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Generation of autonomous oscillations via output feedback

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***Abstract*— In this paper, a newly developed method for generating oscillations in arbitrary order nonlinear systems, is redefined as an output feedback controller. In a recent publication, the state feedback orbitally stabilizing controller has been enhanced to provide strict Lyapunov functions for the oscillating behavior. This property is exploited here in order to prove global asymptotic stability of the output feedback controller in systems up to degree four, and global asymptotic stability with limited initial estimation error in higher–order systems. The output feedback controller is then applied to a fourth–order cascade LC network, while the result may be extrapolated to similar circuits with arbitrary number of LC blocks, which may be regarded as discrete approximations of transmission lines.**

I. INTRODUCTION

The stabilization of oscillations is a classical control problem widely considered in the literature [6], [11], [10]. More recent works [5], [12], [4] address the problem using different methodologies. This work stems from a control design presented in [1], [9], [7], [2], for stabilizing, via state feedback, a class of cascaded nonlinear systems of arbitrary order. In a subsequent work, [3], the domains of attraction (DOA) for this scheme with saturated control were studied. A recent result [8] addresses the robustness problem by providing strict Lyapunov functions for these orbital stabilizing control laws, based on a partition of the state space and solving the Lyapunov equation on each part.

In this work we further benefit from the new strict Lyapunov functions and prove global asymptotic stability (GAS) of the output feedback controller computed with a Luenberger–based estimate of the state. This is a nontrivial result because the original state feedback is highly nonlinear. The GAS result in output feedback extends up to fourth degree systems, while for higher–order we must assume limited initial estimation errors. The closed–loop system obtained is again robustly stable at a desired limit cycle, with strict Lyapunov function. The result is applied to a class of LC cascaded networks, and checked by simulation.

II. TARGET DYNAMICS

In this section we define a simple two-dimensional system $\dot{x} = f(x)$, $x = [x_1, x_2]^\top$, that presents an attractive limit cycle. This system will be used as the target dynamics in a subspace of dimension 2 of the controlled system in subsequent sections. Given a pair of *positive* design parameters $\omega_c \in \mathcal{R}$ and $\mu \in \mathcal{R}$, we will define the target set as the ellipse where the function

$$\Gamma(x_1, x_2) \triangleq \omega_c^2 x_1^2 + x_2^2 - \mu$$

is equal to zero. Now consider the Lyapunov function candidate[1] $V_{0n} = \frac{1}{4}\Gamma^2$. Obviously, the minima of V_{0n} are reached for $\Gamma = 0$, as depicted in Fig. 1 One way to get a dynamical

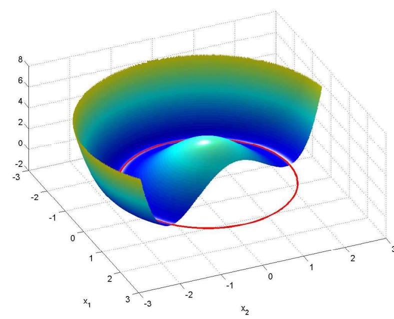

Fig. 1. Minimal set of V_{0n}: the closed curve $\Gamma(x_1, x_2) = 0$.

system with V_{0n} as a Lyapunov function (for which the limit sets are expected to be limit cycles) is to define the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\omega_c^2 x_1 - k_0 \text{sign}(\Gamma) x_2, \qquad (1)$$

with k_0 a tuning parameter. Along its trajectories, we have

$$\dot{V}_{0n} = -k_0 |\Gamma| x_2^2 \leq 0.$$

Remark 1: For $\mu > 0$, the change of variables $z_1 \triangleq \omega_c x_1/\sqrt{\mu}$ and $z_2 \triangleq x_2/\sqrt{\mu}$ and the time scaling $\tau = \omega_c t$, transforms (1) into the canonical form

$$z_1' = z_2 \qquad z_2' = -z_1 - \tilde{k}_0 \tilde{\Gamma} z_2 \qquad (2)$$

where $\tilde{k}_0 = k_0\mu/\omega_c$, $\tilde{\Gamma} = z_1^2 + z_2^2 - 1$ and the (') derivatives are expressed with respect to τ. This corresponds to an oscillation of unitary period and amplitude that will be used for simplicity in subsequent sections.

