Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Exogenous feedback linearization of discrete-time systems

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Abstract— **The goal of this paper is to introduce the notion of exogenous state feedback for discrete-time systems. It is shown that a discrete-time system may be linearizable by exogenous feedback, even if it can not be linearized by endogenous feedback. This property was completely unexpected and constitutes a fundamental difference with respect to the continuous-time case. The theory is illustrated through an example whose structure is similar to the exact discrete-time model of a mobile robot, showing that the above mentioned property concerns not only academic examples, but physical systems as well.**

I. Introduction

The feedback linearization problem is one of the oldest problems of modern control theory and has been addressed by many authors, see [3], [8], [10], [13]. The dynamic feedback linearization problem is often approached through the search of a so-called linearizing output [3], [15]. Roughly speaking, a linearizing output is a function which defines an input-output invertible system with trivial zero dynamics [3], [14], [15]. Therefore, an application of the well known dynamic extension algorithm produces a dynamic compensator which fully linearizes the original system.

Even though the search of a linearizing output may seem to be less general than the dynamic feedback linearization problem, it has been shown that, in the continuous-time case, both problems are equivalent [14].

Moreover, in the continuous-time case it is also known that the dynamic extension required to linearize the system can always be chosen to be endogenous [14]. Recall that a dynamic extension is said to be endogenous if it can be expressed as a function of the original state and a finite number of derivatives of the control variable. An exogenous dynamic feedback is one for which the dynamic extension can not be chosen to be endogenous.

The goal of this paper is to show that, in the discrete-time case, if a system can be linearized by dynamic state feedback, the required dynamic extension can not always be chosen to be endogenous. This property is surprising and was completely unexpected. It constitutes a fundamental difference between continuous-time and discrete-time systems.

Another surprising fact is that the need of exogenous dynamic feedback has been identified for the exact discrete-time model of a mobile robot [16]. This fact shows that exogenous dynamic feedbacks can be required for the control of physical systems, not only for academic examples.

This work was partially supported Ecole Centrale de Nantes and by CONACYT, México, through Grant No. 42093.

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II. Preliminaries

Consider the discrete-time nonlinear system

$$x(t+1) = f(x(t), u(t)),\ x(0) = x_0, \tag{1}$$

where the state $x(t) \in \mathcal{R}^n$, the control $u(t) \in \mathcal{R}^m$ and the mapping $f(\cdot)$ is real analytic. Throughout the paper the following standing assumptions are made:

A1 The dynamics (1) is reversible, i.e.

$$\text{rank}\,\frac{\partial f(x,u)}{\partial x} = n.$$

A2 The m input channels are independent, i.e.

$$\text{rank}\,\frac{\partial f(x,u)}{\partial u} = m.$$

Assumptions A1 and A2 above are rather common in the discrete-time literature [12]. The algebraic framework, that we describe below, was formulated by Grizzle [9] for discrete-time nonlinear systems. This framework is related with Fliess' difference-algebraic approach [7] and has been modified in [3] to end up with an inversive difference field.

At some places, basic facts from Exterior Differential Systems Theory are used. For further details, the reader is referred to [1], [4], [5].

Let $\mathcal{K}$ be the field of meromorphic functions of a finite number of the variables of the following (infinite) set $\{x(0), u(t), t \geq 0\}$. The forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by

$$\delta\varphi[x(0), u(j)] = \varphi[f(x(0), u(0)), u(j+1)].$$

It is always possible to embed $(\mathcal{K}, \delta)$ into an inversive difference overfield $(\mathcal{K}^*, \delta^*)$, called the *inversive closure* [3], [6], [7] of $\mathcal{K}$. By abuse of notation hereinafter we assume that the inversive closure $(\mathcal{K}^*, \delta^*)$ is given and use the same symbol to denote the difference field $(\mathcal{K}, \delta)$ and its inversive closure. Thus $\delta^{-1} : \mathcal{K} \to \mathcal{K}$ is well defined. Sometimes, given $\varphi \in \mathcal{K}$, the abridged notations $\varphi^+(\cdot) = \delta\varphi(\cdot)$ and $\varphi^-(\cdot) = \delta^{-1}\varphi(\cdot)$ are used.

