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Results on input-to-state stability for hybrid systems

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Abstract— **We show that, like continuous-time systems, zero-input locally asymptotically stable hybrid systems are locally input-to-state-stable (LISS). We demonstrate by examples that, unlike continuous-time systems, zero-input locally exponentially stable hybrid systems may not be LISS with linear gain, input-to-state stable (ISS) hybrid systems may not admit any ISS Lyapunov function, and nonuniform ISS hybrid systems may not be (uniformly) ISS. We then provide a strengthened ISS condition as an equivalence to the existence of an ISS Lyapunov function for hybrid systems. This strengthened condition reduces to standard ISS for continuous-time and discrete-time systems. Finally under some other assumptions we establish the equivalence among ISS, several asymptotic characterizations of ISS, and the existence of an ISS Lyapunov function for hybrid systems.**

I. INTRODUCTION

Input-to-state stability (ISS), introduced in [15], is a useful stability notion for studying the robustness of nonlinear control systems affected by noise or disturbances [16], [11], [8]. Some key results related to ISS for continuous-time systems are:

- zero-input local asymptotic stability (0-LAS) implies local input-to-state stability (LISS) [18, Lemma I.2];
- ISS is equivalent to the existence of an ISS Lyapunov function [17];
- ISS has asymptotic characterizations [18, Theorem 1];
- zero-input local exponential stability (0-LES) and local Lipschitz property imply LISS with linear gain [3].

In this paper, we investigate similar statements for hybrid systems.

Hybrid systems are those whose trajectories can flow in continuous time and also jump at discrete instants. The system variables can be dynamical processes (states) and logical processes (modes). In this paper, we will mainly consider a hybrid system that is a combination of a differential equation on a flow set and a difference equation on a jump set. To study (robust) stability of hybrid systems, we will use the solution defined in [6], [7]. This solution notion has been used to establish sequential compactness of solutions and the "upper semicontinuous" dependence of solutions with respect to (w.r.t.) initial conditions [7], and hence to make LaSalle's invariance principle [14] and smooth converse Lyapunov theorems [4] available for hybrid systems. In turn, these results enable the results of the current paper.

For starters, we show the implication from 0-LAS to LISS for hybrid systems (see Theorem 1 in Section IV) by using a result in [7]. However, the combination of the four components of the hybrid system — differential equation, flow set, difference equation, and jump set — may also exhibit complex dynamical behaviors and hence yield different behaviors from those of continuous-time systems. For hybrid systems, we will demonstrate by examples that ISS may not imply the existence of an ISS Lyapunov function, that 0-LES and local Lipschitz property may not imply LISS with linear gain, and that nonuniform ISS (*i.e.* the combination of the asymptotic gain property and global stability) may not imply (uniform) ISS.

It has been shown that ISS is equivalent to the existence of an ISS Lyapunov function for continuous-time systems [17], discrete-time systems [9], and switched systems with arbitrary switching signals [13]. For hybrid systems, it is not hard to show the implication from ISS Lyapunov function to ISS by using a hybrid comparison lemma; hence, the more technical work is to show the converse: under what *additional* conditions does ISS imply the existence of an ISS Lyapunov function? The answers to this question are stated as Theorem 2 and Theorem 3 in Section IV. We provide in Theorem 2 a strengthened ISS condition as an equivalence to the existence of an ISS Lyapunov function for hybrid systems. This strengthened condition reduces to standard ISS for continuous-time systems and discrete-time systems. We present in Theorem 3 some other assumptions to make asymptotic characterizations of ISS available and hence to establish the equivalence between them and the existence of an ISS Lyapunov function for hybrid systems.

The rest of the paper is organized as follows. Section II provides a description of hybrid systems, solutions, and stability concepts. Section III gives the three aforementioned counterexamples. Section IV presents main theorems, whose (sketches of) proofs are provided in appendices so as to make this paper self-contained to some extent. Section V gives conclusions.

Finally, we list the basic definitions and notation:

- $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{N}_+ = \{0, 1, 2, ...\}$.
- $\mathcal{B}$ represents the open unit ball in Euclidean space.
- Given a vector $v = [v_1,\ v_2,\ \cdots,\ v_n] \in \mathbb{R}^n$, v' denotes its transpose, and $|v|$ denotes its Euclidean norm, *i.e.* $|v| = \left(\sum_{i=1}^n |v_i|^n\right)^{1/n}$.
- Given a set $\mathcal{A} \subset \mathbb{R}^n$, the sets $\overline{\mathcal{A}}$ and $\overline{\text{co}}\mathcal{A}$ stand for the closure and the closed convex hull, respectively, of $\mathcal{A}$.
- Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, a point $x \in \mathbb{R}^n$, and a constant $c > 0$, denote $|x|_\mathcal{A} := \min_{y \in \mathcal{A}} |x - y|$ and $\mathcal{A}_{[c]} := \{\xi \in \mathbb{R}^n : |\xi|_\mathcal{A} \leq c\}$.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to

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class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to *class-$\mathcal{K}_\infty$* if, in addition, it is unbounded.

- Denote by α^{-1} the inverse function of $\alpha \in \mathcal{K}$.
- A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to *class-$\mathcal{KL}$* ($\beta \in \mathcal{KL}$) if it satisfies: (i) for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s,t) = 0$, and (ii) for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$.
- A function $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to *class-$\mathcal{KLL}$* ($\gamma \in \mathcal{KLL}$) if, for each $r \geq 0$, $\gamma(\cdot, \cdot, r) \in \mathcal{KL}$ and $\gamma(\cdot, r, \cdot) \in \mathcal{KL}$.

II. Hybrid systems and stability definitions

A. Hybrid systems

Consider hybrid systems $\mathcal{H}_u$ with state x and input u

$$\mathcal{H}_u := \begin{cases} \dot{x} = f(x,u) & \text{for } x \in C, \\ x^+ = g(x,u) & \text{for } x \in D, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $C, D \subset \mathbb{R}^n$. For simplicity of notation, we use the data (f, g, C, D) to represent hybrid system $\mathcal{H}_u$.

