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On the Design of Robust H_∞ Filter-Based Tracking Controller for a Class of Linear Time Delay Systems with Parametric Uncertainties

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***Abstract*— : This paper deals, for the first time, with the design problem of linear robust H_∞ filter-based tracking controller for a class of linear time delay systems subjected to parametric uncertainties. Sweet method is proposed to guarantee the filtering process of the system output at the same time as its robust H_∞ tracking and output model following. Finally, the viability and the efficiency of the proposed method are clearly proved by numerical simulations.**

***Index Terms*— model following, Robust H_∞ tracking control, Filtering process, time-delay systems, parametric uncertainties.**

I. INTRODUCTION

A lot of attention has been devoted to the problem of robust tracking and model following for uncertain linear time-delay systems in the last decade, see for instance [1], [2] and [3] and the reference therein. However, all these works suppose the entire knowledge of the system state for the stability purpose. Also, they all consider the particular class of delayed systems subjected to matched uncertainties. Indeed, based on the matched structure, and by means of the norm tools, all these works consider the boundedness of the further terms corresponding to the uncertainties, perturbations and the delayed terms. Thereafter, the knowledge of these bounds, or just their estimations, as has been made in [3], is exploited in the construction of some types of stabilizing state feedback controllers that may achieve the asymptotic tracking purpose. However, the structure of these controllers which is nonlinear, and often discontinuous, may lead to several problems concerning the implementation purpose. Moreover, all these works suppose the entire knowledge of the system state to ensure the stabilization purpose. However, this is not the case in several practical situations for technical and economical reasons. On the other hand, in the few works where a linear tracking controller has been provided, see for instance [4], only the practical tracking purpose can be achieved, which means that only the boundedness of the tracking error can be guaranteed, and not its asymptotic stability. In this note, for the first time, the problem of robust H_∞ filter-based tracking and model following control for a class of linear time varying delay systems subjected to parametric uncertainties is tackled. Aware of all the aforementioned limitations in the existing works within this framework of study, our major goal is to ensure in the same time the robust H_∞ tracking and the filtering processes, (**RHTFP**), for the more general class of linear time varying delay systems subjected to parametric uncertainties. To the author's best knowledge, this problem has never been done before. Sweet sufficient conditions are provided to ensure the existence and the design of linear and easily implementable robust H_∞ filter-based tracking controller that achieves the (**RHTFP**) goal. Furthermore, a constructive procedure is provided to complete the design purpose. Finally, the validity and efficiency of the proposed approach are clearly proved through some numerical simulations.

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II. PROBLEM STATEMENT AND PRELIMINARIES

We consider a class of uncertain linear time varying delay systems described by the following differential-difference equations:

$$\begin{cases} \dot{x}(t) = \overline{A}x(t) + \overline{A}_d x(t-\tau(t)) + \overline{B}u(t) + \overline{B}_w w(t) \\ x(t) = 0, \quad t \in [-\tau(t), 0] \\ y(t) = \overline{C}x(t) + \overline{C}_d x(t-\tau(t)) + \overline{D}u(t) + \overline{D}_w w(t) \end{cases} \tag{1}$$

where $x(t) \in I\!R^n$ is the state vector, $u(t) \in I\!R^r$ is the input vector, $y(t) \in I\!R^m$ is the output vector, $w(t) \in \mathcal{L}_2^q[0,\infty)$ the external disturbances. $\tau(t)$ represents the time varying lag which is supposed to be a differential equation satisfying $\dot{\tau}(t) < \bar{l} < 1, \quad \forall t \geq 0$. Furthermore, assume that the coefficient matrices are decomposed as follows:

$$\overline{A} = A + \Delta A(t), \quad \overline{A}_d = A_d + \Delta A_d(t) \tag{2}$$
$$\overline{B} = B + \Delta B(t), \quad \overline{C} = C + \Delta C(t) \tag{3}$$
$$\overline{C}_d = C_d + \Delta C_d(t), \quad \overline{D} = D + \Delta D(t) \tag{4}$$
$$\overline{B}_w = B_w + \Delta B_w(t), \quad \overline{D}_w = D_w + \Delta D_w(t) \tag{5}$$

A, A_d, B, B_w, C, C_d, D and D_w are constant matrices with appropriate dimensions, and the uncertainty parts have the following structure:

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) & \Delta B(t) & \Delta B_w(t) \\ \Delta C(t) & \Delta C_d(t) & \Delta D(t) & \Delta D_w(t) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Delta(t) \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \tag{6}$$

where $N_1 \in I\!R^{d\times n}$, $N_2 \in I\!R^{d\times n}$, $N_3 \in I\!R^{d\times r}$ and $N_4 \in I\!R^{d\times q}$ are constant known matrices. $\Delta(t) \in I\!R^{s\times d}$ is an unknown function such that $\Delta^T(t)\Delta(t) < I$. $M_1 \in \mathbb{R}^{n\times s}$, $M_2 \in \mathbb{R}^{m\times s}$ are two constant matrices which are supposed to satisfy the following assumption:

A1) rank $\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \text{rank}[M_2]$

which equivalently means that the kernel of M_2 must be included in that of M_1. Before starting our analysis, we need to introduce some statements and assumptions which will be used subsequently.