III. STRICT LYAPUNOV FUNCTIONS FOR ROBUST STABILIZATION OF OSCILLATIONS

In the previous system, the time derivative of V_{0n} is only negative semi–definite. Hence many of the tools related to strict Lyapunov functions in closed–loop analysis are not available in our framework.

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[1]The subscript 'n' stands for 'nominal'.

A. A strict Lyapunov for second–order systems

In [8], instead of V_{0n}, a strict Lyapunov function suitable robustness analysis is searched for. Via a partition of the state space into two regions determined the sign of the function $\Gamma(x_1, x_2)$, solving the Lyapunov equation on each part and merging the results, the following alternative Lyapunov function is proposed[2]

$$V_{0r} \triangleq \begin{cases} \frac{(x^\top R^+ x - \lambda_{max})^2}{4} & \text{if } x^\top R^+ x > \lambda_{max} \\ \frac{(x^\top R^- x - \lambda_{min})^2}{4} & \text{if } x^\top R^- x < \lambda_{min} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$R^- = \begin{bmatrix} \frac{1}{k_0} + \frac{k_0}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{k_0} \end{bmatrix} > 0, \tag{4}$$

$$R^+ = \begin{bmatrix} \frac{1}{k_0} + \frac{k_0}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{k_0} \end{bmatrix} > 0 \tag{5}$$

and λ_{min}, λ_{max} are the minimum and maximum eigenvalues of both (similar) matrices. Let us also define, for later use, $\eta = [x_1, x_2]^\top$ and

$$R = \begin{bmatrix} \frac{1}{k_0} + \frac{k_0}{2} & \frac{\text{sign}(\Gamma(\eta))}{2} \\ \frac{\text{sign}(\Gamma(\eta))}{2} & \frac{1}{k_0} \end{bmatrix} > 0. \tag{6}$$

The Lyapunov V_{0r} function has, along the trajectories of system (1), the time derivative $\dot{V}_{0r} = -\sqrt{V_{0r}}\|\eta\|^2$, which is strictly negative in the whole state space except the ring–shaped region enclosing the target oscillation where $V_{0r} = 0$. This set, denoted as $\mathcal{S}$ in the following, is illustrated in Fig. 2, as the flat area between the ellipses A and B. The ellipse equations are

$$\begin{aligned} A &: \quad x^\top R^+ x = \lambda_{max} \\ B &: \quad x^\top R^- x = \lambda_{min} \end{aligned} \tag{7}$$

Needless to say, the prescribed behavior can be generated by state feedback in any second–order system of relative degree two (for which the structure (1) is reachable, taking x_1 as the output). Two more significant facts are: (i) the strict Lyapunov function V_{0r} permits to take into account external disturbances, (see Section V); and (ii), the proposed second–order dynamics should be viewed as a subspace of some desired *arbitrary order* oscillating system.

B. Higher–order systems

Indeed, consider a nonlinear n-th order cascaded system of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1(x_1, x_2) + g_1(x_1, x_2) h_1(x_3) \\ \dot{x}_3 &= f_2(x_1 \dots x_3) + g_2(x_1 \dots x_3) h_2(x_4) \\ &\dots \\ \dot{x}_n &= u \end{aligned} \tag{8}$$

[2]The subscript 'r' stands for 'robust'.

with[3] $h_i'(\cdot), g_i(\cdot) \neq 0,\ i = 1 \dots n-2$ on the whole domain of interest. Then, defining the Lyapunov functions

$$V_0 = V_{0n} + V_{0r}, \quad V_i \triangleq V_0 + \frac{1}{2}\sum_{j=1}^{i} z_j^2, \quad i = 1 \dots n-2 \tag{9}$$

where

$$\begin{aligned} z_1 &\triangleq h_1(x_3) - h_1(u_0) \\ z_2 &\triangleq h_2(x_4) - h_2(u_1) \\ &\dots \\ z_{n-2} &\triangleq h_{n-2}(x_n) - h_{n-2}(u_{n-1}) \end{aligned}$$

and based on the iterative application of the backstepping method [1], [2], there is a set of recursively defined control laws of the form

$$\begin{aligned} u_0 &= -x_1 - \frac{2k_0}{\pi}\arctan(\sigma\Gamma) x_2 \\ u_i &= h_{i+1}^{-1}\left(\frac{\tilde{u}_i - f_{i+1}(x_1 \dots x_{i+2})}{g_{i+1}(x_1 \dots x_{i+2})}\right) \end{aligned} \tag{10}$$

