Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Rational stabilizing commutative controllers for unstable plant

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Abstract— **The characteristic locus method provides a systematic way to extend the classical control design techniques to multivariable systems. In addition, the manipulation of the eigenfunctions of the open-loop transfer matrix also allows optimal control problems, usually formulated using H_∞ optimization theory, to be addressed in the same manner as for scalar systems, avoiding the difficulties in the choice of multivariable weights, a problem of multivariable H_∞ design. Furthermore, the relative stability margin objective can also be taken into account by maximizing the minimum distance of the characteristic loci of the open-loop transfer matrix to the critical point. In order to obtain optimal controllers, it is first necessary to guarantee the internal stability of the closed-loop system. In this paper, a complete characterization of the class of stabilizing commutative controllers for continuous-time systems is given and conditions for the existence of these controllers for unstable plants are presented.**

I. INTRODUCTION

The Characteristic Locus Method (CLM) [1] provides a systematic way to design multivariable control systems for plants with the same number of inputs and outputs. Although it is based on the transformation of the design of a multivariable control system in the design of several scalar control systems, it does not make restrictive assumptions such as decoupling or diagonal dominance. This is so because the design of a multivariable control system within the CLM is carried out by using the eigenfunctions of the open-loop transfer matrix which, according to the generalized Nyquist stability criterion [2], defines the stability of the closed-loop system.

The essence of the CLM is to design a commutative controller, *i.e.*, a controller with the same eigenvector and dual-eigenvector matrices (frame) as the plant and to manipulate the controller eigenfunctions so that the closed-loop system is stable and satisfies performance requirements such as tracking, disturbance rejection and good transient response. This poses two serious problems: *(i)* for plants whose frequency response are far from normal at a certain frequency band, the characteristic loci are very sensitive to uncertainty in the parameters of the plant transfer function at these frequencies [3], [4], [5], [6], [7]; *(ii)* except in very special cases, the eigenvector and dual-eigenvector matrices are irrational. Problem *(i)* has been recently tackled [6], [7] by designing a pre-compensator with the view to making the plant as normal as possible at the frequencies of interest, while problem *(ii)* can be circumvented by using as the controller frame some approximation of the plant frame [8], [9], [10], [11].

The manipulation of the eigenfunctions also allows optimal control problems, usually formulated using H_∞ optimization theory, to be addressed in the same manner as for scalar systems, avoiding the difficulties in the choice of multivariable weights. Furthermore, the relative stability margin objective can also be taken into account by maximizing the minimum distance of the characteristic loci of the open-loop transfer matrix to the critical point.

In order to obtain optimal controllers, it is first necessary to guarantee internal stability of the closed-loop system. The problem of finding a stabilizing commutative controller has been been posed in [12]. Using the Youla-Kucera parameterization and the theory of minimal polynomial bases, a parameterization of all stabilizing controllers which commute exactly with the plant has been given for discrete-time systems. The parameterization is based on the calculation of a minimal polynomial basis for the right null space of a certain matrix. However, although in [12] it has been proven that this matrix always has a right null space with dimension greater than or equal to one, the parameterization has not been completely characterized since the nullity of this matrix is not known in advance. In addition, the existence of stabilizing commutative controllers has only been guaranteed in [12] for stable plants.

In this paper, the parameterization presented in [12] is developed for continuous-time systems, leading to a parameterization of all rational stabilizing commutative controllers (RSCC). A complete characterization of this parameterization and all degrees of freedom available are also obtained. Moreover, conditions for the existence of RSCC for unstable plant are presented.

II. MINIMAL POLYNOMIAL BASIS FOR THE RIGHT NULL SPACE OF A POLYNOMIAL MATRIX

Let $\mathbb{R}^{p\times q}[s]$ and $\mathbb{R}^{p\times q}(s)$ denote, respectively, the rings of polynomial and rational matrices of dimension $p\times q$. In addition assume that a matrix $A(s)\in\mathbb{R}^{p\times q}[s]$ ($p<q$ for simplicity) has the following Smith form:

$$\Sigma_A(s)=\begin{bmatrix}\varepsilon_1(s) & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & \varepsilon_2(s) & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & \varepsilon_p(s) & 0 & \cdots & 0\end{bmatrix} \quad (1)$$

where $\varepsilon_k(s)=0$ for $k=p-\nu+1,\ldots,p$. In this case the matrix $A(s)$ is said to have a right null space of dimension

This work was supported by the Brazilian Research Council (CNPq)

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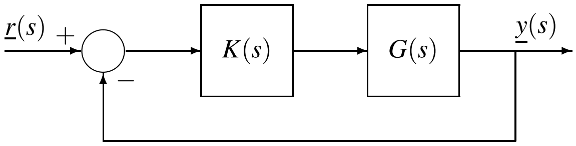

Fig. 1. Block diagram of a negative feedback control system.

$\bar{\nu} = q - p + \nu$, *i.e.* it is always possible to find a set of $\bar{\nu}$ linearly independent polynomial vectors $\underline{f}(s)$ over the field of rational functions such that $A(s)\underline{f}(s) = \underline{0}$. This leads to the definition of minimal polynomial bases [13] for the right null space of $A(s)$ as follows.