Denote by $\mathcal{E}$ the formal vector space spanned by the differentials of the elements of $\mathcal{K}$; that is,

$$\mathcal{E} := \text{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}.$$

The elements of $\mathcal{E}$ are called differential forms of order one, or simply one-forms.

The operators δ and δ^{-1} induce, respectively, the operators $\Delta : \mathcal{E} \to \mathcal{E}$ and $\Delta^{-1} : \mathcal{E} \to \mathcal{E}$ by

$$\begin{array}{rcl}\Delta(\sum_i a_i \mathrm{d}\varphi_i) & \mapsto & \sum_i a_i^+ \mathrm{d}\varphi_i^+, \\ \Delta^{-1}(\sum_i a_i \mathrm{d}\varphi_i) & \mapsto & \sum_i a_i^- \mathrm{d}\varphi_i^-.\end{array}$$

With some abuse of notation, sometimes we write $\omega^+ = \Delta\,\omega$ and $\omega^- = \Delta^{-1}\omega$.

In particular, using the notation above, system (1) can be written simply as

$$x^+ = f(x, u).$$

In order to state our main result, we need to recall some notations. First let us define a sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ by

$$\begin{array}{rcl} \mathcal{H}_1 & = & \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}x\right\}, \\ \mathcal{H}_{k+1} & = & \mathrm{span}_{\mathcal{K}}\left\{\omega \in \mathcal{H}_k \mid \Delta\omega \in \mathcal{H}_k\right\},\ k \geq 1. \end{array} \tag{2}$$

This sequence of subspaces was first introduced in [3] to address different notions of linearizability for discrete-time nonlinear systems. It is clear that the sequence (2) is decreasing. Denote by k^* the least integer such that

$$\mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty. \tag{3}$$

Assume that $\mathcal{H}_\infty = 0$. This is equivalent to assume that system (1) satisfies the accessibility property [3], [12]. This Assumption is natural, as accessibility is a necessary condition for feedback linearizability.

In [3] it has been proved that there exists a set of one-forms $\Omega = \{\omega_1, \ldots, \omega_m\}$ and a list of integers $\{r_1, \ldots, r_m\}$ such that, for $1 \leq k \leq k^*$,

$$\mathcal{H}_k = \mathrm{span}_{\mathcal{K}}\left\{\Delta^j\omega_i, \mid r_i \geq k,\ 0 \leq j \leq r_i - k\right\}. \tag{4}$$

The integer r_i associated to the one-form w_i is called the relative degree of the one-form w_i. A set of one-forms $\Omega = \{\omega_1, \ldots, \omega_m\}$ satisfying (4) is called a **system of linearizing one-forms**.

III. Main results

A. The notion of exogenous output

Definition 3.1 (Exogenous output): An exogenous output (EO) is defined by a p-dimensional mapping

$$y = h(x, u(-j), u(+k)), \tag{5}$$

where $1 \leq j \leq \alpha$ and $0 \leq k \leq \beta$.

Recall that $\mathrm{rank}[\partial f/\partial u] = m$ and $\mathrm{rank}[\partial f/\partial x] = n$. It follows that

$$\mathrm{d}x^- \in \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}x, \mathrm{d}u^-\right\} \tag{6}$$

and conversely

$$\mathrm{d}u^- \in \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}x, \mathrm{d}x^-\right\}. \tag{7}$$

The inclusion (7) implies that an exogenous output can also be represented by

$$y = \bar{h}(x, x(-j), u(+k)), \tag{8}$$

where $1 \leq j \leq \alpha$ and $0 \leq k \leq \beta$. Equations (5) and (8) are different representations of the same mathematical object, and will be used indistinctly, depending on the purpose of the discussion.