where $\sigma > 0 \in I\!R$, $i = 1 \dots n-2$, $k_i > 0$, and

$$\begin{aligned} \tilde{u}_i &= \frac{1}{h_i'(x_{i+2})}\left[h_i'(u_{i-1})\dot{u}_{i-1} - \frac{\partial V_{i-1}}{\partial x_{i+1}} g_i - k_i z_i\right] \\ \dot{u}_i &= \left(\frac{\partial u_i}{\partial(x_1 \dots x_{i+2})}\right)^\top [\dot{x}_1 \dots \dot{x}_{i+2}]^\top, \end{aligned}$$

such that the Lyapunov function

$$V = V_{0n} + V_{0r} + \frac{1}{2} z^T z$$

with $z = [z_1, z_2 \dots z_{n-2}]^\top$, along the trajectories of (8) in closed loop with $u = u_{n-2}$, has time derivative

$$\dot{V} = -k_0|\Gamma|x_2^2 - \sqrt{V_{0r}}\|\eta\|^2 - k_1 z_1^2 - \dots - k_{n-2} z_{n-2}^2$$

that is, we obtain again a strict Lyapunov function in the n-dimensional space .

Remark 2: In u_0, we have replaced the sign$(\cdot)$ function of (1) with arctan$(\sigma\Gamma)$, with $\sigma > 0$ for differentiability, not affecting the nominal stability. The strict Lyapunov function claim is, however, slightly modified, as in a neighborhood of $\Gamma = 0$, we have

$$\begin{aligned} \dot{V}_{0r} &= \sqrt{V_{0r}}\eta^\top \left(\begin{bmatrix} 0 & -1 \\ 1 & -\frac{2}{\pi}\arctan(\alpha\Gamma)k_0 \end{bmatrix} R \right. \\ &+ \left. R \begin{bmatrix} 0 & 1 \\ -1 & -\frac{2}{\pi}\arctan(\alpha\Gamma)k_0 \end{bmatrix} \right) \eta \geq -\sqrt{V_{0r}}\|\eta\|^2 \end{aligned}$$

and the strictly negative term in the derivative becomes *less negative* the closer we move to $\Gamma = 0$; the closed–loop dynamics being less 'robust' in some sense, in that region. Conversely, far away from $\Gamma = 0$ the arctan$(\cdot)$ is practically equal to sign$(\cdot)$ and the last expression becomes an equality. More detailed analysis of the issue analysis has been spared for lack of space.

[3]$h'(\cdot)$ denotes the total derivative of the function with respect to its single argument evaluated at the actual state.

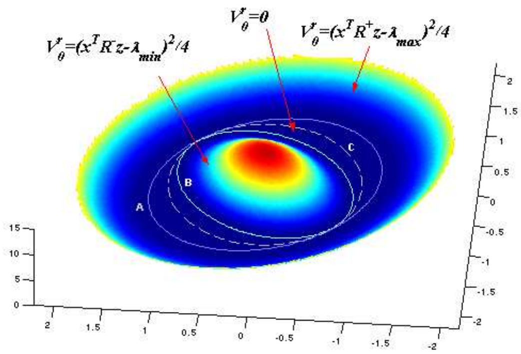

Fig. 2. Function V_{0r}. The transition band $\mathcal{S}$ is limited by the ellipses A and B. C depicts the desired orbit $\Gamma = 0$.

IV. Generation of oscillations via output feedback

The previous result is a full state feedback for generating oscillations in arbitrary order nonlinear systems. In practical applications of large dimension it is desirable if not absolutely necessary to avoid direct measures of a large part of the states and substitute them by an observer estimation. In this section we will address the output feedback problem for arbitrary order systems in Brunovski canonical form,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ &\cdots \\ \dot{x}_n &= v, \end{aligned} \tag{11}$$

we will assume that this structure has been achieved via partial linearization (as in our final example), but we will not consider the effect of computing the partially linearizing state feedback leading to (11) with an estimate of the state.