Definition 1: Let $F(s) = \left[\begin{array}{cccc} \underline{f}_1(s) & \underline{f}_2(s) & \cdots & \underline{f}_{\bar{\nu}}(s) \end{array}\right]$, where $\deg[\underline{f}_i(s)] = \phi_i$, be a polynomial matrix such that $A(s)F(s) = O$. Then $F(s)$ forms a minimal polynomial basis for the right null space of $A(s)$ if and only if $\sum_{i=1}^{\bar{\nu}} \phi_i$ is a minimum. □

With the view to obtaining a minimal polynomial basis for the right null space of $A(s)$, the first step is to compute the nullity of $A(s)$ ($\bar{\nu}$). Notice, from Eq. (1), that the matrix $A(s)$ has rank smaller than the normal rank, denoted here as r, only for a finite number of values of s: the zeros of the invariant polynomials $\varepsilon_i(s)$, $i = 1, \ldots, r$. Therefore, the normal rank of a polynomial matrix can de defined as:

$$r = \max_{s \in \mathbb{C}} \{\rho[A(s)]\}, \tag{2}$$

where $\rho[.]$ denotes the rank of a complex matrix. Once the normal rank r has been computed and supposing that $p < q$, then the nullity can be easily obtained by making $\bar{\nu} = q - r$.

Remark 1: The computation of a minimal polynomial basis for the right null space of a polynomial matrix can be carried out in a straightforward way with the robust algorithm proposed in [14]. □

III. A PARAMETERIZATION OF ALL RATIONAL STABILIZING COMMUTATIVE CONTROLLERS

A. *Problem formulation*

Consider the feedback system of Fig. 1 where $G(s), K(s) \in \mathbb{R}^{m\times m}(s)$ are, respectively, the plant transfer matrix and the controller transfer matrix to be designed. In addition let

$$G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s) \tag{3}$$

be a doubly-coprime factorization of $G(s)$ in $RH_\infty^{m\times m}$ (the ring of all stable transfer matrices in $\mathbb{R}^{m\times m}(s)$). Thus, there exist matrices $X(s), Y(s), \tilde{X}(s), \tilde{Y}(s) \in RH_\infty^{m\times m}$ which satisfy the generalized Bezout identity

$$\left[\begin{array}{cc} \tilde{X}(s) & -\tilde{Y}(s) \\ \tilde{N}(s) & \tilde{M}(s) \end{array}\right] \left[\begin{array}{cc} M(s) & Y(s) \\ -N(s) & X(s) \end{array}\right] = \left[\begin{array}{cc} I & O \\ O & I \end{array}\right]. \tag{4}$$

In [12] the problem of designing a stabilizing commutative controller for discrete-time systems has been posed. In this paper, following the same steps used in [12], a parameterization of all rational stabilizing commutative controllers for continuous-time systems will be obtained, *i.e.*, a controller $K(s)$ such that

$$G(s)K(s) = K(s)G(s) \tag{5}$$

that also internally stabilizes the closed-loop system of Fig. 1. This can be done via Youla-Kucera parameterization [15], [16], which provides a parameterization for the class of all controllers which internally stabilizes the closed-loop system of Fig. 1, as:

$$\begin{array}{rcl} K(s) & = & U(s)V^{-1}(s) = \tilde{V}^{-1}(s)\tilde{U}(s) \\ & = & [Y(s) + M(s)Q(s)][X(s) - N(s)Q(s)]^{-1} \\ & = & [\tilde{X}(s) - Q(s)\tilde{N}(s)]^{-1}[\tilde{Y}(s) + Q(s)\tilde{M}(s)] \end{array} \tag{6}$$

where $Q(s) \in RH_\infty^{m\times m}(s)$ *i.e.* is rational and has all its poles with negative real part. Thus substitution $G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ and $K(s) = U(s)V^{-1}(s) = \tilde{V}^{-1}(s)\tilde{U}(s)$ in (5), results in:

$$\tilde{M}^{-1}(s)\tilde{N}(s)U(s)V^{-1}(s) - \tilde{V}^{-1}(s)\tilde{U}(s)N(s)M^{-1}(s) = O. \tag{7}$$

It is not hard to see that when $\tilde{X}(s)$, $\tilde{Y}(s)$, $X(s)$ and $Y(s)$ are replaced by $\tilde{V}(s)$, $\tilde{U}(s)$, $V(s)$ and $U(s)$, respectively, then the generalized Bezout identity (4) still holds true. Therefore, after some straightforward algebraic calculation, Eq. (7) reduces to:

$$V(s)\tilde{M}(s) - M(s)\tilde{V}(s) = O. \tag{8}$$

Substituting $V(s) = X(s) - N(s)Q(s)$ and $\tilde{V}(s) = \tilde{X}(s) - Q(s)\tilde{N}(s)$ in Eq. (8) above yields:

$$N(s)Q(s)\tilde{M}(s) - M(s)Q(s)\tilde{N}(s) = C(s), \tag{9}$$

where

$$C(s) = X(s)\tilde{M}(s) - M(s)\tilde{X}(s). \tag{10}$$

Finally, writing $Q(s) = \left[\begin{array}{cccc} \underline{q}_1(s) & \underline{q}_2(s) & \cdots & \underline{q}_m(s) \end{array}\right]$ and $C(s) = \left[\begin{array}{cccc} \underline{c}_1(s) & \underline{c}_2(s) & \cdots & \underline{c}_m(s) \end{array}\right]$, where $\underline{q}_i(s)$ and $\underline{c}_i(s)$, $i = 1, 2, \ldots, m$ are the columns of $Q(s)$ and $C(s)$, respectively, then it is not difficult to check that Eq. (9) is equivalent to

$$P(s)\underline{q}(s) = \underline{c}(s), \tag{11}$$

where

$$\begin{array}{rcl} P(s) & = & \tilde{M}^t(s) \otimes N(s) - \tilde{N}^t(s) \otimes M(s) \\ \underline{q}(s) & = & \left[\begin{array}{cccc} \underline{q}_1^t(s) & \underline{q}_2^t(s) & \cdots & \underline{q}_m^t(s) \end{array}\right]^t \\ \underline{c}(s) & = & \left[\begin{array}{cccc} \underline{c}_1^t(s) & \underline{c}_2^t(s) & \cdots & \underline{c}_m^t(s) \end{array}\right]^t \end{array}, \tag{12}$$

and $\otimes$ denotes the Kronecker product. Eqs. (11) and (12) provide a necessary and sufficient condition for the existence of a rational stabilizing commutative controller, namely that, there exists a rational $K(s)$ which stabilizes and commutes with $G(s)$ if and only if there exists a stable vector $\underline{q}(s) \in \mathbb{R}^{m^2}(s)$ such that Eq. (11) is satisfied.