A2) The pair (A,B) is supposed to be stabilizable.

Definition II.1 *(see [5] for more details)*
For a matrix $A \in \mathbb{R}^{m\times n}$ write $A = \begin{bmatrix} A_{*1} & A_{*2} & \cdots & A_{*n} \end{bmatrix}$, where $A_{*j} \in \mathbb{R}^m$, $j = 1, 2, \cdots, n$, then the vector

$$\begin{bmatrix} A_{*1}^T & A_{*2}^T & \ldots & A_{*n}^T \end{bmatrix}^T \in \mathbb{R}^{mn}$$

is said to be the vec-function of A and is written $vec(A)$.

Proposition II.2 *(see [5]) If $A, C \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{m\times n}$ and α, $\beta \in \mathbb{R}$, then*

$$vec(AXB) = (B^T \otimes A)vec(X)$$
$$vec(\alpha A + \beta C) = \alpha vec(A) + \beta vec(C)$$

The problem here is to design an effective control such that the output $y(t)$ can track the reference output $y_m(t)$, while taking into account that the system (1) state is not entirely available for the stabilization purpose. $y_m(t)$ may be generated from a desired model which has the following general form

$$\begin{cases} \dot{x}_m(t) = A_m x_m(t) + B_m u_m(t) \\ y_m(t) = C_m x_m(t) + D_m u_m(t) \end{cases} \quad (7)$$

where $x_m(t) \in \mathbb{R}^{n_m}$, $y_m(t)$ has the same dimension as $y(t)$ and $u_m(t) \in \mathbb{R}^{r_m}$. A_m, B_m, C_m and D_m are known constant matrices with appropriate dimensions.

III. On the existence of robust H_∞ Memoryless tracking filter-based controller

In this section, the problem of the existence of a robust H_∞ memoryless tracking filter-based controller that guarantees both, the filtering of the system (1) output, and its tracking to the model (7) output, will be tackled. We start our analysis by introducing the following transformations

$$x_L(t) = x(t) - Gx_m(t), \quad e_s(t) = y(t) - y_m(t) \quad (8)$$

Then, we deduce from (1) and (7) the following system

$$\begin{cases} \dot{x}_L(t) = \overline{A}x_L(t) + \overline{A}_d x_L(t-\tau(t)) + \overline{B}u(t) \\ \qquad [\overline{A}G - GA_m]x_m(t) + \overline{A}_d G x_m(t-\tau(t)) \\ \qquad -GB_m u_m(t) + \overline{B}_w w(t) \\ e_s(t) = \overline{C}x_L(t) + \overline{C}_d x_L(t-\tau(t)) + \overline{D}u(t) \\ \qquad +[\overline{C}G - C_m]x_m(t) + \overline{C}_d G x_m(t-\tau(t)) \\ \qquad -D_m u_m(t) + \overline{D}_w w(t) \\ x(t) = x_L(t) + Gx_m(t) \end{cases} \quad (9)$$

Our robust H_∞ memoryless filter-based tracking controller candidate is described by

$$\begin{cases} \dot{\widehat{x}}(t) = N\widehat{x}(t) + N_d\widehat{x}(t-\tau(t)) + Bu(t) + N_f[y(t) - Du(t)] \\ u(t) = u_s(t) + u_c(t) \end{cases} \quad (10)$$

where

$$u_s(t) = -F_1\widehat{x}(t), \; u_c(t) = [F_3 + F_1 G]x_m(t) + Qu_m(t) \quad (11)$$

$\widehat{x}(t)$ is the estimate of $x(t)$. $u_s(t)$ is the stabilization state feedback controller, while $u_c(t)$ is the compensation controller. The (**RHTFP**) purpose becomes one of seeking the gain matrices of (10) such that the filtering error $e_f(t) = x(t) - \widehat{x}(t)$ is robustly asymptotically stable, and the tracking error $e_s(t)$ is robustly H_∞ asymptotically stable. To this end, the gain matrices N, N_d, N_f and F_1 can be taking such that the following system

$$\begin{cases} \dot{\overline{x}}(t) = \overline{A}_1\overline{x}(t) + \overline{A}_{d_1}\overline{x}(t-\tau(t)) + \overline{B}_{w_1}w(t) \\ e_s(t) = \overline{C}_1\overline{x}(t) + \overline{C}_{d_1}\overline{x}(t-\tau(t)) + \overline{D}_w w(t) \end{cases} \quad (12)$$

