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Global Finite-Time Stabilization of a Class of Uncertain Nonlinear Systems Using Output Feedback

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Abstract— **In this paper, the problem of global finite-time stabilization by means of output feedback is considered for a class of uncertain nonlinear systems. To solve the problem, we first construct a homogenous observer and controller in a recursive way for a chain of integrators. Then, using the homogeneous domination approach [8], we scale the homogeneous observer and controller with an appropriate choice of gain to render the nonlinear systems globally finite-time stable.**

I. INTRODUCTION

To global stabilize a nonlinear system only using its measurable output has been widely recognized as a challenging problem due to the lack of nonlinear version of "separation principle." Over the past several decades, this problem has been attracting extensive attention from the nonlinear control community and a number of interesting results have been achieved. However, most of the existing works only consider the asymptotic output feedback stabilizer that makes the system states convergent to the equilibrium only when the time goes to the infinity.

In this paper, we focus on the problem of using output feedback to globally stabilize a class of uncertain nonlinear systems in *finite time*, namely, we are interested in finding an output feedback controller that will render the closed-loop systems convergent to the origin in finite time. Compared to the asymptotic stabilization via output feedback, the finite-time stabilization by output feedback is a relatively new problem. In fact, even in the case of finite-time stabilization using *state feedback*, there are very few results in the literature (see [2], [3], [4], [5], [12] and the references therein). In the case when part of the states is not measurable, to stabilize a nonlinear system in finite time only using limited measurable states becomes quite challenging. Recently, some attempts have been made to tackle this difficult problem. In [6], the finite-time stabilization of the double integrator systems was achieved by coupling a finite-time convergent observer with a finite-time control law. Later this result was applied to a robot system in [7]. When the planar system has uncontrollable/unobservable linearization, in [9] an output feedback controller was constructed to globally stabilize the system in finite-time.

Due to the difficulties in dealing with the output feedback finite-time stabilization, the results [6], [9] were developed for some lower-dimensional systems. For the higher-dimensional nonlinear systems, there are two major obstacles preventing us from constructing the global finite-time stabilizer via output feedback, namely: 1) the lack of constructive finite-time convergent observer; and 2) the presence of nonlinearities associated with uncertainties. To overcome the first obstacle, in this paper, we will extend the reduced-order finite-time convergent observer constructed in [9] to higher-dimensional systems, using the recursive design techniques presented in [11]. A homogeneous observer will be constructed step by step, which will be combined with the finite-time state feedback controller. To deal with the uncertain nonlinearities, we adopt a new design method called homogeneous domination approach [8], which will free us from the tedious process in handling the nonlinearities during the observer and controller design procedure. Specifically, the homogeneous domination approach will give us more flexibility in handling the nonlinearities by separating the design in two steps: 1) first a homogeneous output feedback controller is constructed for the nominal linear system; and 2) we then introduce a scaling gain into the homogeneous output feedback controller to dominate the high-order nonlinear functions. By the appropriate choice of the scaling gain, the closed-loop system can be rendered globally stable in finite time.

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II. PROBLEM STATEMENT AND PRELIMINARIES

Throughout this paper, we consider a class of uncertain nonlinear systems described by

$$\dot{\chi}_i = \chi_{i+1} + \phi_i(t, \chi, u), \; i = 1, \cdots, n, \tag{2.1}$$

where $y := \chi_1 \in \mathbb{R}$ and $u := \chi_{n+1} \in \mathbb{R}$ are the system output and control input, respectively. The unknown function $\phi_i(t, \chi, u)$ satisfies the following condition:

Assumption 2.1: For $i = 1, \cdots, n$, there is a constant $\tau \in (-\frac{1}{n}, 0)$ such that $\forall\, t \in \mathbb{R}$, $\chi \in \mathbb{R}^n$, $u \in \mathbb{R}$

$$|\phi_i(t, \chi, u)| \leq c\left(|\chi_1|^{i\tau+1} + \cdots + |\chi_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right), \text{for } c > 0.$$

In what follows, we first adopt a Lyapunov-like Theorem that has been frequently used to determine the finite-time stability of nonlinear systems.

Theorem 2.1: For a continuous system

$$\dot{x}(t) = f(x(t)), \; f(0) = 0, \tag{2.2}$$

suppose there exist a C^1 positive definite and proper function $V\colon \mathbb{R}^n \to \mathbb{R}$, and real numbers $k > 0$ and $\alpha \in (0, 1)$ such that $\dot{V} + kV^\alpha$ is negative semi-definite. Then the origin is a globally finite-time stable equilibrium of (2.2).

In the remainder of the section, we introduce several lemmas that serve as the basis for the development of an output feedback finite-time controller for (2.1). The first two lemmas are the key tools for adding a power integrator.

Lemma 2.1: ([10]) For $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $q \geq 1$ is a constant, the following inequalities hold:

$$|a+b|^q \leq 2^{q-1}|a^q + b^q|, \quad (2.3)$$
$$(|a|+|b|)^{\frac{1}{q}} \leq |a|^{\frac{1}{q}} + |b|^{\frac{1}{q}}. \quad (2.4)$$

If $q \geq 1$ is odd then

$$|a-b|^q \leq 2^{q-1}|a^q - b^q|, \quad (2.5)$$
$$|a^q - b^q| \leq q|a-b||a^{q-1} + b^{q-1}|. \quad (2.6)$$

Lemma 2.2: ([10]) Suppose n and m are two positive real numbers, and $a \geq 0$, $b \geq 0$ and $\pi \geq 0$ are continuous functions. Then, for any constant $c > 0$,

$$a^n b^m \pi \leq c \cdot a^{n+m} + \frac{m}{n+m}\left[\frac{n}{c(n+m)}\right]^{\frac{n}{m}} b^{n+m}\pi^{\frac{n+m}{m}}.$$

The next lemma will be used for $n-1$ times to select observer gains.

Lemma 2.3: ([9]) Let the real number $r \in (0,1)$ be a ratio of odd integers. Then the following inequality holds for any real numbers $0 < \varepsilon < 1$ and t

$$t^r + (1-t)^r + \varepsilon^2 t^{1+r} \geq (2^r - 1)\varepsilon^{1-r}. \quad (2.7)$$

The final lemma shows that a positive definite homogeneous function shares some properties analog to those of a quadratic Lyapunov function used in linear control theory.