The alternative representation (8) can be interpreted as a *standard* output function for an extended system. One possible realization of the extended system can be obtained by adding β pure unit delays *before* the input $u(t)$ and α pure unit delays *after* the state variables $x(t)$. This construction is depicted in Fig. 1 The dimension of the state space of the augmented system depicted in Fig. 1 is $n_e = n + m\beta + n\alpha$.

The main drawback of the realization shown in Fig. 1 is that it is by no means minimal and, in general, possesses unobservable dynamics. In order to avoid this problem, a finer realization of an extended system will be developed. For, some notation is introduced. Define

$$\begin{array}{rcl} \mathcal{Y}_\ell & = & \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y(i), 0 \leq i \leq \ell\right\}, \\ \mathcal{U}_\ell & = & \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}u(i), 0 \leq i \leq \ell\right\}, \\ \mathcal{Y} & = & \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y(i), i \geq 0\right\}, \\ \mathcal{X} & = & \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}x(0)\right\} = \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}x\right\}. \end{array}$$

Let $s = \dim \mathcal{Q}$, where

$$\mathcal{Q} = \frac{\mathcal{Y}_\alpha + \mathcal{X} + \mathcal{U}_{\alpha+\beta}}{\mathcal{X} + \mathcal{U}_{\alpha+\beta}}.$$

Proposition 3.1: The integer s is bounded by $p\alpha$ and represents the minimal number of pure unit delays that must be added after the state variables, so that $y = \bar{h}(x, x(-j), u(+k))$ becomes an standard output.

Proof First note that $\mathrm{d}y_i^{+\alpha} \in \mathcal{X} + \mathcal{U}_{\alpha+\beta}$, for $i = 1, \ldots, p$. Therefore, by definition, the quotient space $\mathcal{Q}$ space has dimension $s \leq p\alpha$.

Let $\mathcal{I}$ and $\mathcal{J}$ be sets of indexes such that

$$\mathcal{B} = \{\mathrm{d}y_i^j \mid i \in \mathcal{I}, 0 \leq j \leq j_i, j_i \in \mathcal{J}\}$$

is a basis of the quotient space $\mathcal{Q}$. Without loss of generality, we can assume that $\mathcal{I} = \{1, \ldots, q\}$ and $\mathcal{J} = \{j_1, \ldots, j_q\}$. By construction, it is clear that $j_i \leq \alpha$, for $i = 1, \ldots, p$ and that $\sum_{i=1}^{q} j_i = s$ Define the functions $\phi_{ij} = y_i^{+(j-1)}$, for $i \in \mathcal{I}$ and $1 \leq j \leq j_i$. Finally, define the extended system of coordinates by $(\xi_1, \xi_2, \xi_3)^T = (x, z, \phi)^T$, where $z = (z_1, \ldots, z_\beta)^T = (u, \ldots, u^{+(\beta-1)})^T$, $v = u^{+\beta}$ and $\phi = (\phi_{11}, \ldots, \phi_{qj_q})^T$.

The dynamics of the extended system are governed by the following difference equations

$$\begin{array}{rcl} \xi_1^+ & = & f(\xi_1, \xi_2), \\ \xi_2^+ & = & A_2\xi_2 + B_2 v, \\ \xi_3^+ & = & A_3\xi_3 + B_3\psi, \end{array} \tag{9}$$

where A_2, A_3, B_2 and B_3 are matrices of appropriate dimensions, the pairs (A_2, B_2) and (A_3, B_3) are in companion form and $\psi = (\phi_{1j_1}^+, \ldots, \phi_{qj_q}^+)^T$.

Finally, by construction, $y = \bar{h}(x, x(-j), u(+k)) = \tilde{h}(x, \phi_{ij}, z) = \tilde{h}(\xi_1, \xi_2, \xi_3)$. •

The realization of the extended system suggested in the proof of Proposition 3.1 requires the addition of only s pure unit delays after some functions of the state variable $x(t)$. This construction is depicted in Fig. 2. The dimension of the state space of the augmented system depicted in Fig. 2 is $n_e = n + m\beta + s$. The practical application of Proposition 3.1 is shown on the example in Section III-D.