where

$$\overline{x}(t) = \begin{bmatrix} x_L(t) \\ e_f(t) \end{bmatrix}, \; \overline{A}_1 = \begin{bmatrix} \overline{A} - \overline{B}F_1 & \overline{B}F_1 \\ \overline{A} - N - N_f\overline{C} & N \end{bmatrix} \quad (13)$$

$$\overline{A}_{d_1} = \begin{bmatrix} \overline{A}_d & 0 \\ \overline{A}_d - N_d - N_f\overline{C}_d & N_d \end{bmatrix} \quad (14)$$

$$\overline{C}_1 = \begin{bmatrix} \overline{C} - \overline{D}F_1 & \overline{D}F_1 \end{bmatrix}, \; \overline{C}_{d_1} = \begin{bmatrix} \overline{C}_d & 0 \end{bmatrix} \quad (15)$$

$$\overline{B}_{w_1} = \begin{bmatrix} \overline{B}_w \\ \overline{B}_w - N_f\overline{D}_w \end{bmatrix} \quad (16)$$

under the following matrix equalities

$$\overline{A} - N - N_f\overline{C} = 0, \; \overline{A_d} - N_d - N_f\overline{C}_d = 0 \quad (17)$$

is robustly H_∞ asymptotically stable, while the gain matrices F_3, G and Q must be taking such that the following matrix equalities hold

$$\overline{A}G - GA_m + \overline{B}F_3 = 0, \; \overline{A}_d G = 0 \quad (18)$$
$$\overline{B}Q - GB_m = 0, \; \overline{C}G - C_m + \overline{D}F_3 = 0 \quad (19)$$
$$\overline{C}_d G = 0, \; \overline{D}Q - D_m = 0 \quad (20)$$

To show that, first of all remark that the matrix equalities given in (17) can be rewritten equivalently after developing terms as follows

$$A - N - N_fC + [M_1 - N_fM_2]\Delta(t)N_1 = 0 \quad (21)$$
$$A_d - N_d - N_fC_d + [M_1 - N_fM_2]\Delta(t)N_2 = 0 \quad (22)$$

and this for all $\Delta(t)$ such that $\Delta^T(t)\Delta(t) < I$. Hence, (21)-(22) can be equivalently transformed, for non trivial cases, into the following matrix equalities

$$A - N - N_fC = 0, \quad A_d - N_d - N_fC_d = 0 \quad (23)$$
$$M_1 - N_fM_2 = 0 \quad (24)$$

Now, at the moment to prove the result mentioned above, let us keep beside the two first matrix equalities given in (23), and use only the matrix equality (24). Indeed, (24) allows us to deduce that filter (10) dynamic can be rewritten as follows

$$\dot{\hat{x}}(t) = N\hat{x}(t) + N_d\hat{x}(t-\tau(t)) + \overline{B}u(t) + N_f[y(t) - \overline{D}u(t)] \quad (25)$$

Hence, it is readily checked that

$$\begin{aligned} \dot{e}_f(t) &= [\overline{A} - N - N_f\overline{C}]x_L(t) + Ne_f(t) \\ &+ [\overline{A}_d - N_d - N_f\overline{C}_d]x_L(t-\tau(t)) + N_d e_f(t-\tau(t)) \\ &+ [\overline{B}_w - N_f\overline{D}_w]w(t) + [\overline{A} - N - N_f\overline{C}]Gx_m(t) \\ &+ [\overline{A}_d - N_d - N_f\overline{C}_d]Gx_m(t-\tau(t)) \end{aligned} \quad (26)$$

Also, after introducing the controller $u(t)$ given in (10) in the system (9), we get

$$\begin{aligned} \dot{x}_L(t) &= [\overline{A} - \overline{B}F_1]x_L(t) + \overline{B}F_1 e_f(t) + \overline{A}_d x_L(t-\tau(t)) \\ &+ [\overline{A}G - GA_m + \overline{B}F_3]x_m(t) + \overline{A}_d Gx_m(t-\tau(t)) \\ &+ [\overline{B}Q - GB_m]u_m(t) + \overline{B}_w w(t) \end{aligned} \quad (27)$$

$$\begin{aligned} e_s(t) &= [\overline{C} - \overline{D}F_1]x_L(t) + \overline{D}F_1 e_f(t) + \overline{C}_d x_L(t-\tau(t)) \\ &+ [\overline{C}G - C_m + \overline{D}F_3]x_m(t) + \overline{C}_d Gx_m(t-\tau(t)) \\ &+ [\overline{D}Q - D_m]u_m(t) + \overline{D}_w w(t) \end{aligned} \quad (28)$$