The solutions to $\mathcal{H}_u$ are defined on hybrid time domains, as used in [6], [7], and [5]. We call a subset $E \subset \mathbb{R}_+ \times \mathbb{N}_+$ a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_{J+1}$. We say E is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid time domain. On each hybrid time domain there is a natural ordering of points: $(t, j) \preceq (s, k)$ if $t + j \leq s + k$. Equivalently, this can be characterized by $t \leq s$ and $j \leq k$.

A *hybrid signal* is a function defined on a hybrid time domain. Specifically, hybrid signal $u : \operatorname{dom} u \mapsto \mathbb{R}^m$ is called a *hybrid input* in this paper. A hybrid signal $x : \operatorname{dom} x \mapsto \mathbb{R}^n$ is called a *hybrid arc* if $x(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc $x : \operatorname{dom} x \mapsto \mathbb{R}^n$ and a hybrid input $u : \operatorname{dom} u \mapsto \mathbb{R}^m$ is a *solution pair* (x,u) to $\mathcal{H}_u$ if

(S0) $\operatorname{dom} x = \operatorname{dom} u$;

(S1) for all $j \in \mathbb{N}_+$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t,j) \in C, \quad \dot{x}(t,j) = f(x(t,j), u(t,j)); \tag{2}$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t,j) \in D, \quad x(t,j+1) = g(x(t,j), u(t,j)). \tag{3}$$

We emphasize from the definition of solution pair that the jump set D (respectively, the flow set C) enables jumps (respectively, flows).

Given any hybrid input, define its supremum norm from $(0, 0)$ to $(t, j) \in \operatorname{dom} u$ as

$$\|u\|_{(t,j)} := \max\left\{ \operatorname*{ess.sup}_{\substack{(s,k)\in \operatorname{dom} u \setminus \Gamma(u),\\ (s,k)\preceq(t,j)}} |u(s,k)|, \sup_{\substack{(s,k)\in\Gamma(u),\\ (s,k)\preceq(t,j)}} |u(s,k)| \right\},$$

where $\Gamma(u)$ denotes the set of all $(s, k) \in \operatorname{dom} u$ such that $(s, k+1) \in \operatorname{dom} u$. When $t+j \to \infty$, $\|u\|_{(t,j)}$ is denoted by $\|u\|_\infty$. We denote by $\mathcal{L}_\infty^m$ the set of hybrid inputs (in $\mathbb{R}^m$) that have finite $\|\cdot\|_\infty$. Throughout this paper, we assume $u \in \mathcal{L}_\infty^m$ for (1).

A solution pair to $\mathcal{H}_u$ is *maximal* if it cannot be extended, and it is *complete* if its hybrid time domain is unbounded. Denote by $\mathcal{S}_u(\xi)$ the set of all maximal solution pairs (x, u) to $\mathcal{H}_u$ with $x(0, 0) = \xi \in \mathbb{R}^n$. By slight abuse of notation, we will use $x(t, j, \xi, u)$ to denote $x(\cdot, \cdot)$ evaluated at $(t, j) \in \operatorname{dom} x$, where $(x, u) \in \mathcal{S}_u(\xi)$. When $u \equiv 0$ and $(x, 0) \in \mathcal{S}_u(\xi)$, we simply say $x \in \mathcal{S}_0(\xi)$ and call x a maximal solution starting from $\xi \in \mathbb{R}^n$. The hybrid system $\mathcal{H}_u$ is *forward complete* if, for each $\xi \in \mathbb{R}^n$, each $(x, u) \in \mathcal{S}_u(\xi)$ is complete.

We impose the following basic conditions for $\mathcal{H}_u$:

Standing Assumption 1: For $\mathcal{H}_u = (f, g, C, D)$,

- f and g are continuous, and f is locally Lipschitz in x uniformly on any compact subset of $\mathbb{R}^m$ [1];
- C and D are closed in $\mathbb{R}^n$, and $C \cup D = \mathbb{R}^n$.

The solution results in [2], [5], [7] tell us that the existence of solutions with $u \in \mathcal{L}_\infty^m$ to the hybrid system (1) is guaranteed by Standing Assumption 1.

B. Stability

Consider a hybrid system $\mathcal{H}_u$ in (1) and let $\mathcal{A}$ be a compact subset of $\mathbb{R}^n$ (throughout this subsection). The set $\mathcal{A}$ is *0-(locally) stable* (0-LS) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\xi \in \mathcal{A}_{[\delta]}$, each solution $x \in \mathcal{S}_0(\xi)$ is complete and satisfies $|x(t, j, \xi, 0)|_\mathcal{A} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} x$; it is *0-attractive* if there exists $\mu > 0$ such that for each $\xi \in \mathcal{A}_{[\mu]}$, each solution $x \in \mathcal{S}_0(\xi)$ is complete and satisfies $\lim_{(t,j)\in\operatorname{dom} x,\ t+j\to\infty} |x(t, j, \xi, 0)|_\mathcal{A} = 0$; and it is *0-input locally asymptotically stable* (0-LAS) if it is both 0-stable and 0-attractive. The set of points $\xi \in \mathbb{R}^n$ such that all solutions in $\mathcal{S}_0(\xi)$ are complete and converge to $\mathcal{A}$ is called the *0-input basin of attraction* for $\mathcal{A}$ and is denoted $\mathcal{B}_\mathcal{A}^0$. From Proposition 6.1(i) in [7], we know that $\mathcal{B}_\mathcal{A}^0$ is **open** (since $C \cup D = \mathbb{R}^n$). The set $\mathcal{A}$ is *0-input globally asymptotically stable* (0-GAS) if $\mathcal{A}$ is 0-LAS and $\mathcal{B}_\mathcal{A}^0 = \mathbb{R}^n$. The set $\mathcal{A}$ is *0-input locally exponentially stable (0-LES)* if there exist $r > 0$, $\lambda > 0$, and $c > 0$ such that, for each $\xi \in \mathcal{A}_{[r]}$, each solution $x \in \mathcal{S}_0(\xi)$ satisfies

$$|x(t, j, \xi, 0)|_\mathcal{A} \leq c|\xi|_\mathcal{A} e^{-\lambda(t+j)} \quad \forall (t, j) \in \operatorname{dom} x.$$