Considering the oscillating pair of variables $\eta = [x_1, x_2]^\top$ as the *output* of the system, we propose a Luenberger state observer of the form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - \eta) \tag{12}$$

where A and B are the matrices of the Brunovski form (11), i.e.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

and L is designed such that $(A+LC)$ is *Hurwitz* (obviously the system is observable), possibly taking into account noise considerations. The exponentially stable error dynamics $(e = \hat{x} - x)$ is then given by

$$\dot{e} = (A + LC)e.$$

For later use we will choose a matrix $Q \in I\!R^4 > 0$ and define $P \in I\!R^4$ as the positive definite solution of the Lyapunov equation

$$(A + LC)^\top P + P(A + LC) = -Q. \tag{13}$$

Our aim is to analyze the stability of system (11) when the orbitally stabilizing state feedback $u_{n-2}(x)$ is computed with estimates of the state, $u_{n-2}(\hat{x})$. This will be done in several steps, one for each of the recursive control laws u_0, u_1, u_2 in the backstepping procedure described in Section III-B.

First, note that $u_0 = u_0(x_1, x_2)$ only depends on the output and hence it is not affected by state estimation errors, i.e. $u_0(\hat{x}) \equiv u_0(x)$.

1) Output feedback in the third order subsystem: First choose Q in (13) such that $[Q]_{(3,3)} > k_1/4 \stackrel{\triangle}{=} q$. Now observe that u_1, computed for system (11), takes the form,

$$\begin{aligned} u_1 &= \dot{u}_0 - \frac{\partial V_0}{x_2} - \frac{\partial V_{0r}}{x_2} - k_1(x_3 - u_0) \\ &= \dot{u}_0 - \Gamma x_2 - \sqrt{V_{0r}}[Rx]_2 - k_1(x_3 - u_0). \end{aligned}$$

Analyzing the functional dependencies, it is clear that u_1 is Lipschitz on x_3 uniformly on (x_1, x_2), i.e.

$$u_1(x_1, x_2, \hat{x}_3) - u_1(x_1, x_2, x_3) \leq k_1|\hat{x}_3 - x_3|.$$

Hence, along the trajectories of the third order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u_1(x_1, x_2, \hat{x}_3) \end{aligned} \tag{14}$$

the Lyapunov function

$$V_{1e} = \frac{\Gamma^2}{4} + V_{0r} + \frac{z_1^2}{2} + pe^2$$

with $e = \hat{x}_3 - x_3$ and $p = [P]_{(3,3)}$ has time derivative

$$\begin{aligned} \dot{V}_{1e} &= -|\Gamma|x_2^2 - \sqrt{V_{0r}}\|\eta\|^2 - k_1 z_1^2 \\ &- \frac{\partial V_1}{\partial x_3}(u_1(\hat{x}) - u_1(x)) - qe^2 \\ &\leq -|\Gamma|x_2^2 - \sqrt{V_{0r}}\|\eta\|^2 - k_1 z_1^2 - z_1 k_1 e - qe^2 \\ &\leq -|\Gamma|x_2^2 - \sqrt{V_{0r}}\|\eta\|^2 - k_1\left(z_1 - \frac{e}{2}\right)^2 \leq 0. \end{aligned}$$

Hence the system asymptotically converges to the largest invariant set where $\dot{V}_{1e} = 0$, i.e.

$$\{\Gamma x_2 \equiv 0 \cap V_{0r} \equiv 0 \cap z_1 \equiv e_2\},$$

and as the error asymptotically uniformly converges to zero, this set is such that $z_1 \equiv 0$, and according to the proof provided in [8], this set is such that $\Gamma \equiv 0$. The convergence of the output feedback controller to the robust set where $V_{0r} = 0$ is obtained, again, with strict Lyapunov function.

2) Output feedback in the fourth–order system: When it comes to apply the feedback $u_2(x)$ with estimated state to the whole system (11) we encounter a major difficulty, the control law is complex and no longer Lipschitz. The orbitally stabilizing feedback in this case is

$$u_2 = \dot{u}_1 - x_3 - u_0 - k_2(x_4 - u_1)$$

and the error in the controller becomes, after some straightforward calculations,

$$u_2(\hat{x}) - u_2(x) = \dot{u}_1(\hat{x}) - \dot{u}_1(x) - e_3 - k_2 e_4 - k_1 e_3$$

where $e_3 = \hat{x}_3 - x_3$ and $e_4 = \hat{x}_4 - x_4$. While the last three terms of u_2 are obviously Lipschitz on (x_3, x_4) the first two are slightly more complex

$$\begin{aligned} \dot{u}_1(\hat{x}) - \dot{u}_1(x) &= \frac{\partial}{\partial x_2}\left[\Gamma x_2 - \sqrt{V_{0r}}[Rx]_2\right] e_3 \\ &= (2x_2^2 + \Gamma + [Rx]_2^2 - \sqrt{V_{0r}}[R]_{22})e_3 \\ &< \sigma_1 \|\eta\|^2 |e_3| \end{aligned}$$

where we have defined the constant

$$\sigma_1 \triangleq \frac{13k_0^2 + 4}{4k_0^2} + \frac{\lambda_{max}(R)}{k_0}.$$