Remark 2: It is important to remark that, although $M(s)$, $N(s)$, $\tilde{M}(s)$, $\tilde{N}(s)$, $X(s)$ and $\tilde{X}(s)$ are rational matrices, it is always possible to form this matrices in such a way that they all have the same denominator polynomial. Thus, it is always possible to assume that $P(s) \in \mathbb{R}^{m^2\times m^2}[s]$ and $\underline{c}(s) \in \mathbb{R}^{m^2}[s]$. □

B. Existence of rational stabilizing commutative controllers

A RSCC $K(s)$ always exists when the plant transfer matrix $G(s)$ is stable. In this case, it can be proven that

$$Q_e(s) = -M^{-1}(s)Y(s) = -\tilde{Y}(s)\tilde{M}^{-1}(s) \tag{13}$$

satisfies the commutativity condition given by Eq. (9) and also belongs to $RH_\infty^{m\times m}$. However, if $G(s)$ is not stable, then $Q_e(s) \notin RH_\infty^{m\times m}$, since the unstable poles of the plant are also unstable zeros of the polynomial denominator of $Q_e(s)$. Thus, in order to deal with the general case of unstable plants, it is necessary to characterize the space generated by all solutions to Eq. (11). Therefore, writing

$$\underline{q}(s) = \frac{1}{d_q(s)}\underline{n}_q(s), \tag{14}$$

where $\underline{n}_q(s) \in \mathbb{R}^{m^2}[s]$ and $d_q(s)$ is a polynomial, and substituting $\underline{q}(s)$, according to Eq. (14), in Eq. (11), yields:

$$P(s)\frac{1}{d_q(s)}\underline{n}_q(s) = \underline{c}(s), \tag{15}$$

which can be written as:

$$T(s)\begin{bmatrix} \underline{n}_q(s) \\ d_q(s) \end{bmatrix} = \underline{0}, \tag{16}$$

where

$$T(s) = \begin{bmatrix} P(s) & -\underline{c}(s) \end{bmatrix}. \tag{17}$$

Therefore, the solutions (stable and unstable) to Eq. (11) will be defined by the right null space of $T(s)$ and will be obtained from linear combinations of the elements of a minimal polynomial basis for the right null space of $T(s)$. Thus, it is important to know in advance the nullity of $T(s)$[1]. In order to do so, the following result must be stated.

Lemma 1: Let $A \in \mathbb{C}^{m\times m}$ be a diagonalizable matrix and let each distinct eigenvalue of A, λ_i, $i = 1,\ldots,l$, have multiplicity μ_i. Then, there are $\sum_{i=1}^{l}\mu_i^2$ linearly independent matrices over the field of complex numbers ($\mathbb{C}$), which commute with respect to multiplication with A.

Proof. Let $A = W\Lambda_A W^{-1}$ be a spectral decomposition of A and suppose that B commutes with A. Therefore

$$W\Lambda_A W^{-1}B = BW\Lambda_A W^{-1} \Rightarrow \Lambda_A(W^{-1}BW) = (W^{-1}BW)\Lambda_A. \tag{18}$$

Denoting $\bar{B} = (W^{-1}BW)$ then, from Eq. (18), it can be seen that B commutes with A if and only if $\bar{B}$ commutes with Λ_A. Since A is diagonalizable, then Λ_A can be written as:

$$\Lambda_A = diag\{\Lambda_{A_i}, i = 1,\ldots,l\}, \tag{19}$$

where $\Lambda_{A_i} = \lambda_i I_{\mu_i}$, with I_{μ_i} denoting the identity matrix of order μ_i. Therefore, it is easy to verify that $\bar{B}$ is block diagonal, namely that, $\bar{B} = diag\{B_i, i = 1,\ldots,l\}$ where each block $B_i \in \mathbb{C}^{\mu_i\times\mu_i}$, and also that Λ_A commutes with $\bar{B}$ if and only if each block B_i commutes with its corresponding block Λ_{A_i}. Since $\Lambda_{A_i} = \lambda_i I_{\mu_i}$ then, defining the following basis $\mathscr{B}_i = \{[\underline{e}_1\ \underline{0}\ldots\underline{0}], [\underline{0}\ \underline{e}_1\ldots\underline{0}],\ldots,[\underline{0}\ \underline{0}\ldots\underline{e}_1],\ldots,[\underline{e}_{\mu_i}\ \underline{0}\ldots\underline{0}], [\underline{0}\ \underline{e}_{\mu_i}\ldots\underline{0}],\ldots,[\underline{0}\ \underline{0}\ldots\underline{e}_{\mu_i}]\}$, where $\underline{e}_i$ denotes the ith column of the identity matrix of order μ_i, it can be seen that there are μ_i^2 linearly independent matrices B_i that commute with Λ_{A_i}. Therefore, because there are l distinct blocks Λ_{A_i}, it is easy to verify that there are $\sum_{i=1}^{l}\mu_i^2$ linearly independent matrices that commute with A. □

From lemma 1 it is possible to obtain the nullity of $P(s)$ from the eigenfunctions of the plant.