Definition 1: System $\mathcal{H}_u$ is (**uniformly**) *input-to-state stable (ISS) w.r.t.* $\mathcal{A}$ if there exist $\gamma \in \mathcal{KLL}$ and $\kappa \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$, each solution pair $(x, u) \in \mathcal{S}_u(\xi)$ satisfies

$$|x(t, j, \xi, u)|_\mathcal{A} \leq \max\left\{\gamma(|\xi|_\mathcal{A}, t, j), \kappa\left(\|u\|_{(t,j)}\right)\right\} \tag{4}$$

for each $(t, j) \in \operatorname{dom} x$.

[1] For each compact $U \subset \mathbb{R}^m$ and each compact $K \subset \mathbb{R}^n$, there exists some constant $L > 0$ such that $|f(x, u) - f(z, u)| \leq L|x - z|$ for all $x, z \in K$ and all $u \in U$.

Definition 2: System $\mathcal{H}_u$ is *locally input-to-state stable (LISS) w.r.t.* $\mathcal{A}$ if there exist $r > 0$, $\gamma \in \mathcal{KLL}$, and $\kappa \in \mathcal{K}$ such that, for each $\xi \in \mathcal{A}_{[r]}$, each solution pair $(x, u) \in \mathcal{S}_u(\xi)$ with $\|u\|_\infty \leq r$ satisfies (4) for each $(t, j) \in \operatorname{dom} x$.

System $\mathcal{H}_u$ is *LISS w.r.t.* $\mathcal{A}$ *with finite gain* if κ in Definition 2 is a linear function.

Like continuous-time and discrete-time systems, we can have Lyapunov characterizations of ISS for hybrid systems (cf. [17, Theorem 1] and [9, Theorem 4]).

Definition 3: A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an *ISS-Lyapunov function w.r.t.* $\mathcal{A}$ for system (1) if there exist α_1, α_2, $\alpha_3 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ such that

$$\alpha_1(|\xi|_\mathcal{A}) \leq V(\xi) \leq \alpha_2(|\xi|_\mathcal{A}) \quad \forall \xi \in \mathbb{R}^n \tag{5}$$

and, for all $\xi \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$ satisfying $|\xi|_\mathcal{A} \geq \rho(|u|)$,

$$\nabla V(\xi) \cdot f(\xi, u) \leq -\alpha_3(|\xi|_\mathcal{A}) \quad \forall \xi \in C, \tag{6}$$

$$V(g(\xi, u)) - V(\xi) \leq -\alpha_3(|\xi|_\mathcal{A}) \quad \forall \xi \in D. \tag{7}$$

Inspired by [8, Lemma 10.4.2], we can have an equivalent definition of ISS-Lyapunov function for (1).

Proposition 1: For system (1), a smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$, satisfying (5) with α_1, $\alpha_2 \in \mathcal{K}_\infty$, is an ISS-Lyapunov function w.r.t. $\mathcal{A}$ if and only if there exist $\hat{\alpha}_3 \in \mathcal{K}_\infty$ and $\hat{\rho} \in \mathcal{K}$ such that, for all $\xi \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\nabla V(\xi) \cdot f(\xi, u) \leq -\hat{\alpha}_3(|\xi|_\mathcal{A}) + \hat{\rho}(|u|) \quad \forall \xi \in C, \tag{8}$$

$$V(g(\xi, u)) - V(\xi) \leq -\hat{\alpha}_3(|\xi|_\mathcal{A}) + \hat{\rho}(|u|) \quad \forall \xi \in D. \tag{9}$$

The next proposition, whose converse does not generally hold (see Example 2 in Subsection III-B), is a corollary of Theorem 2 in Section IV.

Proposition 2: If system (1) has an ISS-Lyapunov function w.r.t. $\mathcal{A}$, then $\mathcal{H}_u$ is ISS w.r.t. $\mathcal{A}$.

Like continuous-time and discrete-time systems, we can also have asymptotic characterizations of ISS for hybrid systems (cf. [18, Theorem 1] and [9, Theorem 4]).

Definition 4: System $\mathcal{H}_u$ has the *asymptotic gain (AG) property w.r.t.* $\mathcal{A}$ if there exists $\kappa \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$, each solution pair $(x, u) \in \mathcal{S}_u(\xi)$ satisfies

$$\limsup_{(t,j) \in \operatorname{dom} x, t+j \to \infty} |x(t, j, \xi, u)|_\mathcal{A} \leq \kappa(\|u\|_\infty).$$

Definition 5: System $\mathcal{H}_u$ has *global stability (GS) w.r.t.* $\mathcal{A}$ if there exists α, $\kappa \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$, each solution pair $(x, u) \in \mathcal{S}_u(\xi)$ satisfies

$$\sup_{(t,j) \in \operatorname{dom} x} |x(t, j, \xi, u)|_\mathcal{A} \leq \max\left\{\alpha(|\xi|_\mathcal{A}), \kappa(\|u\|_\infty)\right\}.$$

Definition 6: System $\mathcal{H}_u$ is *nonuniform ISS w.r.t.* $\mathcal{A}$ if it has the AG property and GS w.r.t. $\mathcal{A}$.

Proposition 3: For system (1), if $\mathcal{H}_u$ is ISS w.r.t. $\mathcal{A}$, then it is nonuniform ISS w.r.t. $\mathcal{A}$.