Therefore, it can be shown that under the matrix equalities given in (17), the augmented state $\begin{bmatrix} x_L(t) \\ e_f(t) \end{bmatrix}$, and the vector $e_s(t)$, are governed by the following closed-loop augmented system

$$\begin{cases} \begin{bmatrix} \dot{x}_L(t) \\ \dot{e}_f(t) \end{bmatrix} = \overline{A}_1 \begin{bmatrix} x_L(t) \\ e_f(t) \end{bmatrix} + \overline{A}_{d_1} \begin{bmatrix} x_L(t-\tau(t)) \\ e_f(t-\tau(t)) \end{bmatrix} + \overline{B}_{w_1} w(t) \\ e_s(t) = \overline{C}_1 \begin{bmatrix} x_L(t) \\ e_f(t) \end{bmatrix} + \overline{C}_{d_1} \begin{bmatrix} x_L(t-\tau(t)) \\ e_f(t-\tau(t)) \end{bmatrix} + \overline{D}_w w(t) \end{cases} \quad (29)$$

Thus, the problem as has been mentioned above is recovered. Now, to render this problem more tractable, let us first try to transform the problem of stability of the system (12) under the linear matrix equalities (23)-(24) into one problem. Firstly, it is well known from the general solution of linear matrix equalities (see[6] for more details) that the matrix equality (24) under the assumption **A1)** is solvable, and its general solution is given by:

$$N_f = \overline{M} + Z_1\overline{N} \quad (30)$$

where Z_1 is an arbitrarily chosen matrix, and

$$\overline{M} = M_1M_2^+, \quad \overline{N} = I - M_2M_2^+ \quad (31)$$

where M_2^+ is the generalized inverse of M_2. Therefore, after including the matrix equalities given in (23) and (30), it is readily checked that the robust H_∞ memoryless filter-based tracking control problem, as has been explained above, becomes one of solving the two following separate problems.

Problem **RHMS** [Robust H_∞ memoryless stabilization];
Find the gain matrices F_1 and Z_1 such that the following closed-loop system

$$\begin{cases} \dot{\overline{x}}(t) = \overline{A}_L\overline{x}(t) + \overline{A}_{d_L}\overline{x}(t-\tau(t)) + \overline{B}_{w_L}w(t) \\ e_s(t) = \overline{C}_L\overline{x}(t) + \overline{C}_{d_L}\overline{x}(t-\tau(t)) + \overline{D}_w w(t) \end{cases} \quad (32)$$

where

$$\overline{A}_L = \begin{bmatrix} \overline{A} - \overline{B}F_1 & \overline{B}F_1 \\ 0 & A - \overline{M}\,C - Z_1\overline{N}\,C \end{bmatrix} \quad (33)$$

$$\overline{A}_{d_L} = \begin{bmatrix} \overline{A}_d & 0 \\ 0 & A_d - \overline{M}\,C_d - Z_1\overline{N}\,C_d \end{bmatrix} \quad (34)$$

$$\overline{C}_L = \begin{bmatrix} \overline{C} - \overline{D}F_1 & \overline{D}F_1 \end{bmatrix}, \quad \overline{C}_{d_L} = \begin{bmatrix} \overline{C}_d & 0 \end{bmatrix} \quad (35)$$

$$\overline{B}_{w_L} = \begin{bmatrix} \overline{B}_w \\ B_w - \overline{M}\,D_w - Z_1\overline{N}\,D_w \end{bmatrix} \quad (36)$$

is robustly H_∞ asymptotically stable. The gain matrices N, N_d and N_f are given by

$$N_f = \overline{M} + Z_1\overline{N}, \quad N = A - N_fC, \quad N_d = A_d - N_fC_d \quad (37)$$

Problem **RMC** [Robust memoryless compensation];
Find the gain matrices F_3, G and Q satisfying the matrix equalities described in (20).

IV. Problem **RHMS** and parametrization of the filter and the stabilization controller $u_s(t)$

In this section, problem **RHMS** will be tackled. There is many ways to deal with this problem in both cases of delay dependent or delay independent approaches. Hereafter, we propose one solution using the delay independent approach. Notice that, admittedly the required criterion of stability will be independent of the delay, but it will be dependent on $\bar{l}$. Now, Let us start by presenting the following result.

Theorem IV.1 *Let F_1 be a matrix such that the matrix $A - BF_1$ is Hurwitz, then the closed-loop system* (32) *is H_∞ asymptotically stable, and thus the problem* **RHMS** *is solvable, if there exist symmetric matrices P_1, P_2, Q_1, Q_2, matrices Q_3 and Y_1 and scalars ε_1 and ε_2 such that the two following linear matrix inequalities LMIS hold*