Lemma 2.4: Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a homogenous function of degree τ with respect to the dilation weight $\Delta = \{r_1, \cdots, r_n\}$. Then the following holds:

1) $\dfrac{\partial V}{\partial x_i}$ is still homogeneous of degree $\tau - r_i$ with r_i being the homogeneous weight of x_i.
2) There is a constant c such that $V(x) \leq c\|x\|_\Delta^\tau$ where $\|x\|_\Delta^\tau = \sqrt{\sum_{i=1}^{n-1}\|x_i\|^{2/r_i}}$. Moreover, if $V(x)$ is positive definite, there is a positive constant $\underline{c}$, such that $\underline{c}\|x\|_\Delta^\tau \leq V(x)$.

III. Main Results

To construct a global output feedback controller for system (2.1), we will employ the homogeneous domination approach introduced in [8]. Specifically, we will first construct a homogeneous output feedback controller for the nominal linear system. Then, we utilize a scaling gain in the controller to dominate the nonlinearities.

As the first step of our design, we consider a linear system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \cdots,\ \dot{x}_n = v,\ y = x_1. \quad (3.1)$$

Theorem 3.1: For any constant $\tau \in (-\frac{1}{n}, 0)$, there is a homogeneous output feedback controller of degree τ rendering system (3.1) globally finite-time stable.

Proof: To prove the result, we first develop a recursive design method to construct a state feedback control law for system (3.1). Then, we construct a homogeneous observer with a set of constant gains to be determined later. Finally, the observer gains will be carefully selected to guarantee that the closed-loop system is globally finite-time stable.

For the simplicity, we assume that $\tau = -\frac{q}{p}$ with an even integer q and an odd integer p. We further denote

$$r_1 = 1, \quad r_i = r_{i-1} + \tau, \quad i = 2, \cdots, n+1. \quad (3.2)$$

I. State Feedback Controller Design

Initial Step: Construct a Lyapunov function $V_1 = \frac{x_1^2}{2}$. Clearly, the virtual controller $x_2^* = -nx_1^{r_2}$ renders

$$\dot{V}_1 \leq -nx_1^{2+\tau} + x_1(x_2 - x_2^*).$$

Inductive Step: Suppose at step $k-1$, there exist a Lyapunov function $V_{k-1}(x_1, \cdots, x_{k-1})$, which is positive definite and proper, and a set of virtual controllers $x_1^*, \cdots, x_k^*$, defined by

$$\begin{array}{lll} x_1^* &= 0 & \xi_1 = x_1^{1/r_1} - x_1^{*1/r_1}, \\ x_j^* &= -\xi_{j-1}^{r_j}\beta_{j-1} & \xi_j = x_j^{1/r_j} - x_j^{*1/r_j},\ j = 2, \cdots, k \end{array} \quad (3.3)$$

with constants $\beta_1 > 0, \cdots,\ \beta_{k-1} > 0$, such that

$$\begin{aligned} \dot{V}_{k-1} \quad \leq \quad & -(n-k+2)(\xi_1^{2+\tau} + \cdots + \xi_{k-1}^{2+\tau}) \\ & + \xi_{k-1}^{2-r_{k-1}}(x_k - x_k^*). \end{aligned} \quad (3.4)$$

In what follows, we will show that (3.4) still holds at step k. To prove the claim, we consider the Lyapunov function

$$V_k(x_1, \cdots, x_k) = V_{k-1}(x_1, \cdots, x_{k-1}) + W_k(x_1, \cdots, x_k)$$

$$W_k(x_1, \cdots, x_k) = \int_{x_k^*}^{x_k} (s^{1/r_k} - x_k^{*1/r_k})^{2-r_k} ds.$$

Hence, the time derivative of V_k is

$$\begin{aligned} \dot{V}_k \leq & -(n-k+2)(\xi_1^{2+\tau} + \cdots + \xi_{k-1}^{2+\tau}) + \xi_{k-1}^{2-r_{k-1}} \\ & \times (x_k - x_k^*) + \frac{\partial W_k}{\partial x_k}\dot{x}_k + \sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l. \end{aligned} \quad (3.5)$$

To estimate the second term in (3.5), using Lemmas 2.1 and 2.2, one has the following estimate:

$$\begin{aligned} \xi_{k-1}^{2-r_{k-1}}(x_k - x_k^*) \quad & \leq \quad |\xi_{k-1}|^{2-r_{k-1}} 2^{1-r_k}|\xi_k|^{r_k} \\ & \leq \quad \frac{1}{2}\xi_{k-1}^{2+\tau} + c_1\xi_k^{2+\tau} \end{aligned} \quad (3.6)$$

for a constant $c_1 > 0$.

To estimate the last term in (3.5), we introduce a proposition whose proof is included in the Appendix.

Proposition 1. There exists a positive constant c_2 such that

$$\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l \leq \frac{1}{2}(\xi_1^{2+\tau} + \cdots + \xi_{k-1}^{2+\tau}) + c_2\xi_k^{2+\tau}. \quad (3.7)$$

Substituting (3.6) and (3.7) into (3.5) yields

$$\begin{aligned} \dot{V}_k \quad \leq \quad & -(n-k+1)(\xi_1^{2+\tau} + \cdots + \xi_{k-1}^{2+\tau}) \\ & + (c_1 + c_2)\xi_k^{2+\tau} + \xi_k^{2-r_k}x_{k+1}. \end{aligned}$$

Clearly, the virtual controller of the form $x_{k+1}^* = -\beta_k\xi_k^{r_{k+1}} = -(n-k+1+c_1+c_2)\xi_k^{r_{k+1}}$ yields

$$\begin{aligned} \dot{V}_k \leq & -(n-k+1)(\xi_1^{2+\tau} + \cdots + \xi_k^{2+\tau}) \\ & + \xi_k^{2-r_k}(x_{k+1} - x_{k+1}^*). \end{aligned} \quad (3.8)$$

This completes the inductive proof.