With this upper bounds, we are able to analyze the trajectories of system (11) in closed–loop with $u_2(\hat{x})$. The Lyapunov function

$$V_{2e} = \frac{\Gamma^2}{4} + V_{0r} + \frac{z_1^2}{2} + \frac{z_2^2}{2} + e^\top P e$$

with $e = [0\ 0\ e_3\ e_4]^\top$ has, along the trajectories the closed–loop system, the time derivative

$$\begin{aligned} \dot{V}_{2e} &= -|\Gamma| x_2^2 - \sqrt{V_{0r}}\|x\|^2 - k_1 z_1^2 - z_1(u_2(\hat{x}) - u_2(x)) \\ &\quad -e^\top Q e \\ &\le -|\Gamma| x_2^2 - \sqrt{V_{0r}}\|x\|^2 - k_1 z_1^2 - k_2 z_2^2 \\ &\quad -|z_2|(1 + k_1 + \sigma_1\|(x_1, x_2)\|^2)|e_3| + k_2|e_4| - e^\top Q e \\ &\le -|\Gamma| x_2^2 - \sqrt{V_{0r}}\|x\|^2 - k_1 z_1^2 - k_2 z_2^2 - |z_2|\sigma_1\|x\|^2 e \\ &\quad -z_2|1 + k_1 + k_2||e| - e^\top Q e. \end{aligned}$$

Defining the following sets in $I\!R^4$, the definition parameterized by the constant $\delta_1 = \sigma_1^2 e_0/(k_2)$ with $e_0 > 0$ some constant of our choice,

$$\begin{aligned} &\Pi_1 : \{(x_1, x_2, z_1, z_2) \in I\!R^4 : V_{0r}(x_1, x_2) > \delta_1\} \\ &\Pi_2 : I\!R^4 - \Pi_1 \end{aligned}$$

we have that, in Π_1

$$\begin{aligned} \dot{V}_{2e} &< -\left[\|x\|^2\ |z_2|\ \|e\|\right] D \left[\|x\|^2\ |z_2|\ \|e\|\right]^\top \\ D &\triangleq \begin{bmatrix} \sigma_1^2 e_0/k_2 & \frac{\sigma_1}{2}\|e\| & 0 \\ \frac{\sigma_1}{2}\|e\| & k_2 & \frac{1+k_1+k_2}{2} \\ 0 & \frac{1+k_1+k_2}{2} & \lambda_{min}(Q) \end{bmatrix} \end{aligned}$$

and with the appropriate choice of Q such that

$$\lambda_{min}(Q) > \frac{(1 + k_1 + k_2)^2}{4k_2}.$$

The matrix D is positive definite for all $\|e(t)\| < e_0$, but, as the error dynamics is exponentially stable, whatever the choice of e_0, this will hold from some time t^* onwards. As a consequence, $\dot{V}_{2e}$ is negative in Π_1 some finite time. On the other hand, in Π_2 we will use the upper bound

$$\begin{aligned} \dot{V}_{2e} &< -[|z_2|\ \|e\|]\, E\, [|z_2|\ \|e\|]^\top \\ E &\triangleq \begin{bmatrix} k_2 & \frac{1+k_1+k_2}{2} + \frac{\sigma_1\|x\|^2}{2} \\ \frac{1+k_1+k_2}{2} + \frac{\sigma_1\|x\|^2}{2} & \lambda_{min}(Q) \end{bmatrix} \end{aligned}$$

but, in Π_2,

$$\sqrt{V_{0r}} \le \delta_1 \Rightarrow \|x\|^2 < \frac{\delta_1 - \lambda_{max}(R^+)}{\lambda_{max}(R^+)} \triangleq \delta_3$$

Hence, again, Q can be chosen (sufficiently large) such that $E > 0$ by setting

$$\lambda_{min}(Q) > \frac{\left(\frac{1+k_1+k_2}{2} + \frac{\sigma_1\|x\|^2}{2}\right)^2}{k_2}.$$

Actually the maximum of both upper bounds should of $\lambda_{min}(Q)$ should be used), and hence $\dot{V}_{2e} < 0$ in Π_2. It is worth to note that

- (i) no parameter tuning has been made in the previous discussion and hence the output feedback controller globally asymptotically stabilizes the system at $\Gamma = 0$ for any three positive values of (k_1, k_2, k_3),
- (ii) the previous proof has makes use of the newly defined strict Lyapunov function in order to prove the positivity of matrix D.