Lemma 2: Let $G(s)$ be the plant transfer matrix. Then, $P(s)$ has normal rank $m^2 - \bar{\nu}$, where $\bar{\nu} = \sum_{i=1}^{l}\mu_i^2$ and μ_i is the multiplicity of the ith eigenfunction of $G(s)$, $\lambda_{g_i}(s)$.

Proof. If $P(s)$ has normal rank $r < m^2$, then there are $\bar{\nu} = m^2 - r$ linearly independent polynomial vectors $\underline{\alpha}(s) \in \mathbb{R}^{m^2}[s]$ such that:

$$\underline{\alpha}^t(s)P(s) = \underline{0}^t. \tag{20}$$

Let $\underline{\alpha}^t(s) = [\underline{\alpha}_1^t(s)\ \underline{\alpha}_2^t(s)\ \ldots\ \underline{\alpha}_m^t(s)]$, where $\underline{\alpha}_i(s) \in \mathbb{R}^m[s]$ and define $A(s) = \begin{bmatrix} \underline{\alpha}_1(s) & \underline{\alpha}_2(s) & \ldots & \underline{\alpha}_m(s) \end{bmatrix}^t$. Therefore, it can be verified that Eq. (20) is satisfied if there is a matrix $A(s)$ that satisfies:

$$\tilde{M}(s)A(s)N(s) - \tilde{N}(s)A(s)M(s) = O. \tag{21}$$

Pre-multiplying Eq. (21) by $\tilde{M}^{-1}(s)$ and post-multiplying it by $M^{-1}(s)$, yields:

$$A(s)N(s)M^{-1}(s) - \tilde{M}^{-1}(s)\tilde{N}(s)A(s) = O. \tag{22}$$

Since $G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s)$ and $G(s) = N_G(s)/d(s)$, where $d(s)$ is the least common denominator of all the elements of $G(s)$ and $N_G(s) \in \mathbb{R}^{m\times m}[s]$, Eq. (22) can be rewritten as:

$$A(s)\frac{1}{d(s)}N_G(s) = \frac{1}{d(s)}N_G(s)A(s). \tag{23}$$

Because $G(s)$ has, by assumption, l distinct eigenfunctions $\lambda_{g_i}(s)$ with multiplicity μ_i, then for an infinite number of frequencies ω_k, $N_G(j\omega_k)$ has l distinct eigenvalues where each one has multiplicity μ_i. Thus, if $j\omega_k$ is not a zero of $d(s)$, then Eq. (23) is satisfied if and only if $A(j\omega_k)$ commutes with $N_G(j\omega_k)$. Since, according to lemma 1, there are $\sum_{i=1}^{l}\mu_i^2$ linearly independent matrices that commutes with $N_G(j\omega_k)$, then for infinite values of ω_k, the nullity of $P(j\omega_k)$ is equal to $\sum_{i=1}^{l}\mu_i^2$. Therefore, in accordance with the definition of normal rank given by Eq. (2), the nullity of $P(s)$ is $\sum_{i=1}^{l}\mu_i^2$. □

Note that the commutativity condition given by Eq. (9) is always verified when $Q(s) = Q_e(s)$ given by Eq. (13). This implies that the vector $\underline{c}(s)$ always belongs to the space generated by the columns of $P(s)$. Therefore, assuming that the polynomial matrix $P(s)$ has nullity $\bar{\nu}$, then $T(s)$, given by Eq. (17), has nullity $\bar{\nu}+1$. Since the nullity of $T(s)$ is already known, the next step is the computation of a basis for the right null space of $T(s)$. Thus, denoting $H(s)$ as the polynomial matrix of dimension $(m^2+1)\times(\bar{\nu}+1)$ whose columns form a minimal polynomial basis for the right null

[1]In [12] the problem of obtaining a stabilizing commutative controller for discrete time systems is also associated with the problem of finding a minimal polynomial basis for a certain polynomial matrix. However, although it has been shown in [12] that this matrix has always nullity greater than or equal to one, the dimension of its right null space has not been determined exactly.

space of $T(s)$, then $T(s)H(s) = O$. This implies that all solutions to Eq. (16) should be of the following form:

$$\left[\begin{array}{c} \underline{n}_q(s) \\ d_q(s) \end{array} \right] = H(s)\underline{\psi}(s), \tag{24}$$

where $\underline{\psi}(s)$ is a polynomial vector. Partitioning $H(s)$ as

$$H(s) = \left[\begin{array}{c} H_t(s) \\ \underline{h}_b^t(s) \end{array} \right], \tag{25}$$

then $d_q(s)$ is given by

$$d_q(s) = \sum_{i=1}^{\bar{\nu}+1} h_{b_i}(s)\psi_i(s), \tag{26}$$

which is a generalized Diophantine equation. Thus Eq. (11) has a stable solution if and only if there exist polynomials $\psi_i(s)$, $i = 1,2,\ldots,\bar{\nu}+1$, such that $d_q(s)$ is a Hurwitz polynomial. Therefore, the problem of finding a rational stabilizing commutative controller for a given plant $G(s)$ turns out to be the problem of finding a Hurwitz polynomial $d_q(s)$.

An important result that relates the poles of the plant transfer matrix to the vector $\underline{h}_b^t(s)$, defined in Eq. (25), is now presented. This result will be used in the sequel to obtain a necessary and sufficient condition for the existence of RSCC.