Therefore, (3.8) is true for $k = 1, 2, \cdots, n$. In other words, when $k = n$, there is a controller $x_{n+1}^* = -\beta_n \xi_n^{r_{n+1}}(x)$ and a Lyapunov function $V_n(x_1, \cdots, x_n)$ such that

$$\dot{V}_n \leq -(\xi_1^{2+\tau} + \cdots + \xi_n^{2+\tau}) + \xi_n^{2-r_n}(v - x_{n+1}^*). \quad (3.9)$$

II. Construction of a Homogeneous Observer

Next, we construct a homogeneous observer

$$\begin{aligned} \dot{\hat{z}}_i &= -\ell_{i-1}\left[\hat{z}_i + \ell_{i-1}\hat{x}_{i-1}\right]^{r_i/r_{i-1}}, \\ \hat{x}_i &= \left[\hat{z}_i + \ell_{i-1}\hat{x}_{i-1}\right]^{r_i/r_{i-1}}, i = 2, \cdots, n \quad (3.10) \end{aligned}$$

with $\hat{x}_1 = x_1$ and gains $l_i > 0, 1 \leq i \leq n-1$ to be determined later. Define $e_i = z_i - \hat{z}_i$ where $z_i = x_i^{r_{i-1}/r_i} - \ell_{i-1}x_{i-1},\ i = 2, \cdots, n$. Hence,

$$e_i = z_i - \hat{z}_i = x_i^{\frac{r_{i-1}}{r_i}} - \hat{x}_i^{\frac{r_{i-1}}{r_i}} - \ell_{i-1}\left(x_{i-1} - \hat{x}_{i-1}\right).$$

A direct calculation yields

$$\dot{e}_i = \frac{r_{i-1}}{r_i} x_i^{\frac{r_{i-1}}{r_i}-1} x_{i+1} - \ell_{i-1}\left(x_i - \hat{x}_i\right) \quad (3.11)$$

where $i = 2, \cdots, n$ and $x_{n+1} = v$.

Construct the following Lyapunov function that is apparently positive definite and proper

$$W(e_2, \cdots, e_n) = \frac{e_2^2}{2} + \frac{e_3^{2/r_2}}{2/r_2} + \cdots + \frac{e_n^{2/r_{n-1}}}{2/r_{n-1}}. \quad (3.12)$$

Clearly, the derivative of (3.12) along (3.11) is

$$\begin{aligned} \dot{W} &= \sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1}\left(\frac{r_{i-1}}{r_i} x_i^{\frac{r_{i-1}}{r_i}-1} x_{i+1}\right) \\ &\quad - \sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1} \ell_{i-1}(x_i - \hat{x}_i). \quad (3.13) \end{aligned}$$

To estimate the terms in (3.13), we introduce the following propositions whose proofs are included in the Appendix.

Proposition 2. There exists a positive constant c_3 such that

$$\begin{aligned} &\sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1}\left(\frac{r_{i-1}}{r_i} x_i^{\frac{r_{i-1}}{r_i}-1} x_{i+1}\right) \\ &\leq \sum_{i=2}^{n} c_3 |e_i|^{\frac{2+\tau}{r_{i-1}}} + \frac{1}{4}(\xi_1^{2+\tau} + \cdots + \xi_n^{2+\tau}). \quad (3.14) \end{aligned}$$

Proposition 3. There exist positive constants c_4, c_5, m, and a $\mathcal{K}_\infty$ function $f(\cdot)$ such that

$$\begin{aligned} &-\sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1} \ell_{i-1}(x_i - \hat{x}_i) \\ &\leq (-f(\ell_{n-1}) + m) e_n^{\frac{2+\tau}{r_{n-1}}} + (-f(\ell_{n-2}) + m) e_{n-1}^{\frac{2+\tau}{r_{n-2}}} \\ &\quad + \cdots + (-f(\ell_1) + m) e_2^{\frac{2+\tau}{r_1}} + c_5 \ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n + r_{n-1})^2}} \xi_n^{2+\tau} \\ &\quad + c_5 \left(\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n + r_{n-1})^2}} + \ell_{n-2}^{\frac{\tau(2+\tau)}{(r_{n-1} + r_{n-2})^2}}\right) \xi_{n-1}^{2+\tau} \\ &\quad + \cdots + c_5 \ell_1^{\frac{\tau(2+\tau)}{(r_2 + r_1)^2}} \xi_2^{2+\tau} \\ &\quad + c_4 \sum_{i=2}^{n} |e_i^{\frac{2}{r_{i-1}}-1}| \ell_{i-1} |\ell_{i-1}(x_{i-1} - \hat{x}_{i-1})|^{\frac{r_i}{r_{i-1}}}. \end{aligned}$$

Substituting Propositions 2 and 3 into (3.13), we have

$$\begin{aligned} &\dot{W}(e_2, \cdots, e_n) \\ &\leq -\hat{f}(\ell_{n-1}) e_n^{\frac{2+\tau}{r_{n-1}}} - \hat{f}(\ell_{n-2}) e_{n-1}^{\frac{2+\tau}{r_{n-2}}} + \cdots \\ &\quad - \hat{f}(\ell_1) e_2^{\frac{2+\tau}{r_1}} + \left(\frac{1}{4} + c_5 \ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n + r_{n-1})^2}}\right) \xi_n^{2+\tau} \\ &\quad + \left(\frac{1}{4} + c_5 \ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n + r_{n-1})^2}} + c_5 \ell_{n-2}^{\frac{\tau(2+\tau)}{(r_{n-1} + r_{n-2})^2}}\right) \xi_{n-1}^{2+\tau} \\ &\quad + \cdots + \left(\frac{1}{4} + c_5 \ell_1^{\frac{\tau(2+\tau)}{(r_2 + r_1)^2}}\right) \xi_2^{2+\tau} + c_4 \sum_{i=2}^{n} |e_i|^{\frac{2}{r_{i-1}}-1} \\ &\quad \times \ell_{i-1} |\ell_{i-1}(x_{i-1} - \hat{x}_{i-1})|^{\frac{r_i}{r_{i-1}}} \quad (3.15) \end{aligned}$$

where $-\hat{f}(\ell_i) = -f(\ell_i) + m + c_3, i = 1, \cdots n-1$.