Proposition 3 is straightforward, but its converse does not generally hold (see Example 3 in Subsection III-C).

A weaker concept than the AG property is the following.

Definition 7: System $\mathcal{H}_u$ has the *limit property w.r.t.* $\mathcal{A}$ if there exists $\kappa \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$, each solution pair $(x, u) \in \mathcal{S}_u(\xi)$ satisfies

$$\inf_{(t,j) \in \operatorname{dom} x} |x(t, j, \xi, u)|_\mathcal{A} \leq \kappa(\|u\|_\infty).$$

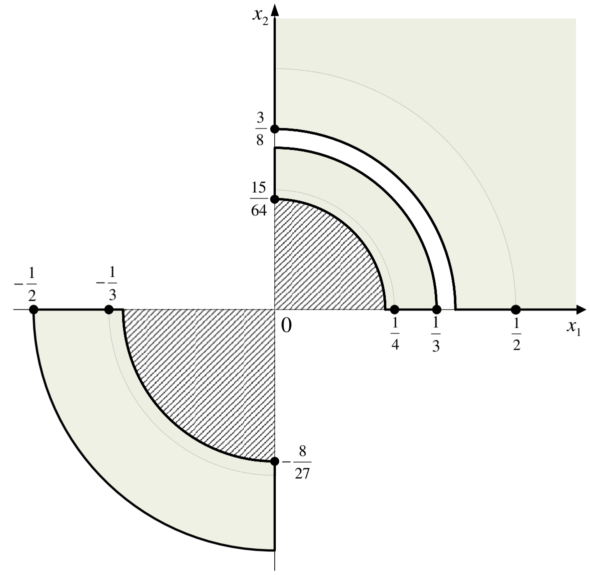

Fig. 1. Example 1 (the lined area is not plotted)

III. Examples

A. *Example 1: 0-LES + Lipschitz $\not\Rightarrow$ LISS with finite gain*

In this subsection, we provide a planar example of hybrid systems, where 0-LES and the local Lipschitz property of f do not imply LISS with finite gain.

Let $x = [x_1,\ x_2]' \in \mathbb{R}^2$ and $u \in \mathbb{R}$. Define

$$f(x, u) := [x_2 + u, \quad -x_1 + u]' ,$$
$$g(x, u) := 0 ,$$
$$D := \overline{D_+ \cup D_- \cup \{0\}} ,$$
$$C := \overline{\mathbb{R}^2 \setminus D} ,$$

where

$$D_+ := \{x : x_1 \geq 0, x_2 \geq 0\} \bigcap \left(\left\{x : |x| \geq \frac{3}{8}\right\} \bigcup \bigcup_{n=1}^{\infty} \left\{x : \frac{1}{2n+2} - \frac{1}{(2n+2)^3} \leq |x| \leq \frac{1}{2n+1}\right\} \right),$$

$$D_- := \{x : x_1 \leq 0, x_2 \leq 0\} \bigcap \bigcup_{n=1}^{\infty} \left\{x : \frac{1}{2n+1} - \frac{1}{(2n+1)^3} \leq |x| \leq \frac{1}{2n}\right\}.$$

Note that f is locally Lipschitz. As shown in Fig. 1, inside the disk with radius $1/2$, the jump set D is partitioned alternatively between the first quadrant and the third quadrant with overlap $\frac{1}{n^3}$ (for example, see grey area in Fig. 1).

Define the hybrid system $\mathcal{H}_u := (f, g, C, D)$ and let $\mathcal{A} := \{0\}$. One can verify $\mathcal{H}_u$ satisfies Standing Assumption 1. When $u = 0$, since no circle in $\mathbb{R}^2$ is a subset of C, we conclude, for each $\xi \in \mathbb{R}^2$, each $x \in \mathcal{S}_0(\xi)$ satisfies

$$|x(t, j, \xi, 0)| \leq e^{2\pi} |\xi| e^{-(t+j)} \quad \forall (t, j) \in \operatorname{dom} x,$$

which means the origin is 0-LES (in a global sense). Now, pick any initial condition $\xi \in C$ with $|\xi| = \frac{1}{n}$. There exists some solution $x(\cdot,\cdot,\xi,u)$, where $(x,u)\in\mathcal{S}_u(\xi)$ and $|u(\cdot,0)|\propto\frac{1}{n^3}$, flowing on C and rotating with a radius of approximately $\frac{1}{n}$ (say $n=3$, then there exists some solution $x(\cdot,\cdot,\xi,u)$ that can rotate with a radius of approximately $\frac{1}{3}$ in the first, second, and fourth quadrant but exactly $\frac{8}{27}$ in the third quadrant). Namely,

$$|x(t,j,\xi,u)|\approx\frac{1}{n}\propto(\|u\|_{(t,j)})^{\frac{1}{3}},\ \forall(t,j)\in\operatorname{dom}x,$$

which means that $\mathcal{H}_u$ can not be LISS with finite gain.

B. Example 2: ISS $\not\Rightarrow$ existence of ISS Lyapunov functions

The following planar example shows the converse of Proposition 2 fails for hybrid systems.