$$\begin{bmatrix} (1.1) & * & * & * & * & * & * & * & * \\ (2.1) & (2.2) & * & * & * & * & * & * & * \\ A_d^TP_1 & 0 & -Q_1 & * & * & * & * & * & * \\ 0 & (4.2) & -Q_2^T & -Q_3 & * & * & * & * & * \\ B_w^TP_1 & (5.2) & 0 & 0 & -\gamma^2 I & * & * & * & * \\ C - DF_1 & DF_1 & C_d & 0 & D_w & -I & * & * & * \\ \varepsilon_1N_1 - \varepsilon_1N_3F_1 & \varepsilon_1N_3F_1 & \varepsilon_1N_2 & 0 & \varepsilon_1N_4 & 0 & -\varepsilon_1 I & * & * \\ (8.1) & M_2^TDF_1 & M_2^TC_d & 0 & M_2^TD_w & 0 & 0 & -\varepsilon_1 I & * \\ N_1 - N_3F_1 & N_3F_1 & N_2 & 0 & N_4 & 0 & 0 & 0 & -\varepsilon_2^{-1} I \end{bmatrix} < 0 \quad (38)$$

$$\begin{bmatrix} -\varepsilon_2^{-1}I & \varepsilon_2^{-1}M_2^T \\ \varepsilon_2^{-1}M_2 & -I \end{bmatrix} < 0 \quad (39)$$

where

$$(1.1) = A^T P_1 + P_1 A - P_1 B F_1 - F_1^T B^T P_1 + (1-\bar{l})^{-1} Q_1 \quad (40)$$

$$(2.1) = F_1^T B^T P_1 + (1-\bar{l})^{-1} Q_2^T \quad (41)$$

$$(2.2) = \Gamma_1^T P_2 + P_2 \Gamma_1 - \Upsilon_1^T Y_1^T - Y_1 \Upsilon_1 + (1-\bar{l})^{-1} Q_3 \quad (42)$$

$$(4.2) = \Gamma_2^T P_2 - \Upsilon_2^T Y_1^T \quad (43)$$

$$(5.2) = \Gamma_3^T P_2 - \Upsilon_3^T Y_1^T \quad (44)$$

$$(8.1) = M_1^T P_1 + M_2^T C - M_2^T D F_1 \quad (45)$$

$$\Gamma_1 = A - \overline{M}\, C,\ \Gamma_2 = A_d - \overline{M}\, C_d,\ \Gamma_3 = B_w - \overline{M}\, D_w \quad (46)$$

$$\Upsilon_1 = \overline{N}\, C,\ \Upsilon_2 = \overline{N}\, C_d,\ \Upsilon_3 = \overline{N}\, D_w \quad (47)$$

Proof: Consider the following Lyapunov functional candidate

$$V(\overline{x}(t)) = \overline{x}^T(t) P \overline{x}(t) + \frac{1}{1-\bar{l}} \int_{t-\tau(t)}^{t} \overline{x}^T(s) Q \overline{x}(s) ds \quad (48)$$

To deal with the prescribed γ attenuation level problem. Our aim is to seek a criterion under which we have $\|H_{e_s w}(s)\|_\infty \leq \gamma$. This problem can be presented otherwise. Seeking a criterion that will ensure that the performance index

$$\int_0^\lambda \left(e_s^T(t) e_s(t) - \gamma^2 w^T(t) w(t) \right) dt < 0 \quad (49)$$

is satisfied $\forall\ \lambda > 0$. Hence, based on the fact that $V(0) = 0$ and $V(\lambda) > 0$, we can state that

$$\begin{aligned} &\int_0^\lambda \left(e_s^T(t) e_s(t) - \gamma^2 w^T(t) w(t) \right) dt \leq \\ &\int_0^\lambda \left(e_s^T(t) e_s(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t) \right) dt \end{aligned} \quad (50)$$

where $\dot{V}(t)$ is the derivative of $V(t)$ with respect to t along the trajectories of (32). Hence, it is readily checked that after developing the terms and applying the properties of $\tau(t)$ we obtain the following condition

$$\int_0^\lambda \left(e_s^T(t) e_s(t) - \gamma^2 w^T(t) w(t) \right) dt \leq \int_0^\lambda X^T(t) \nabla X(t)\, dt \quad (51)$$

where

$$X(t) = \begin{bmatrix} \overline{x}^T(t) & \overline{x}^T(t-\tau(t)) & w^T(t) \end{bmatrix}^T \quad (52)$$

$$\nabla = \begin{bmatrix} (1.1) & * & * \\ \overline{A}_{d_L}^T P + \overline{C}_{d_L}^T \overline{C}_L & \overline{C}_{d_L}^T \overline{C}_{d_L} - Q & * \\ \overline{B}_{w_L}^T P + \overline{D}_w^T \overline{C}_L & \overline{D}_w^T \overline{C}_{d_L} & \overline{D}_w^T \overline{D}_w - \gamma^2 I \end{bmatrix} \quad (53)$$

$$(1.1) = \overline{A}_L^T P + P \overline{A}_L + \overline{C}_L^T \overline{C}_L + (1-\bar{l})^{-1} Q \quad (54)$$