III. Recursive Determination of the Observer Gains

Since $x_2, \cdots, x_n$ are not measurable, we use $\hat{x}_i,\ i = 2, \cdots, n$ to construct the following controller

$$\begin{aligned} v(\hat{x}) &= -\beta_n \xi_n{}^{r_n+\tau}(\hat{x}) = -\beta_n(\hat{x}_n^{\frac{1}{r_n}} + \beta_{n-1}^{\frac{1}{r_{n-1}}}(\hat{x}_{n-1}^{\frac{1}{r_{n-1}}} \\ &\quad + \cdots + \beta_2^{\frac{1}{r_2}}(\hat{x}_2^{\frac{1}{r_2}} + \beta_1 x_1)))^{r_n+\tau} \quad (3.16) \end{aligned}$$

To estimate the redundant term $\xi_n^{2-r_n}(v(\hat{x}) - x_{n+1}^*)$ in (3.9), we first use Lemma 2.1 to have the following estimate

$$\begin{aligned} &|\xi_n^{2-r_n}(v(\hat{x}) - x_{n+1}^*)| \\ &\leq c_6 |\xi_n^{2-r_n}| |\xi_n(\hat{x}) - \xi_n(x)|^{r_n+\tau} \\ &= c_6 |\xi_n^{2-r_n}| |\hat{x}_n^{\frac{1}{r_n}} - x_n^{\frac{1}{r_n}} + \bar{\beta}_{n-1}(\hat{x}_{n-1}^{\frac{1}{r_{n-1}}} - x_{n-1}^{\frac{1}{r_{n-1}}}) \\ &\quad + \cdots + \bar{\beta}_2(\hat{x}_2^{\frac{1}{r_2}} - x_2^{\frac{1}{r_2}})|^{r_n+\tau} \\ &\leq c_7 |\xi_n|^{2-r_n}(|\hat{x}_n^{\frac{1}{r_n}} - x_n^{\frac{1}{r_n}}|^{r_n+\tau} \\ &\quad + |\hat{x}_{n-1}^{\frac{1}{r_{n-1}}} - x_{n-1}^{\frac{1}{r_{n-1}}}|^{r_n+\tau} + \cdots + |\hat{x}_2^{\frac{1}{r_2}} - x_2^{\frac{1}{r_2}}|^{r_n+\tau}) \end{aligned}$$
(3.17)

where $\bar{\beta}_2, \cdots, \bar{\beta}_{n-1}, c_6$ and c_7 are positive constants.

By (2.6), one has

$$\begin{aligned} &c_7 |\xi_n|^{2-r_n} |\hat{x}_i^{\frac{1}{r_i}} - x_i^{\frac{1}{r_i}}|^{r_n+\tau} \\ &\leq c_8 |\xi_n|^{2-r_n} \left[|x_i - \hat{x}_i| |x_i^{\frac{1}{r_i}-1} + \hat{x}_i^{\frac{1}{r_i}-1}|\right]^{r_n+\tau} \\ &\leq c_9 |\xi_n|^{2-r_n} [|x_i - \hat{x}_i| (|x_i|^{\frac{1}{r_i}-1} \\ &\quad + |x_i - \hat{x}_i|^{\frac{1}{r_i}-1})]^{r_n+\tau} \quad (3.18) \end{aligned}$$

for constants c_8 and c_9. By the fact that $x_i^{\frac{1}{r_i}-1} = (\xi_i - \tilde{\beta}_{i-1}\xi_{i-1})^{1-r_i}$, we can apply Lemma 2.2 to (3.18)

$$\begin{aligned} &c_7 |\xi_n|^{2-r_n} |\hat{x}_i^{\frac{1}{r_i}} - x_i^{\frac{1}{r_i}}|^{r_n+\tau} \\ &\leq c_{10} |x_i - \hat{x}_i|^{\frac{2+\tau}{r_i}} + \frac{1}{8}(\xi_i^{2+\tau} + \xi_{i-1}^{2+\tau}) + \frac{1}{8n}\xi_n^{2+\tau} \end{aligned}$$

for a constant c_{10}. With this in mind, (3.17) becomes

$$\begin{aligned} |\xi_n^{2-r_n}(v(\hat{x}) - x_{n+1}^*)| &\leq \frac{1}{4}(\xi_1^{2+\tau} + \cdots + \xi_n^{2+\tau}) \\ &\quad + c_{10} \sum_{i=2}^{n} |x_i - \hat{x}_i|^{\frac{2+\tau}{r_i}} \quad (3.19) \end{aligned}$$

Putting (3.9), (3.15) and (3.19) together yields

$$\begin{aligned}\dot{U} \le & -\hat{f}(\ell_{n-1})e_n^{\frac{2+\tau}{r_{n-1}}} - \hat{f}(\ell_{n-2})e_{n-1}^{\frac{2+\tau}{r_{n-2}}} + \cdots \\ & -\hat{f}(\ell_1)e_2^{\frac{2+\tau}{r_1}} - \left(\frac{1}{2} - c_5\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n+r_{n-1})^2}}\right)\xi_n^{2+\tau} \\ & -\left(\frac{1}{2} - c_5\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n+r_{n-1})^2}} - c_5\ell_{n-2}^{\frac{\tau(2+\tau)}{(r_{n-1}+r_{n-2})^2}}\right)\xi_{n-1}^{2+\tau} \\ & -\cdots - \left(\frac{1}{2} - c_5\ell_1^{\frac{\tau(2+\tau)}{(r_2+r_1)^2}}\right)\xi_1^{2+\tau} + c_4\sum_{i=2}^{n}|e_i|^{\frac{2}{r_{i-1}}-1} \\ & \times\ell_{i-1}|\ell_{i-1}(x_{i-1}-\hat{x}_{i-1})|^{\frac{r_i}{r_{i-1}}} + c_{10}\sum_{i=2}^{n}|x_i-\hat{x}_i|^{\frac{2+\tau}{r_i}}\end{aligned} \tag{3.20}$$

where the Lyapunov function $U = V_n + W$.

By definition of e_i and Lemma 2.1, we have

$$|x_i - \hat{x}_i| \le 2^{1-\frac{r_i}{r_{i-1}}}\left(|e_i| + \ell_{i-1}|x_{i-1}-\hat{x}_{i-1}|\right)^{\frac{r_i}{r_{i-1}}}.$$

Applying the similar argument to $|x_{i-1} - \hat{x}_{i-1}|$ yields

$$\begin{aligned}|x_i - \hat{x}_i| \le\ & 2^{1-\frac{r_i}{r_{i-1}}}(|e_i| + \ell_{i-1}2^{1-\frac{r_{i-1}}{r_{i-2}}}[|e_{i-1}| \\ & +\ell_{i-2}|x_{i-2}-\hat{x}_{i-2}|]^{\frac{r_{i-1}}{r_{i-2}}})^{\frac{r_i}{r_{i-1}}}.\end{aligned}$$

Following the same line to $|x_{i-2} - \hat{x}_{i-2}|, \cdots, |x_2 - \hat{x}_2|$,

$$\begin{aligned}|x_i - \hat{x}_i| \le\ & \lambda|e_i|^{\frac{r_i}{r_{i-1}}} + \rho_{i,i-1}(\ell_{i-1})|e_{i-1}|^{\frac{r_i}{r_{i-2}}} + \cdots \\ & +\rho_{i,2}(\ell_{i-1},\cdots,\ell_2)|e_2|^{\frac{r_i}{r_1}}\end{aligned} \tag{3.21}$$

where λ is a positive constant and $\rho_{i,j}(\cdot)$ is a polynomial function of its variables.