Let $x=[x_1,\ x_2]'\in\mathbb{R}^2$ and $u=[u_1,\ u_2]'\in\mathbb{R}^2$. Define

$$\begin{aligned}
f(x,u)&:=[|u_1-u_2|-1,\quad u_1-u_2]'\ ,\\
g(x,u)&:=0\ ,\\
C&:=\{x\in\mathbb{R}^2: x_1\ge 0, x_2=0\}\ ,\\
D&:=\mathbb{R}^2\ .
\end{aligned}$$

One can verify that the hybrid system $\mathcal{H}_u:=(f,g,C,D)$ satisfies Standing Assumption 1 and is ISS w.r.t. $\mathcal{A}:=\{0\}$. Nevertheless, $\mathcal{H}_u$ does not admit an ISS Lyapunov function. Otherwise, one could pick $u=[2,\ 0]'$ and $\upsilon=[0,\ 2]'$; then for any $\xi\in C$ satisfying $|\xi|\ge\max\{\rho(|u|),\rho(|\upsilon|)\}$, where $\rho\in\mathcal{K}$ comes from Definition 3, using (6) we have,

$$\nabla V(\xi)\cdot(f(\xi,u)+f(\xi,\upsilon))\le-2\alpha_3(V(\xi)).\qquad(10)$$

On the other hand, define the function $\widetilde{f}:\mathbb{R}^2\to\mathbb{R}^2$ by letting $\widetilde{f}(\xi):=f(\xi,u)+f(\xi,\upsilon)=[2,\ 0]'$ for each $\xi\in\mathbb{R}^2$ and define a new hybrid system $\widetilde{\mathcal{H}}:=(\widetilde{f},0,C,D)$ (without inputs), and then we can have some solution to $\widetilde{\mathcal{H}}$ flowing on C and blowing up. This contradicts the combination of (10) and (5).

C. Example 3: nonuniform ISS $\not\Rightarrow$ ISS

The following planar example shows the converse of Proposition 3 fails for hybrid systems.

Let $x=[x_1,\ x_2]'\in\mathbb{R}^2$ and $u\in\mathbb{R}$. Define a periodic function $\psi:\mathbb{R}\to\mathbb{R}$ by letting $\psi(x_1):=|x_1-2n|^3$ for each integer n and each $x_1\in[2n-1,2n+1]$. Define

$$\begin{aligned}
f(x,u)&:=[\cos(\theta(x,u)),\quad\sin(\theta(x,u))]'\ ,\\
g(x,u)&:=0\ ,\\
C&:=\overline{\cup_{n=1}^{\infty}(C_+^n\cap C_-^n)}\ ,\\
D&:=\overline{\mathbb{R}^2\setminus C}\ ,
\end{aligned}$$

where

$$\begin{aligned}
\theta(x,u)&:=\left[-\frac{\pi}{2}+\arcsin\psi(x_1)\right]\sin\left(\frac{u}{2}\right),\\
C_+^n&:=\{x: x_1\ge 2n-1 \text{ and } x_1-2n\le x_2\le 2n-x_1\}\,,\\
C_-^n&:=\bigcup_{\ell=1}^{\infty}\left[2n-\frac{1}{2\ell},2n-\frac{1}{2\ell+1}\right]\times\mathbb{R}.
\end{aligned}$$

Note that f is locally Lipschitz and that $f(x,u)$ has no convex property w.r.t. u.

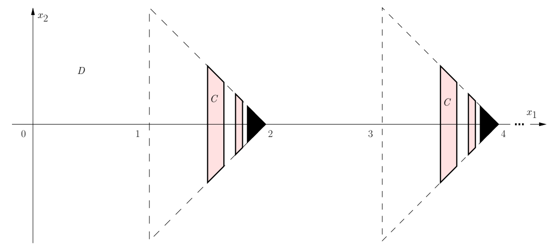

Fig. 2. Example 3 (the dark area is not plotted)

Let $\mathcal{A}:=\{0\}$ and define the hybrid system $\mathcal{H}_u:=(f,g,C,D)$, which indeed satisfies Standing Assumption 1. Note that $f(x,u)\ne 0$ and $\cos(\theta(x,u))\in[0,1]$ for all x and u and that $\cos(\theta(x,u))=0$ only for $\sin(u/2)=\pm1$ and $x_1=2n$, where n is an arbitrary integer. These imply that $|x_1(\cdot,0,\xi,u)|$ increases as long as $x(\cdot,0,\xi,u)$ flows on C. Since C is defined as a union of isolated trapezoids (for example, see the grey area), each $x(\cdot,\cdot,\xi,u)$ with $\xi\in C$ will flow to the boundary of C in finite time and then jump to the origin. Therefore, we conclude that, for each $\xi\in\mathbb{R}^2$, each $(x,u)\in\mathcal{S}_u(\xi)$ satisfies $|x(t,j,\xi,u)|\le 2|\xi|$ for all $(t,j)\in\operatorname{dom}x$ and $\lim_{(t,j)\in\operatorname{dom}x,\ t+j\to\infty}|x(t,j,\xi,u)|=0$, which establish GS and the AG property for $\mathcal{H}_u$. Nevertheless, $\mathcal{H}_u$ is not ISS. Suppose there exist $\gamma\in\mathcal{KLL}$ and $\kappa\in\mathcal{K}$ such that (4) holds. Then pick two positive integers n and ℓ such that $n>\kappa(\pi)$ and $\gamma(2n,\ell,0)\le 1$. Consider $\xi=\left[2n-\frac{1}{2\ell},\quad 0\right]'$ and pick $u(\cdot,0)\in\{-\pi,\pi\}$ in such a way to assure $|x_2(\cdot,0,\xi,u)|\le\frac{1}{2\ell+1}$. Consequently, we have $\dot{x}_1=|x_1-2n|^3$. Define $z:=2n-x_1$ gives the differential equation $\dot{z}=-z^3$, which takes time $t^*=2\ell+\frac{1}{2}$ to decrease from $z(0)=\frac{1}{2\ell}$ to $z(t^*)=\frac{1}{2\ell+1}$. In particular, $|x(t^*,0,\xi,u)|>\max\{\kappa(\pi),1\}$, which contradicts (4).

IV. Main Results

A. 0-LAS implies LISS

Inspired by a result on LISS for continuous-time systems [18, Lemma I.2], we propose the following implication from 0-LAS to LISS.

Theorem 1: If the compact set $\mathcal{A}\subset\mathbb{R}^n$ is 0-LAS for $\mathcal{H}_u$, then $\mathcal{H}_u$ is LISS w.r.t. $\mathcal{A}$.