Let us first rewrite the matrices of the system (32) as follows

$$\overline{A}_L = A_L + M_{1_L} \Delta(t) N_{1_L},\ \overline{A}_{d_L} = A_{d_L} + M_{1_L} \Delta(t) N_{2_L} \quad (55)$$

$$\overline{C}_L = C_L + M_2 \Delta(t) N_{1_L},\ \overline{C}_{d_L} = C_{d_L} + M_2 \Delta(t) N_{2_L} \quad (56)$$

$$\overline{B}_{w_L} = B_{w_L} + M_{1_L} \Delta(t) N_4 \quad (57)$$

where

$$A_L = \begin{bmatrix} A - BF_1 & BF_1 \\ 0 & A - \overline{M}\, C - Z_1 \overline{N}\, C \end{bmatrix} \quad (58)$$

$$A_{d_L} = \begin{bmatrix} A_d & 0 \\ 0 & A_d - \overline{M}\, C_d - Z_1 \overline{N}\, C_d \end{bmatrix} \quad (59)$$

$$C_L = \begin{bmatrix} C - DF_1 & DF_1 \end{bmatrix},\ C_{d_L} = \begin{bmatrix} C_d & 0 \end{bmatrix} \quad (60)$$

$$B_{w_L} = \begin{bmatrix} B_w \\ B_w - \overline{M}\, D_w - Z_1 \overline{N}\, D_w \end{bmatrix} \quad (61)$$

$$M_{1_L} = \begin{bmatrix} M_1 \\ 0 \end{bmatrix},\ N_{1_L} = \begin{bmatrix} N_1 - N_3 F_1 & N_3 F_1 \end{bmatrix} \quad (62)$$

$$N_{2_L} = \begin{bmatrix} N_2 & 0 \end{bmatrix} \quad (63)$$

Hence, it is readily checked that ∇ can be equivalently rewritten as follows

$$\begin{aligned} &\begin{bmatrix} A_L^T P + P A_L + (1-\bar{l})^{-1} Q & P A_{d_L} & P B_{w_L} \\ A_{d_L}^T P & -Q & 0 \\ B_{w_L}^T P & 0 & -\gamma^2 I \end{bmatrix} + \Lambda_1^T \Lambda_1 \\ &+ \Lambda_2^T \Delta^T(t) \Lambda_3 + \Lambda_3^T \Delta(t) \Lambda_2 + \Lambda_2^T \Delta^T(t) M_2^T M_2 \Delta(t) \Lambda_2 \end{aligned} \quad (64)$$

where

$$\begin{aligned} \Lambda_1 &= \begin{bmatrix} C_L & C_{d_L} & D_w \end{bmatrix},\ \Lambda_2 = \begin{bmatrix} N_{1_L} & N_{2_L} & N_4 \end{bmatrix} \\ \Lambda_3 &= \begin{bmatrix} M_{1_L}^T P + M_2^T C_L & M_2^T C_{d_L} & M_2^T D_w \end{bmatrix} \end{aligned} \quad (65)$$

Now, it is readily checked that

$$\Lambda_2^T \Delta^T(t) \Lambda_3 + \Lambda_3^T \Delta(t) \Lambda_2 \leq \varepsilon_1 \Lambda_2^T \Lambda_2 + \varepsilon_1^{-1} \Lambda_3^T \Lambda_3 \quad (66)$$

where ε_1 is a positive non nul scalar. Also, let us introduce the following condition

$$M_2^T M_2 < \varepsilon_2 I \quad (67)$$

where ε_2 is a positive non nul scalar, and let $P = diag\{P_1, P_2\}$ and $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}$. Hence, by means of all these facts and thereafter applying the Schur complement lemma, and finaly expanding the resulting inequality and taking $Y_1 = P_2 Z_1$, we get the matrix inequality (38) which by choosing and seeting F_1 such that the matrix $A - BF_1$ is Hurwitz becomes LMI. This choice of F_1 is due to the fact that the (1.1) entry must be negative definite. On the other hand, condition (67) can be transformed equivalently using the Schur complement lemma into the linear matrix inequality (39). Therefore, the problem **RHMS** is solvable by means of LMIs (38) and (39). ■

V. Problem RMC and Parametrization of the Compensation Controller $u_c(t)$

In this section, the problem **RMC** will be tackled and a complete parametrization of the compensation controller $u_c(t)$ in (11) will be presented. This is equivalent to find the gain matrices F_3, G and Q which satisfy the matrix equalities (20).

A. On the existence of a solution to the matrix equalities (20)

Hereafter, a necessary and sufficient condition of the solvability of the matrix equalities (20) will be presented. The result is stated into the following corollary.