By using (3.21) and Lemma 2.2, it is clear that

$$\begin{aligned}& c_4\sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1}\ell_{i-1}|\ell_{i-1}(x_{i-1}-\hat{x}_{i-1})|^{\frac{r_i}{r_{i-1}}} \\ & +c_{10}\sum_{i=2}^{n}|x_i-\hat{x}_i|^{\frac{2+\tau}{r_i}} \\ & \le \mu e_n^{\frac{2+\tau}{r_{n-1}}} + \hat{\rho}_{n-1}(\ell_{n-1})e_{n-1}^{\frac{2+\tau}{r_{n-2}}} \\ & \quad +\hat{\rho}_{n-2}(\ell_{n-1},\ell_{n-2})e_{n-2}^{\frac{2+\tau}{r_{n-3}}} + \cdots \\ & \quad +\hat{\rho}_2(\ell_{n-1},\cdots,\ell_2)e_2^{\frac{2+\tau}{r_1}}\end{aligned} \tag{3.22}$$

where $\hat{\rho}_i(\ell_{n-1},\cdots,\ell_i), i = 2,\cdots,n-1$ are continuous functions of $\ell_i,\cdots,\ell_{n-1}$ and μ is a positive constant.

Applying (3.22) to (3.20), we have

$$\begin{aligned}\dot{U} \le & -(\hat{f}(\ell_{n-1})-\mu)e_n^{\frac{2+\tau}{r_{n-1}}} - (\hat{f}(\ell_{n-2}) - \\ & \hat{\rho}_{n-1}(\ell_{n-1}))e_{n-1}^{\frac{2+\tau}{r_{n-2}}} - \cdots - (\hat{f}(\ell_1) - \hat{\rho}_2(\ell_{n-1}, \\ & \cdots,\ell_2))e_2^{\frac{2+\tau}{r_1}} - \left(\frac{1}{2} - c_5\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n+r_{n-1})^2}}\right)\xi_n^{2+\tau} \\ & -\left(\frac{1}{2} - c_5\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n+r_{n-1})^2}} - c_5\ell_{n-2}^{\frac{\tau(2+\tau)}{(r_{n-1}+r_{n-2})^2}}\right)\xi_{n-1}^{2+\tau} \\ & -\cdots - \left(\frac{1}{2} - c_5\ell_1^{\frac{\tau(2+\tau)}{(r_2+r_1)^2}}\right)\xi_1^{2+\tau}.\end{aligned} \tag{3.23}$$

With the help of (3.23), we are ready to choose appropriate ℓ_i. First, we select a gain ℓ_{n-1} such that $\hat{f}(\ell_{n-1}) - \mu \ge 1/4$. Because τ is negative, we can also make gain ℓ_{n-1} large enough to satisfy $c_5\ell_{n-1}^{\tau(2+\tau)/(r_n+r_{n-1})^2} \le 1/8$.

By now we have fixed ℓ_{n-1}. Next, we choose ℓ_{n-2} which is large enough to satisfy the following conditions

$$\begin{aligned}\hat{f}(\ell_{n-2}) - \hat{\rho}_{n-1}(\ell_{n-1}) &\ge 1/4, \\ c_5\ell_{n-2}^{\frac{\tau(2+\tau)}{(r_{n-1}+r_{n-2})^2}} &\le 1/8.\end{aligned}$$

Following the same line, at step $i = n-2,\cdots,1$, we choose ℓ_i based on the fixed gains $\ell_{i+1}\cdots,\ell_{n-1}$

$$\begin{aligned}\hat{f}(\ell_i) - \hat{\rho}_{i+1}(\ell_{n-1},\cdots,\ell_{i+1}) &\ge 1/4, \\ c_5\ell_i^{\frac{\tau(2+\tau)}{(r_{i+1}+r_i)^2}} &\le 1/8.\end{aligned}$$

Under the above choice of gains, it is easy to verify that

$$\dot{U} \le -\frac{1}{4}(\xi_1^{2+\tau} + \cdots + \xi_n^{2+\tau} + e_2^{\frac{2+\tau}{r_1}} + \cdots + e_n^{\frac{2+\tau}{r_{n-1}}}). \tag{3.24}$$

The construction of U indicates that U is positive definite and proper with respect to

$$(x_1,\cdots,x_n,\hat{z}_2,\cdots,\hat{z}_n)^T =: X. \tag{3.25}$$

Moreover, it is straightforward to verify that the closed-loop system (3.1)-(3.10)-(3.16), which can be rewritten as the following compact form

$$\dot{X} = F(X) = (x_2,\cdots,x_n,v,\dot{\hat{z}}_2,\cdots,\dot{\hat{z}}_n)^T, \tag{3.26}$$

is homogeneous of degree τ with the dilation weight

$$\Delta = (\underbrace{1,\cdots,(n-1)\tau+1,}_{\text{for } x_1,\cdots,x_n}\ \underbrace{1,\cdots,(n-2)\tau+1}_{\text{for } \hat{z}_2,\cdots,\hat{z}_n}).$$

In addition, it can be shown that U is homogeneous of degree 2. By Lemma 2.4, there is a constant $\bar{c}_2$ such that

$$U(X) \le \bar{c}_2\|X\|_\Delta^2 \tag{3.27}$$

where $\|X\|_\Delta = \sqrt{\sum_{i=1}^{2n-1}\|X_i\|^{2/r_i}}$. Similarly, since the right hand side of (3.24) is homogeneous of degree $2+\tau$, by Lemma 2.4 there is a constant $\bar{c}_1 > 0$ such that

$$\frac{\partial U}{\partial X}F(X) \le -\bar{c}_1\|X\|_\Delta^{2+\tau}. \tag{3.28}$$

Putting (3.27) and (3.28) together yields $\dot{U} + kU^{\frac{2+\tau}{2}} \le 0$ for a positive constant $k > 0$. Therefore, by Theorem 2.1 with $\alpha = \frac{2+\tau}{2} < 1$ the closed-loop system (3.1)-(3.10)-(3.16) is globally finite-time stable. ∎

Remark 3.1: In the case when τ is any negative real number, we are still able to design a homogenous controller globally stabilizing the system (3.1) with necessary modification to preserve the sign of function $[\cdot]^{r_i}$. Specifically, for any real number $r_i > 0$, we define $[\cdot]^{r_i} = \text{sign}(\cdot)|\cdot|^{r_i}$. With the help of this function, we are able to design the controller without requiring even numerator of τ.