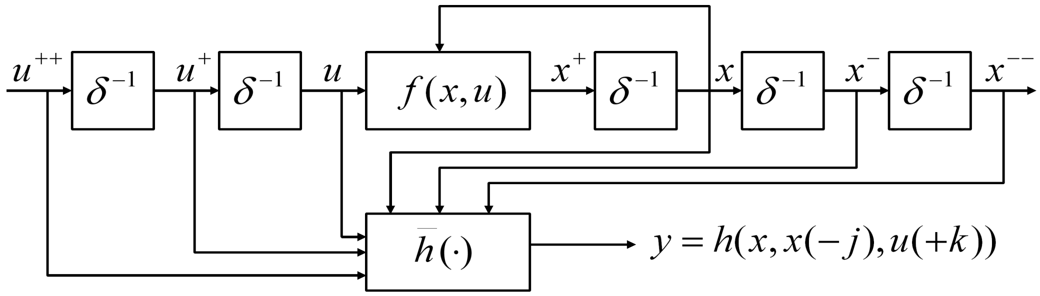

Fig. 1. Realization of extended system

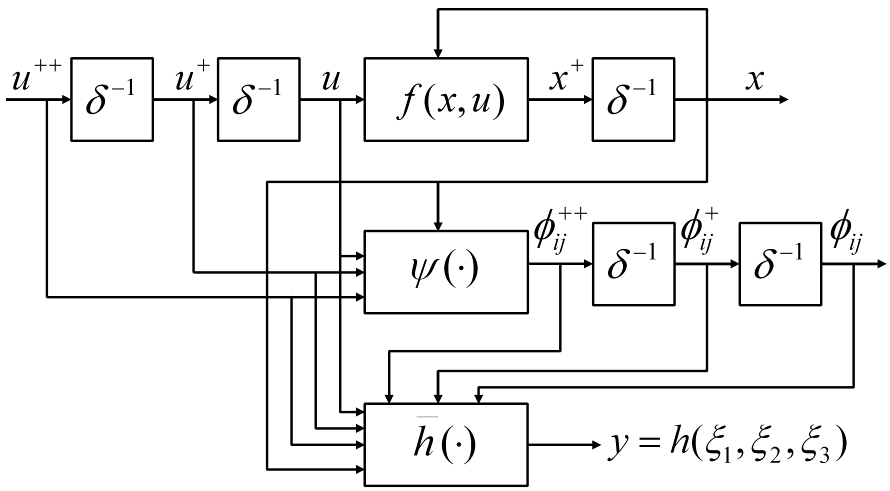

Fig. 2. Minimal realization of extended system

B. Transformal operators

Let $\mathcal{K}[\Delta]$ denote the set of polynomials in the operator Δ with coefficients in $\mathcal{K}$. One can give $\mathcal{K}$ the structure of a noncommutative ring with the addition defined in the usual manner and the multiplication defined by the noncommutative operation

$$\Delta\, p = p^{+}\Delta,\ \forall\, p \in \mathcal{K},$$

which corresponds to operators composition. Let $\mathcal{K}^{m\times m}[\Delta]$ denote the set of $m \times m$ matrices whose entries belong to $\mathcal{K}[\Delta]$. The set $\mathcal{K}^{m\times m}[\Delta]$ is also a noncommutative ring. The elements of $\mathcal{K}^{m\times m}[\Delta]$ are called matrix transformal operators.

Let $\mathcal{E}^m$ denote the $\mathcal{K}$-vector space spanned by m-tuples of one-forms. Every matrix transformal operator $P(\Delta) \in \mathcal{K}^{m\times m}[\Delta]$ defines a mapping from $P(\Delta) : \mathcal{E}^m \to \mathcal{E}^m$ following the usual rules of matrix multiplication.