Ccorrolary V.1 *The matrix equalities described in* (20) *are solvable if and only if the following rank condition holds*

$$rank\begin{bmatrix} \Psi & \Omega \end{bmatrix} = rank[\Psi] \tag{68}$$

where

$$\Psi = \begin{bmatrix} I_{n_m}\otimes A - A_m^T\otimes I_n & I_{n_m}\otimes B & 0 \\ I_{n_m}\otimes A_d & 0 & 0 \\ -B_m^T\otimes I_n & 0 & I_{r_m}\otimes B \\ I_{n_m}\otimes C & I_{n_m}\otimes D & 0 \\ I_{n_m}\otimes C_d & 0 & 0 \\ 0 & 0 & I_{r_m}\otimes D \\ I_{n_m}\otimes N_1 & I_{n_m}\otimes N_3 & 0 \\ I_{n_m}\otimes N_2 & 0 & 0 \\ 0 & 0 & I_{r_m}\otimes N_3 \end{bmatrix} \tag{69}$$

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & vec(C_m)^T & 0 & vec(D_m)^T & 0 & 0 & 0 \end{bmatrix}^T \tag{70}$$

Proof: To show that, remark at the outset that the matrix equalities (20) can be rewritten after developing terms as follows

$$\begin{bmatrix} AG - GA_m + BF_3 \\ A_dG \\ BQ - GB_m \end{bmatrix} + M_1\Delta(t)\begin{bmatrix} N_1G + N_3F_3 \\ N_2G \\ N_3Q \end{bmatrix} = 0$$
$$\begin{bmatrix} CG - C_m + DF_3 \\ C_dG \\ DQ - D_m \end{bmatrix} + M_2\Delta(t)\begin{bmatrix} N_1G + N_3F_3 \\ N_2G \\ N_3Q \end{bmatrix} = 0 \tag{71}$$

Therefore since the two matrix equalities in (71) must be satisfied for all $\Delta(t)$ such that $\Delta^T(t)\Delta(t) < I$. For the non trivial cases, they hold if and only if

$$\begin{array}{l} AG - GA_m + BF_3 = 0, \ \ A_dG = 0, \ \ BQ - GB_m = 0 \\ CG - C_m + DF_3 = 0, \ \ C_dG = 0, \ \ DQ - D_m = 0 \\ N_1G + N_3F_3 = 0, \ \ N_2G = 0, \ \ N_3Q = 0 \end{array} \tag{72}$$

Thus, by means of the tools described in proposition II.2, the matrix equalities (72) can be equivalently rewritten as follows:

$$\Psi\begin{bmatrix} \text{vec}(G) \\ \text{vec}(F_3) \\ \text{vec}(Q) \end{bmatrix} = \Omega \tag{73}$$

where Ψ and Ω are described by (69) and (70). which is a system of linear equations for the $n_m(n+r)+r_mr$ components of blkcol$\{\text{vec}(G), \text{vec}(F_3), \text{vec}(Q)\}$. Therefore, from the general solution of linear matrix equalities (see[6] for more details), there exists a solution to (73) if and only if (68) holds. This concludes the proof of corollary V.1. ■

B. On the design of the robust H_∞ Memoryless tracking compensator $u_c(t)$

The satisfaction of condition (68) allows us to deduce the general solution of (73):

$$\begin{bmatrix} \text{vec}(G) \\ \text{vec}(F_3) \\ \text{vec}(Q) \end{bmatrix} = \Psi^+\Omega + (I - \Psi^+\Psi)Z_2 \tag{74}$$

where Z_2 is an arbitrarily chosen matrix, Ψ^+ is the generalized inverse of Ψ. Hence, the gain matrices G, F_3 and Q can be found by unstacking vec(G), vec(F_3) and vec(Q), respectively.

VI. Numerical Simulations

In this section, we give an example to illustrate the validity and the efficiency of the obtained results. Let us consider an unstable system with the same structure as in (1), where

$$A = \begin{bmatrix} -1 & 2 & 0 \\ -2 & 0 & 3 \\ 5 & 0 & -4 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.1 \end{bmatrix}$$
$$A_d = \begin{bmatrix} 0.2 & 0.1 & -0.3 \\ 0.1 & 0.3 & -0.4 \\ -0.5 & 0.2 & 0.3 \end{bmatrix}, \quad M_2 = 0.2$$
$$B = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}, C_d = \begin{bmatrix} 0.1 & 0 & -0.1 \\ 0.1 & 0.2 & -0.3 \end{bmatrix}$$
$$D = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, D_w = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, N_1 = \begin{bmatrix} 0.2 & 0.3 & -0.5 \end{bmatrix}$$
$$N_2 = \begin{bmatrix} -0.2 & 0.5 & -0.3 \end{bmatrix}, \quad N_3 = 0, \quad N_4 = 0.1$$