Lemma 3: If $\underline{h}_b^t(s_0) = \underline{0}^t$, then s_0 is a pole of the plant.

Proof. Note that $Q_e(s) = -M^{-1}(s)Y(s)$ satisfies Eq. (9) and, consequently, the rational vector $\underline{q}_e(s)$, formed according to Eq. (12) from $Q_e(s)$, satisfies Eq. (11). Thus, writing $\underline{q}_e(s) = \frac{1}{d_{q_e}(s)}\underline{n}_{q_e}(s)$, it follows that the zeros of $d_{q_e}(s)$ are poles of $G(s)$. Therefore, if any value of $s = s_0$ is such that $\underline{h}_b^t(s_0) = \underline{0}^t$, then s_0 is a pole of the plant. □

From lemma 3, a necessary and sufficient condition for the existence of RSCC can now be stated.

Theorem 1: Let $G(s)$ be the plant transfer matrix. Then, there exist RSCC for $G(s)$ if and only if $\underline{h}_b^t(s_0) \neq \underline{0}^t$, for all s_0 equal to an unstable pole of the plant.

Proof. Note that there is no $\psi_i(s)$, $i = 1,2,\ldots,\bar{\nu}+1$, such that $d_q(s)$ is a Hurwitz polynomial if and only if the greatest common divisor of $h_{b_i}(s)$ for $i = 1,\ldots,\bar{\nu}+1$, $\chi(s)$, has an unstable zero, namely that, $\chi(s_0) = 0$ and s_0 has positive real part. This means that there is no RSCC if and only if $\underline{h}_b^t(s_0) = \underline{0}^t$ and s_0 has positive real part. According to lemma 3, if $\underline{h}_b^t(s_0) = \underline{0}^t$ then s_0 is a pole of the plant. Therefore, there is no RSCC if and only if $\underline{h}_b^t(s_0) = \underline{0}^t$ and s_0 is an unstable pole of the plant. □

Although theorem 1 leads to a necessary and sufficient condition for the existence of RSCC, it does not associate any property of the plant with the existence of RSCC. In addition, this condition has a high computational cost since it requires the computation of a minimal polynomial basis for the polynomial matrix $T(s)$, forming the matrix $H(s)$, and then checking the values of the vector $\underline{h}_b^t(s_0)$ for each unstable pole of the plant $s = s_0$. Therefore, it will be more interesting to find conditions, based explicitly on the plant transfer matrix, for the existence of RSCC that are simpler to be verified then the one stated in theorem 3. From lemma 2 and theorem 1, a sufficient condition for the existence of RSCC can be stated. This condition will be used in the sequel to obtain a sufficient condition for the existence of RSCC based explicitly on the plant transfer matrix.

Theorem 2: If $P(s_0)$ has rank $m^2 - \bar{\nu}$, where $\bar{\nu}$ is the nullity of $P(s)$, for all unstable pole of the plant, s_0, then there exist a RSCC.

Proof. Suppose that there is no RSCC for the plant, *i. e.*, according to theorem 1, $\underline{h}_b^t(s_0) = \underline{0}^t$ for some s_0 equal to an unstable pole of the plant and let $H(s)$ be the polynomial matrix obtained from a minimal polynomial basis for the right null space of $T(s)$. Therefore,

$$T(s_0) \left[\begin{array}{c} H_t(s_0) \\ \underline{0}^t \end{array} \right] = O \Rightarrow P(s_0)H_t(s_0) = O. \tag{27}$$

Since the columns of $H(s)$ form a minimal polynomial basis, then $H(s)$ is irreducible [17], *i.e.*, it has full column rank for all s and therefore $H_t(s_0)$ has rank $\bar{\nu}+1$. In accordance with lemma 2, $P(s)$ has nullity $\bar{\nu}$, and then, in order for Eq. (27) to have a solution, it is necessary that $P(s_0)$ has nullity greater than or equal to $\bar{\nu}+1$, which means that $P(s_0)$ must have rank smaller than the normal rank of $P(s)$. Therefore, if $P(s_0)$ has rank $m^2 - \bar{\nu}$, then it is not possible that $H_t(s_0)$ has rank $\bar{\nu}+1$, which leads to $\underline{h}_b^t(s_0) \neq \underline{0}^t$. □

Theorem 2 provides a sufficient condition for the existence of RSCC which is satisfied for a large class of plants as will be stated in the following theorem.

Theorem 3: If $N_G(s_0)$ has rank m and if the distinct eigenvalues of $N_G(s_0)$ have multiplicity equal to the multiplicity of the corresponding eigenfunctions of $N_G(s)$, for all s_0 equal to an unstable pole of the plant, then there exist a RSCC.

Proof. If $N_G(s_0)$ has rank m, then the factor $(s-s_0)$ does not belong to any invariant polynomial of $N_G(s)$. Thus, it is easy to verify that $M(s_0) = \tilde{M}(s_0) = O$, and therefore, according to Eq. (9), (10) and (12), $P(s_0) = O$ and $\underline{c}(s_0) = \underline{0}$. This implies that Eq. (11) can be rewritten as:

$$(s-s_0)P_1(s)\underline{q}(s) = (s-s_0)\underline{c}_1(s). \tag{28}$$

Thus, the problem of finding a solution $\underline{q}(s)$ to Eq. (28) is equivalent to the problem of finding a solution to

$$P_1(s)\underline{q}(s) = \underline{c}_1(s). \tag{29}$$

Suppose, without loss of generality, that s_0 is a pole of $G(s)$ with multiplicity m. Then, define

$$G_1(s) = (s-s_0)G(s) = \frac{(s-s_0)}{d(s)}N_G(s) = \frac{1}{d_1(s)}N_G(s) \tag{30}$$