A transformal operator $P(\Delta) \in \mathcal{K}^{m\times m}[\Delta]$ is said to be a unit of the ring $\mathcal{K}^{m\times m}[\Delta]$ or simply to be a unimodular operator if there exists another transformal operator $Q(\Delta) \in \mathcal{K}^{m\times m}[\Delta]$ such that

$$\forall\,\Omega \in \mathcal{E}^m,\ Q(\Delta)\circ P(\Delta)\,(\Omega) = \Omega,$$

or, equivalently, $Q(\Delta) \circ P(\Delta) = I_m$, the identity matrix in $\mathcal{R}^{m\times m}$. If such operator $Q(\Delta) \in \mathcal{K}^{m\times m}[\Delta]$ exists, it is called a left-inverse of the operator $P(\Delta) \in \mathcal{K}^{m\times m}[\Delta]$. For the sake of simplicity, the symbol $\circ$ is often dropped.

Some useful properties of unimodular operators are those given by the following couple of technical results.

Proposition 3.2: The only units of the noncommutative ring $\mathcal{K}[\Delta]$ are the nonzero functions $p \in \mathcal{K}$.

Proposition 3.3: Let $P(\Delta) \in \mathcal{K}^{m\times m}[\Delta]$ be a unimodular operator. Then, its inverse is unique and two-sided. Therefore, the inverse can be denoted by $P^{-1}(\Delta)$.

The noncommutative rings $\mathcal{K}[\Delta^{-1}]$ and $\mathcal{K}^{m\times m}[\Delta^{-1}]$ are defined by replacing Δ by Δ^{-1} in the previous discussion.

Consider the system (1) and suppose that an m-dimensional output function $y = h(x)$ has been defined. Let $U(\Delta) \in \mathcal{K}^{m\times m}[\Delta]$ and $D(\Delta^{-1}) \in \mathcal{K}^{m\times m}[\Delta^{-1}]$ be fixed transformal operators. These operators define new output functions given respectively, by $\tilde{y} = U(\Delta)y$ and $\bar{y} = D(\Delta^{-1})y$. In the rest of this paper we will be interested in the action that two particular classes of transformal operators have on the structure of a system with a given output. Namely, the class of unimodular operators $\mathcal{M}(\Delta) \subset \mathcal{K}^{m\times m}[\Delta]$, and the class of diagonal operators $\mathcal{D}(\Delta^{-1}) \subset \mathcal{K}^{m\times m}[\Delta^{-1}]$ defined by:

$$\begin{aligned} \mathcal{D}(\Delta^{-1}) \quad = \quad & \big\{ D(\Delta^{-1}) \in \mathcal{K}^{m\times m}[\Delta^{-1}] \mid \\ & D(\Delta^{-1}) = \mathrm{diag}\,(\Delta^{-\rho_i}),\ \rho_i \geq 0 \big\}\,. \end{aligned} \tag{10}$$

The output function $\bar{y} = D(\Delta^{-1})y$ defined by a diagonal operator $D(\Delta^{-1}) \in \mathcal{D}(\Delta^{-1})$ is an exogenous output, in the sense of the previous section. In the rest of this paper it will be assumed that the original system (1) is provided with a minimal extension such that the output $\bar{y}$ becomes a standard output for the augmented system. From the Proof of Proposition 3.1, it turns out that the minimal extension consists of s pure unit delays after the state variable $x(t)$ and no pure unit delays are needed before the control input $u(t)$. For latter use, define

$$\begin{aligned}\bar{\mathcal{Y}} &= \text{span}_{\mathcal{K}}\{\mathrm{d}\bar{y}(k), k \geq 0\},\\ \Phi &= \text{span}_{\mathcal{K}}\{\mathrm{d}\phi_{ij}, i \in \mathcal{I}, 0 \leq j \leq j_i, j_i \in \mathcal{J}\},\end{aligned}$$

where ϕ_{ij}, $\mathcal{I}$ and $\mathcal{J}$ have been defined in the Proof of Proposition 3.1.

