles and well-posedness in optimization and calculus of variations,”SIAM J. Control Optim., vol. 38, no.2, pp. 566–581, 2000.

[2] A. L. Dontchev and T. Zolezzi, Well-posed optimization problems, ser. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 1993, vol. 1543.

[3] T. Zolezzi, “Wellposed problems of the calculus of variations for nonconvex integrals,” J. Convex Anal., vol. 2, no. 1-2, pp. 375–383, 1995.

[4] H. Attouch and R. Wets, “Quantitative stability of variational systems. I. The epigraphical distance,” Trans. Amer. Math. Soc., vol. 328, no. 2, pp. 695–729, 1991.

[5] ——, “Quantitative stability of variational systems. II. A framework for nonlinear conditioning,” SIAM J. Optim., vol. 3, no. 2, pp. 359–381, 1993.

[6] G. Beer and R. Lucchetti, “Convex optimization and the epi-distance topology,” Trans. Amer. Math. Soc., vol. 327, no. 2, pp. 795–813, 1991.

[7] S. Villa, “AW-convergence and well-posedness of non convex functions,” J. Convex Anal., vol. 10, no. 2, pp. 351–364, 2003.

[8] M. A. Sychëv, “Necessary and sufficient conditions in theorems of semicontinuity and convergence with a functional,” Mat. Sb., vol. 186, no. 6, pp. 77–108, 1995.

[9] C. Olech, “The Lyapunov theorem: its extensions and applications,” in Methods of nonconvex analysis (Varenna, 1989), ser. Lecture Notes in Math. Berlin: Springer, 1990, vol. 1446, pp. 84–103.

[10] A. Visintin, “Strong convergence results related to strict convexity,” Comm. Partial Differential Equations, vol. 9, no. 5, pp. 439–466, 1984.

[11] A. Cellina and S. Zagatti, “A version of Olech’s lemma in a problem of the calculus of variations,” SIAM J. Control Optim., vol. 32, no. 4, pp. 1114–1127, 1994.

[12] A. Cellina and S. Perrotta, “On minima of radially symmetric functionals of the gradient,” Nonlinear Anal., vol. 23, no. 2, pp. 239–249, 1994.

[13] G. Crasta, “Existence, uniqueness and qualitative properties of minima to radially symmetric non-coercive non-convex variational problems,” Math. Z., vol. 235, no. 3, pp. 569–589, 2000.

[14] S. Villa, “Well-posedness of nonconvex integral functionals,” SIAM J. Control Optim., vol. 43, no. 4, pp. 1298–1312 (electronic), 2004/05.

[15] S. Bertirotti, “Wellposedness in the calculus of variations,” J. Convex Anal., vol. 7, no. 2, pp. 299–318, 2000.

[16] S. Walczak, “On the continuous dependence on parameters of solutions of the Dirichlet problem. II. The case of saddle points,” Acad. Roy. Belg. Bull. Cl. Sci. (6), vol. 6, no. 7-12, pp. 263–273, 1995.

[17] ——, “On the continuous dependence on parameters of solutions of the Dirichlet problem. I. Coercive case,” Acad. Roy. Belg. Bull. Cl. Sci. (6), vol. 6, no. 7-12, pp. 247–261, 1995.

[18] T. Zolezzi, “Well-posed optimization problems for integral functionals,” J. Optim. Theory Appl., vol. 31, no. 3, pp. 417–430, 1980.

[19] ——, “Well-posedness criteria in optimization with application to the calculus of variations,” Nonlinear Anal., vol. 25, no. 5, pp. 437–453, 1995.

[20] A. Cellina, “On minima of a functional of the gradient: sufficient conditions,” Nonlinear Anal., vol. 20, no. 4, pp. 343–347, 1993.

[21] ——, “On minima of a functional of the gradient: necessary conditions,” Nonlinear Anal., vol. 20, no. 4, pp. 337–341, 1993.

[22] G. Friesecke, “A necessary and sufficient condition for nonattainment and formation of microstructure almost everywhere in scalar variational problems,” Proc. Roy. Soc. Edinburgh Sect. A, vol. 124, no. 3, pp. 437–471, 1994.

[23] I. Ekeland and R. Temam, Convex analysis and variational problems. Amsterdam: North-Holland Publishing Co., 1976, translated from the French, Studies in Mathematics and its Applications, Vol. 1.