where $d_1(s_0) \neq 0$. Thus, $G_1(s)$ can be written as:

$$G_1(s) = N(s)M_1^{-1}(s) = \tilde{M}_1^{-1}(s)\tilde{N}(s), \tag{31}$$

where $M_1(s) = \frac{1}{s-s_0}M(s)$ and $\tilde{M}_1(s) = \frac{1}{s-s_0}\tilde{M}(s)$. Following the same steps as in the proof of lemma 2 it can be seen that the problem of finding the rank of $P_1(s_0)$ is equivalent to the problem of finding all linearly independent matrices $A(s_0)$ such that:

$$\tilde{M}_1(s_0)A(s_0)N(s_0) - \tilde{N}(s_0)A(s_0)M_1(s_0) = O. \tag{32}$$

Since $d_1(s_0) \neq 0$ then $M_1(s_0)$ and $\tilde{M}_1(s_0)$ are invertible. Thus, pre-multiplying Eq. (32) by $\tilde{M}_1^{-1}(s_0)$ and post-multiplying by $M_1^{-1}(s_0)$ yields:

$$A(s_0)N(s_0)M_1^{-1}(s_0) - \tilde{M}_1^{-1}(s_0)\tilde{N}(s_0)A(s_0) = O, \tag{33}$$

and, according to Eqs. (31) and (30), Eq. (33) can be rewritten as:

$$A(s_0)\frac{1}{d_1(s_0)}N_G(s_0) - \frac{1}{d_1(s_0)}N_G(s_0)A(s_0) = O. \tag{34}$$

Therefore, all possible matrices $A(s_0)$ which satisfy Eq. (32) must commute with $N_G(s_0)$. Since the eigenvalues of $N_G(s_0)$ have the same multiplicity as the eigenfunctions of $N_G(s)$, then the rank of $P_1(s_0)$ is equal to $m^2 - \bar{\nu}$, where $\bar{\nu}$ denotes the nullity of $P(s)$ and, according to theorem 2, this guarantee the existence of a RSCC. □

Theorem 3 leads to a simple way to ascertain in advance the existence of RSCC: for all unstable poles of the plant, s_0, one must compute the eigenvalues of $N_G(s_0)$. If, for instance, the eigenvalues of $N_G(s_0)$ are distinct and different from zero, then there exist a RSCC and one can use the parameterization presented in Eq. (24) to obtain all rational and stable vectors $\underline{q}(s)$ that satisfy Eq. (11).

C. General solution and characterization of the degrees of freedom

The general solution to the problem of finding a polynomial matrix $Q(s) \in RH_\infty$, which leads to a rational stabilizing commutative controller $K(s)$, is now presented.

Theorem 4: Let $G(s) \in \mathbb{R}^{m\times m}(s)$ and suppose that $G(s)$ satisfies the conditions given by theorem 1 for the existence of a rational stabilizing commutative controller. Then, the class of all rational stabilizing commutative controller can be parameterized by a rational, proper and stable transfer matrix $Q(s)$ whose columns $\underline{q}_i(s)$, $i = 1, 2, \ldots, m$ are obtained as follows:

$$\begin{bmatrix} \underline{q}_1(s) \\ \underline{q}_2(s) \\ \vdots \\ \underline{q}_m(s) \end{bmatrix} = \frac{1}{d_q(s)} H_t(s)\underline{\psi}(s) \tag{35}$$

where

i) $H(s) = \begin{bmatrix} H_t(s) \\ \underline{h}_b^t(s) \end{bmatrix}$ is a $m^2 + 1 \times (\bar{\nu}+1)$ polynomial matrix whose columns form a minimal polynomial basis for the right null space of the matrix $T(s) = \begin{bmatrix} P(s) & -\underline{c}(s) \end{bmatrix}$ defined in Eq. (17);

ii) $\underline{\psi}(s)$ is a $(\bar{\nu}+1)$ dimensional vector whose entries are polynomials, being the degrees of freedom available on the general solution, which are deployed to obtain a Hurwitz polynomial $d_q(s) = \sum_{i=1}^{\bar{\nu}+1} h_{b_i}(s)\psi_i(s)$, where $h_{b_i}(s)$ are the entries of vector $\underline{h}_b(s)$. □

IV. EXAMPLE

In order to illustrate the parameterization presented in this paper, let

$$G(s) = \frac{1}{d(s)} \begin{bmatrix} -56s & -47s+2 \\ -50s-2 & -42s \end{bmatrix}.$$

where $d(s) = (s-1)(s+2)$. According to theorem 3, if for all unstable pole of the plant $s = s_0$, $N_G(s_0)$ has rank m and its distinct eigenvalues have multiplicity equal to the multiplicity of the eigenfunctions of $N_G(s)$, then there exist RSCC for $G(s)$. It is easy to see that this plant has only one unstable pole at 1 whose multiplicity depends on the rank of $N_G(1)$. It can be easily verified that the eigenvalues of $N_G(1)$ are equal to $-97,877$ and $-0,123$, which implies that the multiplicity of the unstable pole of $G(s)$ is equal to 2. In addition, since the eigenvalues of $N_G(1)$ are distinct and different from zero, then the conditions of theorem 3 are satisfied and the existence of rational stabilizing commutative controllers for $G(s)$ is guaranteed.