The output function $\tilde{y} = U(\Delta)y$ defined by an unimodular operator $U(\Delta) \in \mathcal{D}(\Delta)$ depends, in general, of $\{u(k) \mid 0 \leq k \leq \beta\}$ for some nonnegative integer β. In the rest of this paper it will be assumed that the original system (1) is provided with a suitable extension such that the output $\tilde{y}$ becomes an standard output for the augmented system. From the Proof of Proposition 3.1, it turns out that the extension consists of $m\beta$ pure unit delays before the control variable $u(t)$ and no pure unit delays are needed after the state variable $x(t)$. For latter use, define $\tilde{\mathcal{Y}} = \text{span}_{\mathcal{K}}\{\mathrm{d}\tilde{y}(k), k \geq 0\}$, and $\mathcal{Z} = \text{span}_{\mathcal{K}}\{\mathrm{d}z_i, 1 \leq i \leq \beta\}$, where $z = (z_1, \ldots, z_\beta)^T$ has been defined in the Proof of Proposition 3.1. Note that, by definition, $\mathcal{Z} = \mathcal{U}_\beta \subset \mathcal{U}$.

Theorems 3.1 and 3.2 below describe the effect of unimodular and diagonal operators on the structure of a given system. A definition of the rank of a discrete time system can be found in [7], [9].

Theorem 3.1: Consider the system (1) together with a fixed m-dimensional output function $y = h(x)$. Let $U(\Delta) \in \mathcal{M}(\Delta)$ be an arbitrary unimodular operator. Consider now system (1) with a new output function defined by $\tilde{y} = U(\Delta)y$. Then the following assertions do hold:

1) The rank of system (1) with respect to the output $\tilde{y}$ equals the rank of system (1) with respect to the output $y = h(x)$.
2) The spaces $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ satisfy $\mathcal{X} \cap \mathcal{Y} = \mathcal{X} \cap \tilde{\mathcal{Y}}$

Sketch of Proof The Proof is similar to [2]. •

Theorem 3.2: Consider system (1) together with a fixed m-dimensional output function $y = h(x)$. Let $D(\Delta^{-1}) = \text{diag}(-\rho_i) \in \mathcal{D}(\Delta^{-1})$, $\rho_i \geq 0$ be an arbitrary diagonal operator. Consider now system (1) with a new output function defined by $\bar{y} = D(\Delta^{-1})y$. Define $s = \sum_{i=1}^m \rho_i$. Then the following assertions do hold:

1) There is an extended system of dimension $n + s$ such that $\bar{y}$ becomes a standard output.
2) The rank of the extended system with respect to the output $\bar{y}$ equals the rank of system (1) with respect to the output $y = h(x)$.
3) The space $\bar{\mathcal{Y}}$ satisfies $\bar{\mathcal{Y}} \cap (\mathcal{X} \oplus \Phi) = (\mathcal{Y} \cap \mathcal{X}) \oplus \Phi$

Sketch of Proof

Point 1 This is a special case of Proposition 3.1.

Point 2 Note that the scalar components of the output functions y and $\bar{y}$ are related by $\bar{y}_i = y_i^{-\rho_i}$ or, equivalently, by $y_i = \bar{y}_i^{+\rho_i}$. Suppose that the original output function satisfies a nontrivial difference equation of the following form

$$R(y_i^{+j}, 0 \leq i \leq m, 0 \leq j \leq q_i) = 0.$$

Then, the new output function $\bar{y}$ satisfies the following difference equation

$$R(\bar{y}_i^{+\rho_i+j}, 0 \leq i \leq m, 0 \leq j \leq q_i) = 0.$$

Therefore the rank of the system can not increase under the action of $D(\Delta^{-1})$. A symmetric argument completes the Proof of this Point.