Proof: See Appendix I. ■

Remark 1: The proof of Theorem 1 does not require the Lipschitz condition but only the continuity of f.

B. Existence of ISS Lyapunov function

As Example 2 in Subsection III-B shows, ISS does not imply the existence of an ISS Lyapunov function for hybrid systems. The main reason behind this is that the solutions to the differential inclusion $\dot{x}\in f(x,\varepsilon\overline{\mathcal{B}})$ may not be dense any more in the solutions to $\dot{x}\in\overline{\text{co}}f(x,\varepsilon\overline{\mathcal{B}})$ on the flow set C, which, unfortunately, may not be $\mathbb{R}^n$ for hybrid systems (cf. the Relaxation Theorem for differential inclusions, Theorem 10.4.4 in [1]). In order to achieve ISS characterizations, one may require nice behaviors on the flow and jump set boundaries (like Theorem 2), or one may require $f(x,u)$ to have a convex property w.r.t. u (like Theorem 3 in the next subsection).

Theorem 2: Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. For (1), $\mathcal{H}_u$ admits an ISS-Lyapunov function w.r.t. $\mathcal{A}$ if and only if there exists a continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_+$ such that $\mathcal{A} = \{\xi : \sigma(\xi) = 0\}$ and the hybrid system $\mathcal{H}_{u\sigma} := (f, g, C_\sigma, D_\sigma)$ is forward complete and ISS w.r.t. $\mathcal{A}$, where

$$C_\sigma := \{x \in \mathbb{R}^n : (x + \sigma(x)\overline{\mathcal{B}}) \cap C \neq \emptyset\},$$
$$D_\sigma := \{x \in \mathbb{R}^n : (x + \sigma(x)\overline{\mathcal{B}}) \cap D \neq \emptyset\}.$$

Sketch of proof: See Appendix II. ■

Remark 2: If $C = \mathbb{R}^n$ and $D = \emptyset$, then $C_\sigma = C$ and hence Theorem 2 for $\mathcal{A} = \{0\}$ becomes the equivalence between ISS and ISS Lyapunov function for continuous-time systems (see the equivalence 1⇔2 of Theorem 1 in [17]). Similarly, if $D = \mathbb{R}^n$ and $C = \emptyset$, then Theorem 2 for $\mathcal{A} = \{0\}$ becomes the one for discrete-time systems (see the equivalence 1⇔4 of Theorem 1 in [9]).

Remark 3: The property of forward completeness does not necessarily carry over from $\mathcal{H}_u$ to $\mathcal{H}_{u\sigma}$. Consider Example 2 by redefining $f(x, u) := \left[x_1^3|u_1 - u_2|, \quad u_1 - u_2\right]'$. Clearly, $\mathcal{H}_u$ is forward complete, but $\mathcal{H}_{u\sigma}$ is not: one can find $x(\cdot, 0)$ to flow in C_σ and blow up (in the x_1 coordinate) in finite time, where (x, u) is a maximal solution pair starting from C_σ and $u(\cdot, 0)$ is chosen appropriately.

C. Asymptotic characterizations of ISS

For continuous-time systems, nonuniform ISS is equivalent to ISS even without assuming that $f(x, u)$ has a convex property w.r.t. u (see [18, Theorem 1]), but there is no such equivalence for hybrid systems (see Example 3 in Subsection III-C). If $f(x, u)$ is assumed with a convex property w.r.t. u, then asymptotic characterizations of ISS will carry over from continuous-time systems to hybrid systems. The following theorem provides a hybrid version of [18, Theorem 1] and [9, Theorem 1].

Theorem 3: Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. For (1), assume that $\mathcal{H}_u$ is forward complete and that $f(x, \varepsilon\overline{\mathcal{B}}) = \overline{\text{co}} f(x, \varepsilon\overline{\mathcal{B}})$ for each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$. Then the following statements are equivalent:

1) $\mathcal{H}_u$ is ISS w.r.t. $\mathcal{A}$;
2) $\mathcal{H}_u$ is nonuniform ISS w.r.t. $\mathcal{A}$;
3) $\mathcal{H}_u$ has the AG property w.r.t. $\mathcal{A}$ and the set $\mathcal{A}$ is 0-LS for $\mathcal{H}_u$;
4) $\mathcal{H}_u$ satisfies the limit property w.r.t. $\mathcal{A}$ and the set $\mathcal{A}$ is 0-LS for $\mathcal{H}_u$;
5) $\mathcal{H}_u$ satisfies the AG property and is LISS w.r.t. $\mathcal{A}$;
6) $\mathcal{H}_u$ admits an ISS-Lyapunov function w.r.t. $\mathcal{A}$.

Sketch of proof: See Appendix III. ■

V. Conclusions

We have demonstrated similarities and differences between ISS results for continuous-time systems and hybrid systems. We have investigated conditions to guarantee Lyapunov and asymptotic characterizations of ISS for hybrid systems. These characterizations parallel what has been developed previously for continuous-time and discrete-time systems.

Appendix I
Proof of Theorem 1

Suppose $\mathcal{H}_u$ is 0-LAS with the 0-input basin of attraction $\mathcal{B}^0_{\mathcal{A}}$. Then Theorem 6.2 in [7] implies the existence of $c > 0$, a proper indicator ω (see definition in [4]) for $\mathcal{A}$ on $\mathcal{B}^0_{\mathcal{A}}$, and a continuous $\gamma \in \mathcal{KLL}$ such that $\omega(\eta) = |\eta|_{\mathcal{A}}$ for all $\eta \in \mathcal{A}_{[c]}$ and, for each $\xi \in \mathcal{B}^0_{\mathcal{A}}$, each $x \in \mathcal{S}_0(\xi)$ satisfies

$$\omega(x(t, j, \xi, 0)) \leq \gamma(\omega(\xi), t, j) \quad \forall (t, j) \in \operatorname{dom} x.$$

Furthermore, using Theorem 6.2 in [7], we have the following claim.