The first step to find a RSCC for $G(s)$ is to compute $N(s)$, $M(s)$, $\tilde{N}(s)$, $\tilde{M}(s)$, $X(s)$, $Y(s)$, $\tilde{X}(s)$ and $\tilde{Y}(s) \in \mathbb{R}H_\infty$ satisfying the generalized Bezout identity and, in the sequel, to form, according to Eqs. (12*a*) and (12*c*), the polynomial matrix $P(s)$ and the polynomial vector $\underline{c}(s)$. Once $P(s)$ and $\underline{c}(s)$ have been computed, the next step is to form the matrix $T(s) = [P(s) \ -\underline{c}(s)]$ and via minimal polynomial basis to compute $H(s)$ such that $T(s)H(s) = O$. According to lemma 2, the nullity of $P(s)$ is $\bar{\nu} = \sum_{i=1}^{l} \mu_i^2$, where μ_i, $i = 1, \ldots, l$ denotes the multiplicity of each one of the l distinct eigenfunctions of $G(s)$. Thus, since $N_G(1)$ has two distinct eigenvalues, then $G(s)$ has also two distinct eigenfunctions and the nullity of $P(s)$ is $\bar{\nu} = 2$. Therefore, the nullity of $T(s)$ is equal to 3 $(\bar{\nu}+1)$, which means that the algorithm for the computation of a minimal polynomial basis for the right null space of $T(s)$ must stop when three polynomial vectors are obtained. Using the algorithm proposed in [14] for the computation of $H(s)$ one obtains:

$$H(s) = \begin{bmatrix} -0.4284 & 0.1552s+0.4077 & 0.3992s-0.1471 \\ 0.5100 & -0.1849s-0.4824 & -0.4753s+0.1758 \\ 0.4794 & -0.1738s-0.4423 & -0.4467s+0.1805 \\ -0.5712 & 0.2070s+0.5234 & 0.5322s-0.2160 \\ 0.0148 & 0.0007s-0.0011 & -0.0107s+0.0090 \end{bmatrix}.$$

According to theorem 4, the class of all RSCC can be parameterized by the matrix $Q(s)$, obtained from $H(s)$ and with the degrees of freedom, $\underline{\psi}(s)$, chosen such that $d_q(s) = 0.0148\psi_1(s) + (0.0007s - 0.0011)\psi_2(s) + (-0.0107s + 0.0090)\psi_3(s)$ is a Hurwitz polynomial. For instance, suppose that ψ_1, ψ_2 and ψ_3 must be chosen such that $d_q(s) = s + 15$. A solution to this generalized Diophantine equation is $\psi_1 = 1064.8$, $\psi_2 = -109.5$ and $\psi_3 = -100$ which leads to the following matrix $Q(s) \in \mathbb{R}_\infty^{m\times m}$:

$$Q(s) = \frac{1}{s+15}\begin{bmatrix} -56.9184s - 486.1498 & 63.7144s + 540.8471 \\ 67.7812s + 578.2878 & -75.8972s - 643.9797 \end{bmatrix}. \tag{36}$$

Substituting $Q(s)$, given by Eq. (36), in the Youla-Kucera parameterization (Eq. 6), yields:

$$K(s) = N_K(s)M_K^{-1}(s) \tag{37}$$

where

$$N_K(s) = \begin{bmatrix} n_{k_{11}}(s) & n_{k_{12}}(s) \\ n_{k_{21}}(s) & n_{k_{22}}(s) \end{bmatrix} \text{ and } M_K(s) = \begin{bmatrix} m_{k_{11}}(s) & m_{k_{12}}(s) \\ m_{k_{21}}(s) & m_{k_{22}}(s) \end{bmatrix}$$

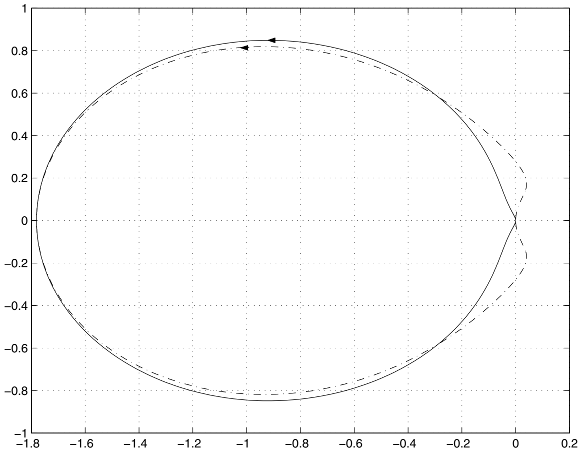

Fig. 2. Characteristic loci of the open-loop transfer matrix $G(s)K(s)$. Characteristic locus number 1 (dash-dotted line) and number 2 (solid line).

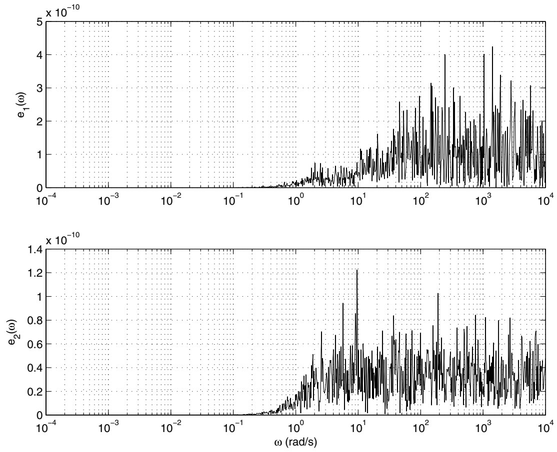

Fig. 3. Percent commutativity error between the characteristic loci of $G(s)K(s)$ and the product of the characteristic loci of $G(s)$ and $K(s)$ at each frequency ω.