3) Output feedback in higher–order systems: The output feedback controller for generating oscillations in systems with higher order than four is more involved. We will consider an n-th order system in the form (11) and the corresponding control law $u = u_{n-2}(x)$ computed according to Section III-B. Define also the output $\eta = [x_1, x_2]^\top$ and the state observer (12).

Proposition 4.1: For every initial condition $x(0)$ there is a set of controller parameters $(k_i, i = 0 \ldots n-2)$ and a maximum initial estimate error e_{max} such that the system in closed–loop with the observer–based controller $u_2(\hat{x})$, with initial estimation error $\|e(0)\| \le e_{max}$ asymptotically converges to the periodic orbit described by $\{\Gamma = 0, z = 0\}$ (except trajectories starting at the origin).

Proof: Define $e = [0\ 0\ e_3, e_4 \ldots e_n]^\top$ with $e_i = \hat{x}_i - x_i$, $i = 3 \ldots n$. Define also $z_k = x_{k+2} - u_{k-1}$, $k = 1 \ldots n-2$, and $z = [z_1 \dots z_{n-2}]^\top$. Then, the Lyapunov function

$$V_{(n-2)e} = \frac{\Gamma^2}{4} + V_{0r} + \frac{z^\top z}{2} + e^\top P e$$

has, along the closed–loop trajectories, the time derivative

$$\begin{aligned}\dot V_{(n-2)e} &= -|\Gamma| x_2^2 - \sqrt{V_{0r}}\|\eta\|^2 - k_1 z_1^2 \cdots - k_{n-2} z_{n-2}^2 \\ &- z_{n-2}(u_{n-2}(\hat x) - u_{n-2}(x)).\end{aligned}$$

But, $u_{n-2}(x)$ has the form

$$u_{n-2} = \underbrace{\dot u_{n-3} - \frac{\partial V_{n-3}}{\partial x_{n-1}}}_{u^a_{n-2}} - \underbrace{k_{n-2}(x_n - u_{n-3})}_{u^b_{n-2}},$$

where the decomposition is such that u^a_{n-2} is independent of k_{n-2}. As u_{n-2} has continuous partial derivatives, it is Lipschitz on any bounded set. Hence, choosing some $e_0 > 0$ arbitrarily and defining the set

$$\Omega : \{(x, e) \in I\!R^{2n-2} : V_{(n-2)e}(x, e) < V_{(n-2)e}(x(0), e_0)\},$$

which is, obviously, bounded, there are positive constants β_1 and β_2, *independent of the choice* of k_{n-2} such that

$$\begin{aligned}u_{n-3}(\hat x) - u_{n-3}(x) &< \beta_1\|e\| \;\; \forall (x, e) \in \Omega \\ u^a_{n-2}(\hat x) - u^a_{n-2}(x) &< \beta_2\|e\| \;\; \forall (x, e) \in \Omega.\end{aligned}$$

Then, inside Ω, the following holds

$$u_{n-2}(\hat x) - u_{n-2}(x) < (\beta_2 + (k_{n-2} + 1)\beta_1)\|e\|,$$

and this implies

$$\begin{aligned}\dot V_{(n-2)e} &< -[|z_{n-2}|\ \|e\|]\, F\, [|z_{n-2}|\ \|e\|]^\top \\ F &\triangleq \begin{bmatrix} k_{n-2} & \frac{\beta_2 + (k_{n-2}+1)\beta_1}{2} \\ \frac{\beta_2 + (k_{n-2}+1)\beta_1}{2} & \lambda_{min}(Q)\end{bmatrix}.\end{aligned}$$

This suggests two possibilities

- $\exists k_{n-2} : F \geq 0$. Then, for that value of k_{n-2}, Ω is a domain of attraction without any limitation on the initial error (in Ω).
- $\nexists k_{n-2} : F \geq 0$. Then, there is some constant $\gamma > 0$ such that the substitution $Q \to \gamma Q$ in F makes this matrix positive definite. However, in order to not to affect the definition of Ω and the constants β_1, β_2 depending on it, the initial maximum allowable error should be scaled by $\sqrt{\gamma}$, with respect to the given choice, i.e. $e_{max} = e_0/\sqrt{\gamma}$.