The Proof of Point 3 is a little more involved and is omitted here. •

C. The notion of exogenous linearizing output

In the rest of this paper we will be concerned with the class of dynamic compensators of the following type:

$$\begin{aligned}\chi^+ &= \alpha(x, \chi, v)\\ u &= \gamma(x, \chi, v),\end{aligned} \tag{11}$$

where $\chi \in \mathcal{R}^s$ is the state of the compensator and $v \in \mathcal{R}^m$ is a new control variable. The dynamic compensator (11) is said to be regular if the closed loop system (1)-(11) with input v and output u is invertible.

The discrete-time system (1) is said to be linearizable by dynamic state feedback if there exists a regular dynamic compensator of the type (11) and a change of coordinates $\zeta = \beta(x, \chi)$ such that in new coordinates the dynamics of the closed loop system (1)-(11) is governed by the difference equation

$$\zeta^+ = A\zeta + Bv,$$

where $\zeta \in \mathcal{R}^{n+s}$, $v \in \mathcal{R}^m$, A and B are matrices of appropriate dimensions and the pair (A, B) is in Brunovsky canonical form.

As we have anticipated, the dynamic state feedback linearization problem is often addressed through the search of a so-called linearizing output.

Definition 3.2 (Exogenous linearizing output): A m-dimensional output function

$$y = h(x, u(-j), u(+k)) = \bar{h}(x, x(-j), u(+k)) \tag{12}$$

is said to be an exogenous linearizing output if it satisfies the following conditions:

1) The system (1) together with the output function (12) is invertible.
2) The space $\bar{\mathcal{Y}}$ satisfies $\bar{\mathcal{Y}} \cap \mathcal{X} = \mathcal{X}$.

The existence of an exogenous linearizing output is a sufficient condition for dynamic feedback linearization, as it is stated by the following result.

Theorem 3.3: Consider system (1) and suppose it admits an exogenous linearizing output $y = h(x, u(-j), u(+k)) = \bar{h}(x, x(-j), u(+k))$. Then, system (1) is linearizable by dynamic state feedback.

Sketch of Proof Proposition 3.1 guarantees that there exists an extended system such that y becomes an standard output. Point 3 of Theorem 3.2 implies that the extended system exhibits trivial zero dynamics. From this point, an application of the well known dynamic extension algorithm [11] suffices to construct a dynamic compensator that fully linearizes the system. •

The existence of an exogenous linearizing output can be characterized in terms a system of linearizing one-forms.

Theorem 3.4: Let $\Omega = (\omega_1, \ldots, \omega_m)^T$ be a system of linearizing one-forms. There exists a exogenous linearizing output $y = h(x, u(-j), u(+k)) = \bar{h}(x, x(-j), u(+k))$ if and only if there exist transformal operators $U_1(\Delta), U_2(\Delta) \in \mathcal{M}(\Delta)$ and $\Lambda(\Delta^{-1}) \in \mathcal{D}(\Delta^{-1})$ such that

$$\Theta = U_1(\Delta)\Lambda(\Delta^{-1})U_2(\Delta)\Omega, \qquad (13)$$

with $\mathrm{d}\Theta = 0$. In that case, we have $\Theta = (\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_m)^T$ and $y = (\zeta_1, \ldots, \zeta_m)^T$ is an exogenous linearizing output.

The Proof of Theorem 3.4 is rather technical and is omitted here.

Remark 3.1: Theorem 3.4 is not fully constructive since there is no general procedure to construct $U_1(\Delta)$ and $U_2(\Delta)$ or to characterize their existence. However, conceptually, it provides a complete solution to the search of exogenous linearizing outputs. Moreover, it is interesting to note that when $\Lambda(\Delta^{-1}) = I_m$, the identity operator, the conditions given by Theorem 3.4 reduce to the conditions of existence of a endogenous linearizing output, already identified in [3].

mimics completely analogous results obtained for flat continuous time systems.