and the model to be tracked has the same structure as in (7), where

$$A_m = \begin{bmatrix} -1 & -3 \\ -1 & 1 \end{bmatrix}, \quad B_m = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$
$$C_m = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad D_m = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Now let us choose F_1 such that the matrix $A - BF_1$ is Hurwitz, for instance:

$$F_1 = \begin{bmatrix} -1 & -1 & -0.5 \end{bmatrix}$$

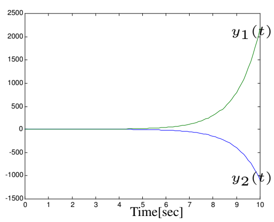
Now after solving the LMIs (38) and (39) we obtain the following results Thus, we deduce the gain matrices of the robust H_∞ memoryless filter-based tracking controller (10)

$$N = \begin{bmatrix} -260.59 & 2 & -64.3975 \\ -348.8336 & 0 & -82.7084 \\ -6.1357 & 0 & -6.5339 \end{bmatrix}$$
$$N_d = \begin{bmatrix} -12.8795 & 12.9795 & -0.1 \\ -17.4417 & 17.4417 & 0 \\ -1.1068 & 0.7068 & 0.4 \end{bmatrix}$$
$$N_f = \begin{bmatrix} 195.1925 & -64.3975 \\ 261.1252 & -85.7084 \\ 8.6018 & -2.5339 \end{bmatrix}$$

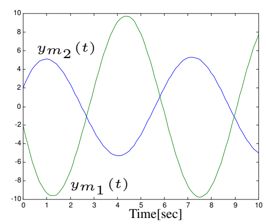
Finally, it is readily checked that the rank condition given in corollary V.1 is satisfied, and the two rank terms are equal to 9. Hence, following the steps derived in the last section to design the feedforward part we obtain the following results

$$G = \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad Q = -1.$$

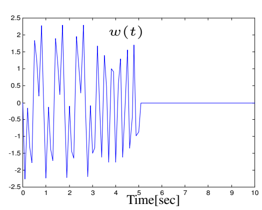
The obtained results are valide for any given value of the delay $\tau(t)$ in $I\!R^+$, such that $\dot{\tau}(t) < 0.64$. The simulation results are depicted in figures (a), (b), (c), (d), (e), (f), (g), (h) and (i). The minimal γ attenuation level of the external disturbances in the tracking process obtained is $\gamma_{min} = 0.671$.



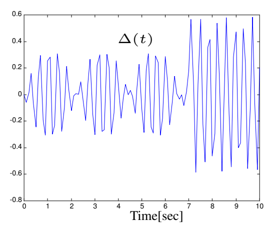
(a) The system output with u(t)=0



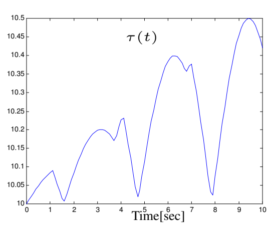
(b) The model output



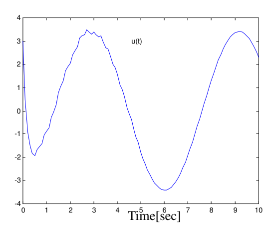
(c) The external disturbances



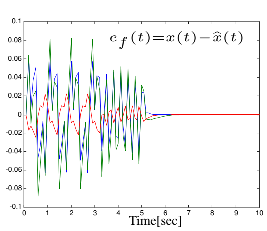
(d) The uncertainty



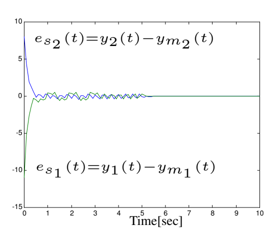
(e) The variable delay with $\dot{\tau}(t) < 0.64$



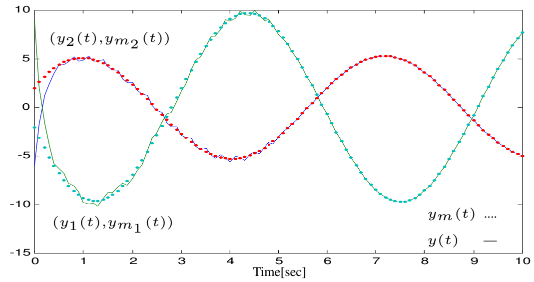
(f) The robust H_∞ filter-based tracking controller



(g) The filtering error



(h) The tracking error



(i) The system output under the filter-based tracking controller and the model output

VII. Conclusion

For the first time, the problem of the design of robust H_∞ filter-based tracking controller for a class of linear time varying delay systems with parametric uncertainties and external disturbances has been investigated. Two focal point constitute the major contribution of this paper: First, a large class of uncertain time delay systems subjected to parametric uncertainties is concerned, and the system state is supposed to be not entirely known. Second, the proposed approach succeeds in carrying out a simple and easily implementable robust H_∞ memoryless filter-based tracking controller.

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