Then, the initial condition $x(0), e(0)$ with $\|e(0)\| < e_{max}$ belongs to Ω, and this set is a domain of attraction of $\Gamma = 0$. This completes the proof. ∎

V. Output Feedback in LC cascade circuits

In this section we will consider the ladder structure (LC network) illustrated in Fig. 3. We will first derive the state feedback that generates robust oscillations and then we will solve the *output feedback* problem for the proposed example, taking into account that the resulting controller can be extrapolated to arbitrary order systems.

The state vector of this system is $y = [y_1, y_2, y_3, y_4]^\top$ where (y_1, y_3) represent, respectively, the voltage drops across the capacitors C_1, C_3, and (y_2, y_4), are the electrical currents through the inductors L_2 and L_4. With these coordinates, the closed–loop dynamics are expressed as

$$\dot y = Ay + BV_{in} \tag{15}$$

where $B = [0\ 0\ 0\ 1]^\top$ and

$$A = \begin{bmatrix} 0 & \frac{1}{C_1} & 0 & 0 \\ \frac{1}{L_2} & 0 & \frac{1}{L_2} & 0 \\ 0 & -\frac{1}{C_3} & 0 & \frac{1}{C_3} \\ 0 & 0 & -\frac{1}{L_4} & -R_5 \end{bmatrix}$$

A. State feedback controller

In order to produce a robust oscillation at the *output* $V_{out} = y_1$ with state feedback via the method described in III-B, the system is required to be in strict cascade form. This is achieved by deriving the output four times with respect to time: as the system has strictly relative degree four, it can be transformed via a trivial pre–feedback into

$$y_1^{(4)} = v \tag{16}$$

where v is the new input. Under these conditions, we define the new coordinates $(x_1, x_2, x_3, x_4) \triangleq (\dot y_1, \ddot y_1, y_1^{(3)}, y_1^{(4)})$. From (16), the system has the canonical Brunovsky form (11), thus allowing to easily compute the robust oscillating control law. In the first step we define the control law for the first second–order subsystem

$$u_0 = -x_1 - k_0 \mathrm{sign}(\Gamma) x_2$$

The Lyapunov function for the (x_1, x_2) subsystem when x_3 equals u_0 exactly are both V_{0n} (nominal) and V_{0r} (robust). The sum $V_{0n} + V_{0r}$ will be used for backstepping. Defining z_1 as the deviation of x_3 with respect to u_0, i.e. $z_1 \triangleq x_3 - u_0$, and $V_1 = V_{0n} + V_{0r} + z_1^2/2$, the application of Eq. (10) yields the pseudo–control law to–be–tracked by x_3,

$$\begin{aligned}u_1 &= \left(-1 + \frac{2\alpha\, x_1\, x_2}{1 + \alpha^2\Gamma^2}\right) x_2 \\ &+ \left(\frac{2\alpha\, {x_2}^2}{1 + \alpha^2\Gamma^2} + \arctan(\alpha\,\Gamma)\right) x_3 \\ &- \Gamma x_2 - \sqrt{V_{0r}}[0\ 1] R(x) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - k_1 z_1\end{aligned}$$

where $R(x)$ is defined as in (6). For the last step of backstepping we define

$$z_2 \triangleq x_4 - u_1 \qquad V_2 \triangleq V_{0n} + V_{0r} + \frac{z_1^2}{2} + \frac{z_2^2}{2} \tag{17}$$

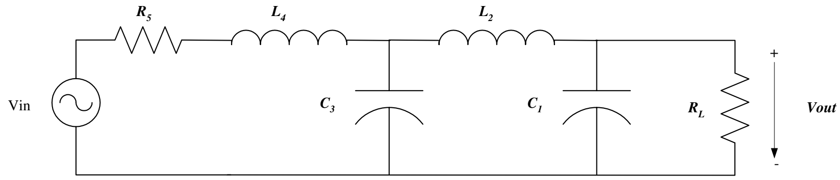

Fig. 3. RLC ladder circuit.