Claim 1: For each $\varepsilon > 0$ and each compact set $K \subset \mathcal{B}^0_{\mathcal{A}}$, there exists $\delta > 0$ such that the solutions x_δ to the hybrid inclusion $\mathcal{H}_\delta := (f_\delta, g_\delta, C, D)$ with initial condition $\xi \in K$ satisfy, for all $(t, j) \in \operatorname{dom} x_\delta$,

$$\omega(x_\delta(t, j, \xi)) \leq \gamma(\omega(\xi), t, j) + \varepsilon,$$

where

$$f_\delta(\xi) := \overline{\text{co}}\{v \in \mathbb{R}^n : v = f(\xi, u), u \in \delta\overline{\mathcal{B}}\},$$
$$g_\delta(\xi) := \{v \in \mathbb{R}^n : v = g(\xi, u), u \in \delta\overline{\mathcal{B}}\}.$$

Given any $\rho \in \mathcal{K}_\infty$ satisfying $\rho(r) \geq \gamma(r, 0, 0) \geq r$ for each $r \geq 0$, let $K = \mathcal{A}_{[\rho^{-1}(c)]}$ and, without loss of generality, let $\alpha \in \mathcal{K}$ be such that Claim 1 holds with $\delta = \alpha(\varepsilon)$. Define $r := \min\{\rho^{-1}(c),\ \sup_{\varepsilon > 0}\ \alpha(\varepsilon)\}$. Then $\kappa := \alpha^{-1}$ is a class-$\mathcal{K}$ function on $[0, r)$. Using Claim 1, we have $r > 0$, $\gamma \in \mathcal{KLL}$, and $\kappa \in \mathcal{K}$ for Definition 2.

Appendix II
Sketch of proof of Theorem 2

Given any $\alpha \in \mathcal{K}_\infty$ and any continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_+$, define set-valued mappings F_o, G, F, F_σ, $G_\sigma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as follows:

$$F_o(x) := \{v \in \mathbb{R}^n : v = f(x, u), u \in \alpha(|x|_{\mathcal{A}})\overline{\mathcal{B}}\},$$
$$G(x) := \{v \in \mathbb{R}^n : v = g(x, u), u \in \alpha(|x|_{\mathcal{A}})\overline{\mathcal{B}}\},$$
$$F(x) := \overline{\text{co}} F_o(x),$$
$$F_\sigma(x) := \overline{\text{co}} F((x + \sigma(x)\overline{\mathcal{B}}) \cap C) + \sigma(x)\overline{\mathcal{B}},$$
$$G_\sigma(x) := \{v : v \in g + \sigma(g)\overline{\mathcal{B}}, g \in G((x + \sigma(x)\overline{\mathcal{B}}) \cap D)\}.$$

A. Necessity

Let $V : \mathbb{R}^n \to \mathbb{R}_+$ and $\rho \in \mathcal{K}$ come from Definition 3. Pick $\widetilde{\rho} \in \mathcal{K}_\infty$ to majorize ρ. Define the function $\alpha := \widetilde{\rho}^{-1}$.

First, we use the ISS Lyapunov function V to show that $\mathcal{A}$ is globally asymptotically stable for the hybrid inclusion $\mathcal{H} := (F, G, C, D)$. Then the combination of Corollary 2 and Theorem 6 in [4] implies the existence of a continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_+$ and a smooth function $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_+$ such that $\mathcal{A} = \{x : \sigma(x) = 0\}$ and $\widetilde{V}$ is a smooth Lyapunov function (see [4, Definition 2]) for the perturbed hybrid inclusion $\mathcal{H}_\Sigma := (F_\sigma, G_\sigma, C_\sigma, D_\sigma)$.

Now consider the hybrid system $\mathcal{H}_{u\sigma} := (f, g, C_\sigma, D_\sigma)$, and note the solution relationship between $\mathcal{H}_{u\sigma}$ and $\mathcal{H}_\Sigma$. With the properties of $\widetilde{V}$ and the following lemma, which is a hybrid version of Comparison Principle (cf. [12, Lemma 4.3] and [10, Lemma 4.4]), we can follow similar arguments in the proof of [17, Lemma 2.14] to establish the necessity in Theorem 2.

Lemma 1 (Hybrid Comparison Principle): For each $\alpha \in \mathcal{K}$, there exists $\gamma_\alpha \in \mathcal{KLL}$ with the following properties: if a hybrid arc $z : \operatorname{dom} z \mapsto \mathbb{R}_+$ satisfies $\dot{z}(t,j) \leq -\alpha(z(t,j))$ and $z(t,j+1) - z(t,j) \leq -\alpha(z(t,j))$, then $z(t,j) \leq \gamma_\alpha(z(0,0),t,j) \leq z(0,0)$ for each $(t,j) \in \operatorname{dom} z$.

B. Sufficiency

Let $\gamma \in \mathcal{KLL}$ and $\kappa \in \mathcal{K}$ come from Definition 1. Pick $\bar{\kappa} \in \mathcal{K}_\infty$ to majorize κ. Define $\alpha(s) := \bar{\kappa}^{-1}(\frac{s}{2})$ for all $s \geq 0$. Consider $\mathcal{H}_{o\sigma} := (F_o, G, C_\sigma, D_\sigma)$. Then the ISS assumption of $\mathcal{H}_{u\sigma}$ implies each maximal solution x to $\mathcal{H}_{o\sigma}$ starting from any $\xi \in \mathbb{R}^n$ satisfy, for each $(t,j) \in \operatorname{dom} x$,

$$|x(t,j,\xi)|_{\mathcal{A}} \leq \max\left\{\gamma(|\xi|_{\mathcal{A}},t,j), \frac{1}{2}\sup_{\substack{(s,k)\in\operatorname{dom} x,\\ (s,k)\preceq(t,j)}} |x(s,k,\xi)|_{\mathcal{A}}\right\},$$

which immediately gives the uniform stability of $\mathcal{A}$ for $\mathcal{H}_{o\sigma}$. Furthermore, with the routine small-gain arguments, one can use the inequality above to establish the uniform attractivity of $\mathcal{A}$ for $\mathcal{H}_{o\sigma}$. Then using Proposition 1 in [4] we conclude that $\mathcal{A}$ is globally asymptotically stable for $\mathcal{H}_{o\sigma}$, which makes the following lemma applicable.