Based on the homogeneous output feedback controller constructed in Theorem 3.1, we are now ready to design the output feedback finite-time stabilizer for system (2.1).

Theorem 3.2: Under Assumption 2.1, there is an output feedback controller rendering system (2.1) globally finite-time stable.

Proof: Under the new coordinates $x_i = \frac{\chi_i}{L^{i-1}}$, $i = 1, \cdots, n$, $v = \frac{u}{L^n}$ with $L > 1$, (2.1) can be rewritten as

$$\begin{aligned} \dot{x}_i &= Lx_{i+1} + \phi_i(\cdot)/L^{i-1}, \; i = 1, \cdots, n-1, \\ \dot{x}_n &= Lv + \phi_n(\cdot)/L^{n-1}. \end{aligned} \qquad (3.29)$$

Next, we construct the observer with a scaling gain L

$$\begin{aligned} \dot{\hat{z}}_i &= -L\ell_{i-1}\left[\hat{z}_i + \ell_{i-1}\hat{x}_{i-1}\right]^{r_i/r_{i-1}}, \\ \hat{x}_i &= \left[\hat{z}_i + \ell_{i-1}\hat{x}_{i-1}\right]^{r_i/r_{i-1}}, i = 2, \cdots, n. \end{aligned} \qquad (3.30)$$

We design v using the construction of (3.16). Specifically

$$\begin{aligned} v(\hat{x}) &= -\beta_n(\hat{x}_n^{\frac{1}{r_n}} + \beta_{n-1}^{\frac{1}{r_{n-1}}}(\hat{x}_{n-1}^{\frac{1}{r_{n-1}}} + \cdots \\ &\quad + \beta_2^{\frac{1}{r-2}}(\hat{x}_2^{\frac{1}{r_2}} + \beta_1 x_1)))^{r_n+\tau}. \end{aligned} \qquad (3.31)$$

Hence, adopting the same Lyapunov function U used in Theorem 3.1, it can be concluded from (3.28) that

$$\dot{U} \leq -L\bar{c}_1\|X\|_{\Delta}^{2r_n} + \frac{\partial U}{\partial X}(\phi_1(\cdot), \cdots, \frac{\phi_n(\cdot)}{L^{n-1}}, 0 \cdots, 0)^T. \qquad (3.32)$$

Under the changes of coordinates $\chi_i = L^{i-1}x_i$ and $u = L^n v$, we deduce from Assumption 2.1 the following relation

$$|\phi_i(t, \chi, u)| \leq c\left(|x_1|^{\frac{i\tau+1}{1}} + \cdots + |L^{i-1}x_i|^{\frac{i\tau+1}{(i-1)\tau+1}}\right).$$

Due to the fact that $L > 1$, we can conclude that

$$\left|\frac{\phi_i(.)}{L^{i-1}}\right| \leq cL^{1-\frac{1}{(i-1)\tau+1}}(|x_1|^{\frac{r_i+\tau}{r_1}} + \cdots + |x_i|^{\frac{r_i+\tau}{r_i}}). \qquad (3.33)$$

Recall that for $i = 1, \cdots, n-1$, $\dfrac{\partial U}{\partial X_i}$ is homogeneous of degree $2 - r_i$. By Lemma 2.4, we know that

$$\left|\frac{\partial U}{\partial X_i}\right|\left(|x_1|^{\frac{r_i+\tau}{r_1}} + |x_2|^{\frac{r_i+\tau}{r_2}} + \cdots + |x_i|^{\frac{r_i+\tau}{r_i}}\right) \qquad (3.34)$$

is homogeneous of degree $2 + \tau$.

With (3.33) and (3.34) in mind, by Lemma 2.4 we can find a positive constant m_i such that

$$\frac{\partial U}{\partial X_i}\frac{\phi_i(\cdot)}{L^{i-1}} \leq m_i L^{1-\frac{1}{(i-1)\tau+1}}\|X\|_{\Delta}^{2+\tau}. \qquad (3.35)$$

Substituting (3.35) into (3.32) yields

$$\dot{U} \leq -L(\bar{c}_1 - \sum_{i=1}^{n} m_i L^{-\frac{1}{(i-1)\tau+1}})\|X\|_{\Delta}^{2+\tau}. \qquad (3.36)$$

Apparently, when L is large enough the right hand side of (3.36) is negative definite. Moreover, using the same argument as in the proof of Theorem 3.1, we have

$$\dot{U} + k_1 U^{\frac{2+\tau}{2}} \leq 0, \text{ for a constant } k_1 > 0.$$

Thus, the closed-loop system is globally finite-time stable. ∎

Example 3.1: Consider a 3-dimensional system

$$\begin{aligned} \dot{x}_1 &= x_2, \; \dot{x}_2 = x_3 + d(t)\sin(x_2), \\ \dot{x}_3 &= u + \ln(1 + x_3^2)/8, \; |d(t)| < 1. \end{aligned} \qquad (3.37)$$

It is easy to see that Assumption 2.1 holds for $\tau = -\frac{2}{13}$, $c = 1$. Therefore, by Theorem 3.2, we can construct an output feedback controller. Specifically, we choose

$$\begin{aligned} \dot{\hat{z}}_2 &= -L\ell_1\left[\hat{z}_2 + \ell_1 x_1\right]^{r_2/r_1}, \dot{\hat{z}}_3 = -L\ell_2\left[\hat{z}_3 + \ell_2\hat{x}_2\right]^{r_3/r_2} \\ u &= -L^3 b_3(\hat{x}_3^{\frac{1}{r_3}} + b_2(\hat{x}_2^{\frac{1}{r_2}} + b_1 x_1)))^{r_3+\tau} \end{aligned} \qquad (3.38)$$

A large enough gain L will enable controller (3.38) to render the system (3.37) globally finite-time stable.