to obtain the actual control law to be applied in (16)

$$
\begin{aligned}
u &= \ddot{u}_1 - \frac{\partial V_1}{\partial z_1} = \left(\left(\left(\frac{2\alpha\, x_2}{1+\alpha^2\Gamma^2} - \frac{8\alpha^3 {x_1}^2 x_2\,\Gamma}{\left(1+\alpha^2\Gamma^2\right)^2}\right) x_2\right.\right. \\
&+ \left.\left(-\frac{8\alpha^3 {x_2}^2\Gamma\, x_1}{\left(1+\alpha^2\Gamma^2\right)^2} + \frac{2\alpha\, x_1}{1+\alpha^2\Gamma^2}\right) x_3\right) x_2 \\
&+ \left(\left(-\frac{8\alpha^3 {x_2}^2\Gamma\, x_1}{\left(1+\alpha^2\Gamma^2\right)^2} + \frac{2\alpha\, x_1}{1+\alpha^2\Gamma^2}\right) x_2 - 1\right. \\
&+ \left.\frac{2\alpha\, x_1\, x_2}{1+\alpha^2\Gamma^2} + \left(\frac{6\alpha\, x_2}{1+\alpha^2\Gamma^2} - \frac{8\alpha^3 {x_2}^3\Gamma}{\left(1+\alpha^2\Gamma^2\right)^2}\right) x_3\right) x_2 \\
&+ \left(\frac{2\alpha\, {x_2}^2}{1+\alpha^2\Gamma^2} + \arctan\left(\alpha\,\Gamma\right)\right) x_4 - z_1 - k_2 z_2 \qquad (18)
\end{aligned}
$$

The Lyapunov function V_2 is such that along the trajectories of the closed–loop system

$$\dot{V}_2 \sim -|\Gamma| x_2^2 - \sqrt{V_{0r}}\|\eta\|^2 - k_1 z_1^2 - k_2 x_2^2 < 0$$

where $\eta = [x_1, x_2]^\top$ and the '$\sim$' sign recalls that the equation is not exact (thought not affecting the sign) in an arbitrarily small region in the boundary of $\mathcal{S}$ (see Remark 2). If we ignore this fact of little impact, the inequality is strict for all $x \in I\!R^4 - \{\mathcal{S} \cap (x = 0)\}$.

Remark 3 (Extension to higher order LC networks): The LC network treated could be extended to an arbitrary number of LC blocks. This suggest a procedure for inducing robust oscillations in transmission lines, for which the circuit of Fig. 3 can be seen as a discrete approximation.

VI. Simulation Results

Figure 4 shows the results of a simulation with $\omega_c = 1$, $\mu = 1$, $\sigma = 10$, $k_0 = 4$, $k_1 = 5$, $k_2 = 10$ and

$$L = \begin{bmatrix} -1.3938 & -0.7608 \\ -0.7608 & -2.2542 \\ -0.5212 & -2.3300 \\ -0.1691 & -0.9856 \end{bmatrix}.$$

It can be seen that variables x_1 and x_2 reach the desired oscillation (top), the estimation error tending to zero (bottom).

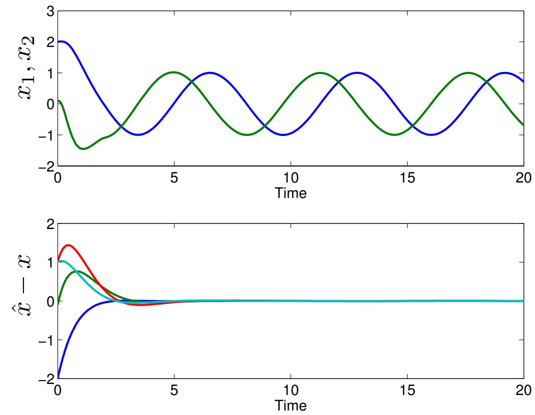

Fig. 4. Simulation results

VII. Conclusions

In this work we extended a recent method for generation of autonomous oscillations in arbitrary order systems, by replacing the state feedback controller by an output feedback computed with state estimates. For the new output feedback orbitally stabilizing controller, global asymptotic stability proofs have been provided for systems up to degree four, while the extension to arbitrary order systems has only been achieved for limited initial errors.

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