Lemma 2: If there exists a continuous function $\sigma : \mathcal{X} \to \mathbb{R}_+$ such that $\mathcal{A} = \{x : \sigma(x) = 0\}$ is globally asymptotically stable for $\mathcal{H}_{o\sigma}$, then $\mathcal{A}$ is also globally asymptotically stable for the hybrid system $\mathcal{H} := (F, G, C, D)$.

Now, using Theorem 1 in [4] we have a smooth Lyapunov function w.r.t. $\mathcal{A}$ for $\mathcal{H}$. Defining the function $\rho := \alpha^{-1}$ for Definition 3, we establish the sufficiency in Theorem 2.

APPENDIX III
SKETCH OF PROOF OF THEOREM 3

The implications 1⇒2, 2⇒3, and 3⇒4 are obvious. The implication 6⇒1 comes from Proposition 2.

Next we show 5⇒6. Let $r > 0$, $\gamma \in \mathcal{KLL}$, and $\kappa_1 \in \mathcal{K}$ come from Definition 2. Let $\kappa_2 \in \mathcal{K}$ come from Definition 4. Pick $\bar{\kappa} \in \mathcal{K}_\infty$ to majorize κ_1 and κ_2. Define $\alpha(s) := \bar{\kappa}^{-1}(\frac{s}{2})$ for all $s \geq 0$. Consider the hybrid system $\mathcal{H}_o := (F_o, G, C, D)$, where the set-valued mappings F_o and G are defined in Appendix II. Note from assumptions that $F_o(x)$ is convex for each $\xi \in \mathbb{R}^n$.

Using the AG property w.r.t. $\mathcal{A}$, following similar arguments to the proof of [18, Lemma II.1], and using the properties of $\bar{\kappa}$ and α, we can show that, each maximal solution x to $\mathcal{H}_o$ starting from any $\xi \in \mathbb{R}^n$ satisfies

$$\limsup_{\substack{(t,j)\in\operatorname{dom} x,\\ t+j\to\infty}} |x(t,j,\xi)|_{\mathcal{A}} \leq \frac{1}{2} \limsup_{\substack{(t,j)\in\operatorname{dom} x,\\ t+j\to\infty}} |x(t,j,\xi)|_{\mathcal{A}},$$

which gives the global attractivity of $\mathcal{A}$ for $\mathcal{H}_o$.

Using the LISS property of $\mathcal{H}_u$, we can establish the local stability of $\mathcal{A}$ for $\mathcal{H}_o$. Using [7, Proposition 6.1(iii)], we conclude that $\mathcal{A}$ is (uniformly) globally asymptotically stable for $\mathcal{H}_o$. Using Theorem 1 in [4] we have a smooth Lyapunov function w.r.t. $\mathcal{A}$ for $\mathcal{H}_o$. Defining the function $\rho := \alpha^{-1}$ for Definition 3 establishes the implication 5⇒6.

Finally, we follow similar arguments in [18] to show 4⇒5. Without loss of generality, let $\kappa_* \in \mathcal{K}_\infty$ come from Definition 7. If $\|u\|_\infty = 0$, then combining the 0-LS of $\mathcal{A}$ and the limit property gives 0-LAS of $\mathcal{A}$ for $\mathcal{H}_u$, and then Theorem 1 gives LISS w.r.t. $\mathcal{A}$ for $\mathcal{H}_u$.

If $r := \max\{|\xi|_{\mathcal{A}}, \kappa_*(\|u\|_\infty)\} > 0$, then define

$$\theta(r) := \sup\{|x(t,j,\xi,u)|_{\mathcal{A}} \;:\; \xi \in \mathcal{A}_{[2r]},\ (x,u) \in \mathcal{S}_u(\xi),\\ (t,j) \in \operatorname{dom} x,\ \|u\|_\infty \leq \kappa_*^{-1}(r)\}.$$

The limit property assumption gives the existence of $(s,k) \in \operatorname{dom} x$ such that $x(s,k,\xi,u) \in \mathcal{A}_{[3r/2]}$. Then we can use the following lemma to conclude that $\theta(r) < \infty$ and hence choose $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ as a nondecreasing function.

Lemma 3: Let U be a compact subset of $\mathbb{R}^m$. Let K_1 and and K_2 be compact subsets of $\mathbb{R}^n$ such that $K_1 + \epsilon\overline{\mathcal{B}} \subset K_2$, where $\epsilon > 0$. Assume, for each $\xi \in K_2$ and each $(x,u) \in \mathcal{S}_u(\xi)$ with $u(\cdot,\cdot) \in U$, there exists $(s,k) \in \operatorname{dom} x$ such that $x(s,k,\xi,u) \in K_1$. Then the (infinite horizon) reachable set starting from K_2 is bounded.

Combining the property of θ and the LISS w.r.t. $\mathcal{A}$ gives GS w.r.t. $\mathcal{A}$, say with $\hat{\alpha}, \hat{\kappa} \in \mathcal{K}$ for Definition 5. Then defining the function $\kappa := \max\{\hat{\alpha} \circ \kappa_*, \hat{\kappa}\}$ for Definition 4 and following similar arguments to the proof of [18, Lemma I.4], we can obtain the AG property w.r.t. $\mathcal{A}$ for $\mathcal{H}_u$ and hence establish 4⇒5.

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