The simulation is shown in Figure 1.

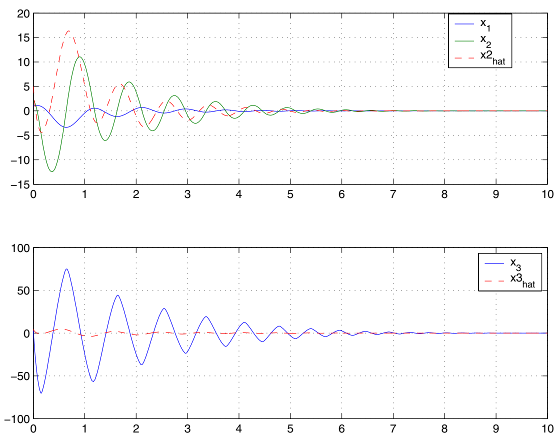

Figure 1: $(x_1, x_2, x_3, \hat{z}_2, \hat{z}_3)(0) = (1, 2, 3, 5, 4)$ and $L = 4, l_1 = 5, l_2 = 2, b_1 = 0.5, b_2 = 1, b_3 = 2$.

IV. Conclusions

This paper addresses the problem of global finite-time stabilization by output feedback for a class of uncertain nonlinear systems. A global finite-time stabilizer by output feedback was constructed using the homogeneous domination approach. Specifically, we first constructed a homogenous output feedback controller for the nominal linear system. Then, a scaling gain was introduced into the homogeneous output feedback controller to dominate the uncertain nonlinear functions. The appropriate choice of the scaling gain renders the closed-loop systems globally finite-time stable.

Appendix

This section collects the proofs of Propositions 1-3.

Proof of Proposition 1: By the construction of W_k, we have

$$\begin{aligned} &\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l \\ &= \sum_{l=1}^{k-1}(r_k - 2)\frac{\partial x_k^{*\frac{1}{r_k}}}{\partial x_l}\int_{x_k^*}^{x_k}(s^{1/r_k} - x_k^{*1/r_k})^{1-r_k}ds\dot{x}_l \\ &\leq a_k|\xi_k|\left|\sum_{l=1}^{k-1}\frac{\partial(\hat{\beta}_k x_{k-1}^{\frac{1}{r_{k-1}}} + \cdots + \hat{\beta}_2 x_1^{\frac{1}{r_1}})}{\partial x_l}x_{l+1}\right| \\ &\leq \hat{a}_k|\xi_k|\left|\sum_{l=1}^{k-1}x_l^{\frac{1}{r_l}-1}x_{l+1}\right| \end{aligned} \qquad (A.1)$$

where $a_k, \hat{a}_k, \hat{\beta}_k, \cdots \hat{\beta}_2$ are positive constants.

Note that by the change of coordinates (3.3)

$$x_{i+1} = (\xi_{i+1} - \beta_i^{\frac{1}{r_{i+1}}}\xi_i)^{r_{i+1}} \leq h(|\xi_{i+1}| + |\xi_i|)^{r_{i+1}} \quad \text{(A.2)}$$

for a positive constant h. By using (A.2) and Lemma 2.2,

$$\begin{aligned} \sum_{l=1}^{k-1} \frac{\partial W_k}{\partial x_l}\dot{x}_l &\leq \breve{a}_k|\xi_k| \sum_{l=1}^{k-1} (|\xi_l| + |\xi_{l-1}|)^{1-r_l}(|\xi_{l+1}| + |\xi_l|)^{r_{l+1}} \\ &\leq \frac{1}{2}(\xi_1^{2+\tau} + \cdots + \xi_{k-1}^{2+\tau}) + c_2\xi_k^{2+\tau}. \end{aligned}$$

where $\breve{a}_k$ and c_2 are positive constants.

Proof of Proposition 2: By using (A.2) and Lemma 2.2,

$$\begin{aligned} &\sum_{i=2}^{n} e_i^{\frac{2-r_{i-1}}{r_{i-1}}} \left(\frac{r_{i-1}}{r_i} x_i^{\frac{r_{i-1}}{r_i}-1} x_{i+1}\right) \\ &\leq \bar{h}\sum_{i=2}^{n} |e_i|^{\frac{2-r_{i-1}}{r_{i-1}}} (|\xi_i| + |\xi_{i-1}|)^{-\tau}(|\xi_{i+1}| + |\xi_i|)^{r_{i+1}} \\ &\leq \sum_{i=2}^{n} c_3|e_i|^{\frac{2+\tau}{r_{i-1}}} + \frac{1}{4}(\xi_1^{2+\tau} + \cdots + \xi_n^{2+\tau}) \end{aligned}$$

where $\bar{h}$ and c_3 are positive constants.

Proof of Proposition 3: By definition of z_i, $\hat{z}_i$ and (2.5)

$$\begin{aligned} &-\sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1} \ell_{i-1}(x_i - \hat{x}_i) \\ &= -\sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1} \ell_{i-1}[(z_i + \ell_{i-1}x_{i-1})^{\frac{r_i}{r_{i-1}}} \\ &\quad -(\hat{z}_i + \ell_{i-1}x_{i-1})^{\frac{r_i}{r_{i-1}}}] - \sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1}\ell_{i-1} \\ &\quad \times \left[(\hat{z}_i + \ell_{i-1}x_{i-1})^{\frac{r_i}{r_{i-1}}} - (\hat{z}_i + \ell_{i-1}\hat{x}_{i-1})^{\frac{r_i}{r_{i-1}}}\right] \\ &\leq -\sum_{i=2}^{n} e_i^{\frac{2-r_{i-1}}{r_{i-1}}} \ell_{i-1}\left[(x_i^{\frac{r_{i-1}}{r_i}})^{\frac{r_i}{r_{i-1}}} - (x_i^{\frac{r_{i-1}}{r_i}} - e_i)^{\frac{r_i}{r_{i-1}}}\right] \\ &\quad + c_4\sum_{i=2}^{n} |e_i|^{\frac{2}{r_{i-1}}-1}\ell_{i-1}|\ell_{i-1}(x_{i-1} - \hat{x}_{i-1})|^{\frac{r_i}{r_{i-1}}} \quad \text{(A.3)} \end{aligned}$$