D. Example

Consider the following academic example, whose structure is derived from the structure of the exact discrete-time model of a wheeled mobile robot [16]. However, by no means the variables of this example have any physical interpretation since it has been designed to capture, in a simplified way, the robot structural properties under interest.

$$\begin{array}{rcl} x_1^+ & = & x_1 + u_1 \cos u_2 \\ x_2^+ & = & x_2 + u_1 \sin u_2 \\ x_3^+ & = & x_3 + u_2. \end{array} \qquad (14)$$

Compute

$$\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}} \{\omega_1\},$$

where $\omega_1 = \sin u_2^- \mathrm{d}x_1 - \cos u_2^- \mathrm{d}x_2 + u_1^- \mathrm{d}x_3$, and

$$\mathcal{H}_1 = \mathrm{span}_{\mathcal{K}} \left\{\omega_1, \omega_1^+, \omega_2\right\}$$

where $\omega_2 = \mathrm{d}x_3$ for instance.

Let $a = x_1 \cos u_2^- + x_2 \sin u_2^-$. The conditions in Theorem 3.4 are fulfilled and the one-forms

$$\Theta = \left[\begin{array}{cc} 1 & (a-u_1)\Delta - a \\ 0 & 1 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ 0 & \Delta^{-1} \end{array}\right] \left(\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right). \qquad (15)$$

are exact. Equation (15) is a special case of equation (13) where U_2 is the identity matrix. Thus, an exogenous output y is computed as

$$\mathrm{d}y = \left[\begin{array}{cc} 1 & (a-u_1) - a\Delta^{-1} \\ 0 & \Delta^{-1} \end{array}\right] \left(\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right), \qquad (16)$$

.

The exogenous linearizing output is obtained as

$$\begin{array}{rcl} y_1 & = & x_1 \sin(x_3 - x_3^-) - x_2 \cos(x_3 - x_3^-), \\ y_2 & = & x_3^-. \end{array} \qquad (17)$$

From equation (14) it follows that $x_3^- = x_3 - u_2^-$. Therefore, an alternative representation of the exogenous output (17) is given by

$$\begin{array}{rcl} y_1 & = & x_1 \sin(u_2^-) - x_2 \cos(u_2^-), \\ y_2 & = & x_3 - u_2^-. \end{array} \qquad (18)$$

The application of Proposition 3.1 is as follows. First, note that $\alpha = 1$, $\beta = 0$. Therefore, it follows that

$$\mathcal{Q} \sim \mathrm{span}_{\mathcal{K}} \left\{\mathrm{d}u_2^-\right\} = \mathrm{span}_{\mathcal{K}} \left\{\mathrm{d}x_3^-\right\}.$$

According to Theorems 3.3 and 3.4, system (14) is linearizable by exogenous dynamic feedback. Note that, adding a pure unit delay after the state variable x_3, the exogenous output function (17) become a standard output for the extended system. More precisely, consider the exogenous extension

$$\xi^+ = x_3. \qquad (19)$$

Finally, the dynamic compensator dynamic compensator that fully linearizes the system (14)-(19) is given by

$$u_1 = \frac{v_1 - x_1 \sin(v_2 - z) + x_2 \cos(v_2 - z)}{\sin(v_2 + x_3)} \qquad (20)$$

$$u_2 = z - x_3 \qquad (21)$$

$$z^+ = v_2 \qquad (22)$$

where $v = (v_1, v_2)$ is a new control input.

IV. Conclusions

The notions of exogenous output and exogenous feedback have been introduced for discrete-time nonlinear systems. These notions allow to establish new sufficient conditions for dynamic feedback linearization. The new conditions include, as a particular case, the previously known conditions for the existence of a so-called (endogenous) linearizing output. The results have been applied to a discrete-time system whose dynamics mimics that of a wheeled mobile robot.

The fact that a discrete-time system can be rendered linear by exogenous feedback constitutes a fundamental difference with respect to the continuous-time setting. The complete characterization of discrete time systems which can be fully linearized by general dynamic compensation remains an open problem.

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