$$
\begin{aligned}
n_{k_{11}}(s) &= -19.72522s^2 - 184.23543s - 505.12541\\
n_{k_{12}}(s) &= 6.68943s^3 + 77.84707s^2 + 307.8451s + 368.43548\\
n_{k_{21}}(s) &= 23.15039s^2 + 209.45136s + 577.78226\\
n_{k_{22}}(s) &= -7.74881s^3 - 85.8247s^2 - 341.60247s - 414.28443,
\end{aligned}
$$

$$
\begin{aligned}
m_{k_{11}}(s) &= 16.883s^2 + 287.35275s + 324.53726\\
m_{k_{12}}(s) &= -10.70724s^3 - 194.4218s^2 - 411.79439s - 232.70256\\
m_{k_{21}}(s) &= 14.77263s^2 + 251.96508s + 283.53603\\
m_{k_{22}}(s) &= -9.46018s^3 - 172.03686s^2 - 365.42515s - 206.81084.
\end{aligned}
$$

It is easy to verify that all poles of $K(s)$ are stable; thus, because the plant has two unstable poles, then for the internal stability of the closed-loop system it is necessary that the characteristic loci of $G(s)K(s)$, encircle the critical point $-1+j0$ twice in an anti-clockwise direction; this is actually so as shown in Fig. 2.

Consider now the following measure of commutativity

$$
e_i(\omega)(\%) = \frac{|\lambda_{q_i}(j\omega) - \lambda_{g_i}(j\omega)\lambda_{k_i}(j\omega)|}{|\lambda_{q_i}(j\omega)|}100\%,\ i = 1,2,
$$

which represents the percent error between the characteristic loci of $G(s)K(s)$ ($\lambda_{q_i}(j\omega)$) and the product of the characteristic loci of $G(s)$ ($\lambda_{g_i}(j\omega)$) and $K(s)$ ($\lambda_{k_i}(j\omega)$) at each frequency ω. It is clear from Fig. 3 that the percent error for both characteristic loci is less than $1.4\times 10^{-10}\%$ at all frequencies which shows that $G(j\omega)$ and $K(j\omega)$ actually commute.

V. CONCLUSIONS AND FUTURE WORKS

In this paper a parameterization of all rational stabilizing commutative controller for continuous-time systems and a complete characterization of the degrees of freedom available on this parameterization are presented. A necessary and sufficient condition for the existence of rational stabilizing commutative controllers for unstable plants is also given. In addition, a sufficient condition is presented which shows that for a large class of plants there exists rational stabilizing commutative controllers.

The example used in this paper to illustrate the parameterization of all stabilizing commutative controllers suggests that there are sufficient degrees of freedom in this parameterization in order to consider other control objectives besides stability.

REFERENCES

[1] A. G. J. MacFarlane and J. J. Belletruti, "The characteristic locus design method," *Automatica*, vol. 9, pp. 575–588, 1973.

[2] A. G. J. MacFarlane and I. Postlethwaite, "The generalized nyquist stability criterion and multivariable root loci," *International Journal of Control*, vol. 25, pp. 81–127, 1977.

[3] J. C. Doyle and G. Stein, "Multivariable feedback design: Concepts for a classic/modern synthesis," *IEEE - Transactions on Automatic Control*, vol. 26, pp. 4–16, 1981.

[4] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*. Oxford: Clarendon Press, 1965.

[5] I. Postlethwaite, "Sensitivity of the characteristic gain loci," *Automatica*, vol. 18, pp. 709–712, 1982.

[6] J. C. Basilio and J. A. Sahate, "A normalizing precompensator for the design of effective and reliable commutative controllers," *International Journal of Control*, vol. 73, pp. 1280–1297, 2000.

[7] M. V. Moreira and J. C. Basilio, "Design of normalising precompensators via alignment of output-input principal directions," in *44th IEEE Conference on Decision and Control and European Control Conference (accepted for presentation)*, 2005.

[8] A. G. J. MacFarlane and B. Kouvaritakis, "A design technique for linear multivariable feedback systems," *International Journal of Control*, vol. 25, pp. 837–874, 1977.

[9] D. J. Cloud and B. Kouvaritakis, "Commutative controllers revisited: Parallel computation, a new lease of life," *International Journal of Control*, vol. 45, pp. 1335–1370, 1987.

[10] B. Kouvaritakis and J. C. Basilio, "Bi-causal eigenvector sequences and the design of causal commutative controllers," *International Journal of Control*, vol. 59, pp. 1173–1189, 1994.

[11] J. C. Basilio and B. Kouvaritakis, "Design of causal reversed-frame-normalizing controllers using bicausal expansions," *International Journal of Control*, vol. 66, pp. 1–14, 1997.

[12] J. C. Basilio, "Multivariable generalised Nyquist design," Ph.D. dissertation, University of Oxford, 1995.

[13] D. G. Forney, "Minimal bases of rational vector spaces, with applications to multivariable linear systems," *SIAM Journal of Control*, vol. 13, pp. 493–520, 1975.

[14] J. C. Basilio and M. V. Moreira, "A robust solution of the generalized polynomial Bezout identity," *Linear Algebra and its Applications*, vol. 385, pp. 287–303, 2004.

[15] D. C. Youla, H. A. Jabr, and J. J. Bongiorno, "Modern Wiener-Hopf design of optimal controllers - Part II: the multivariable case," *IEEE Transactions on Automatic Control*, vol. 21, pp. 319–338, 1976.

[16] V. Kucera, *Discrete linear control: the polynomial equation approach*. Chichester, U. K.: John Wiley & Sons, 1979.

[17] T. Kailath, *Linear Systems*. Prentice-Hall, 1980.