for a positive constant c_4. When $e_i \neq 0$, substituting $t = \frac{x_i^{\frac{r_{i-1}}{r_i}}}{e_i}$, $\varepsilon = \ell_{i-1}^{\frac{-1}{1+\frac{r_i}{r_{i-1}}}}$ and $r = \frac{r_i}{r_{i-1}}$ into (2.7) yields

$$\begin{aligned} &-\sum_{i=2}^{n} e_i^{\frac{2-r_{i-1}}{r_{i-1}}} \ell_{i-1}\left[(x_i^{\frac{r_{i-1}}{r_i}})^{\frac{r_i}{r_{i-1}}} - (x_i^{\frac{r_{i-1}}{r_i}} - e_i)^{\frac{r_i}{r_{i-1}}}\right] \\ &\leq \sum_{i=2}^{n} \ell_{i-1}^{\frac{\tau}{r_i+r_{i-1}}} x_i^{1+\frac{r_{i-1}}{r_i}} e_i^{\frac{2-2r_{i-1}}{r_{i-1}}} - \sum_{i=2}^{n} f(\ell_{i-1})e_i^{\frac{\tau+2}{r_{i-1}}} \end{aligned} \quad \text{(A.4)}$$

where $f(\ell_{i-1}) = (2^{\frac{r_i}{r_{i-1}}} - 1)\ell_{i-1}^{\frac{2r_i}{r_i+r_{i-1}}}$. In addition, when $e_i = 0$, (A.4) holds automatically. By using (A.2) and Lemma 2.2 $(r_{i-1} + r_i + 2 - 2r_{i-1} = 2 + \tau)$, we have

$$\begin{aligned} &\sum_{i=2}^{n} \ell_{i-1}^{\frac{\tau}{r_i+r_{i-1}}} x_i^{1+\frac{r_{i-1}}{r_i}} e_i^{\frac{2-2r_{i-1}}{r_{i-1}}} \\ &\leq h'\sum_{i=2}^{n} \ell_{i-1}^{\frac{\tau}{r_i+r_{i-1}}} (|\xi_i| + |\xi_{i-1}|)^{r_{i-1}+r_i} e_i^{\frac{2-2r_{i-1}}{r_{i-1}}} \\ &\leq m(e_2^{\frac{2+\tau}{r_1}} + \cdots + e_n^{\frac{2+\tau}{r_{n-1}}}) + c_5\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n+r_{n-1})^2}}\xi_n^{2+\tau} \\ &\quad + c_5\left(\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n+r_{n-1})^2}} + \ell_{n-2}^{\frac{\tau(2+\tau)}{(r_{n-1}+r_{n-2})^2}}\right)\xi_{n-1}^{2+\tau} \\ &\quad + \cdots + c_5\ell_1^{\frac{\tau(2+\tau)}{(r_2+r_1)^2}}\xi_2^{2+\tau} \end{aligned} \quad \text{(A.5)}$$

where h', m and c_5 are positive constants.

Combining (A.3), (A.4) and (A.5), we have

$$\begin{aligned} &-\sum_{i=2}^{n} e_i^{\frac{2}{r_{i-1}}-1}\ell_{i-1}(x_i - \hat{x}_i) \\ &\leq (-f(\ell_{n-1}) + m)e_n^{\frac{2+\tau}{r_{n-1}}} + (-f(\ell_{n-2}) + m)e_{n-1}^{\frac{2+\tau}{r_{n-2}}} \\ &\quad + \cdots + (-f(\ell_1) + m)e_2^{\frac{2+\tau}{r_1}} + c_5\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n+r_{n-1})^2}}\xi_n^{2+\tau} \\ &\quad + c_5\left(\ell_{n-1}^{\frac{\tau(2+\tau)}{(r_n+r_{n-1})^2}} + \ell_{n-2}^{\frac{\tau(2+\tau)}{(r_{n-1}+r_{n-2})^2}}\right)\xi_{n-1}^{2+\tau} \\ &\quad + \cdots + c_5\ell_1^{\frac{\tau(2+\tau)}{(r_2+r_1)^2}}\xi_2^{2+\tau} \\ &\quad + c_4\sum_{i=2}^{n} |e_i|^{\frac{2}{r_{i-1}}-1}\ell_{i-1}|\ell_{i-1}(x_{i-1} - \hat{x}_{i-1})|^{\frac{r_i}{r_{i-1}}}. \end{aligned}$$

REFERENCES

[1] S. P. Bhat and D. S. Bernstein, Finite-time stability of homogeneous systems, *Proc. Amer. Control Conf*, 2513-2514, 1997.

[2] S. P. Bhat and D. S. Bernstein, Continuous finite-time stabilization of the translational and rotational double integrators, *IEEE TAC*, 43(5):678-682, 1998.

[3] S. P. Bhat and D. S. Bernstein, Finite-time stability of continuous autonomous systems, *SIAM J. Contr. Opt*, 38(3):751-766, 2000.

[4] V. T. Haimo, Finite-time controllers, *SIAM J. Control Optim*, 24(4):760-770, 1986.

[5] Y. Hong, Finite-time stabilization and stabilizability of a class of controllable systems. *System and Control Letters*, 46(4):231-236, 2002.

[6] Y. Hong, J. Huang, and Y. Xu, On an output feedback finite-time stabilization problem. *IEEE TAC*, 46(2):305-309, 2001.

[7] Y. Hong, Y. Xu, and J. Huang, Finite-time control for robot manipulators. *Systems and Control Letters*, 46(4): 243 - 253, 2002.

[8] C. Qian, A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems, to appear in *Proc of 24th Amer. Control Conference*, June 2005.

[9] C. Qian and J. Li, Global finite-time stabilization by output feedback for planar systems without observable linearization, *IEEE TAC*, 50(6) 885-890, 2005.

[10] C. Qian and W. Lin, A continuous feedback approach to global strong stabilization of nonlinear systems, *IEEE TAC,* 46(7):1061-1079, 2001.

[11] C. Qian and W. Lin, Recursive observer design and nonsmooth output feedback stabilization of inherently nonlinear systems, *Proc of IEEE 43th Conference on Decision and Control*, 4927-4932, 2004.

[12] E. R. Rang, Isochrone families for second order systems, *IEEE TAC*, 8(1): 64-65, 1963.