Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Results on robust stabilization of asymptotically controllable systems by hybrid feedback

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Abstract— **We show that a nonlinear system that is asymptotically controllable to a compact set can be stabilized by a hybrid feedback that is robust to small measurement noise, actuator error, and external disturbances. The construction of such a feedback hinges upon recasting a stabilizing patchy feedback in the hybrid framework. Auxiliary results on generic robustness of stability in hybrid systems are given.**

I. INTRODUCTION

In nonlinear control system theory, numerous methods, say backstepping, forwarding, feedback linearization, and passivation, exist to design locally or globally asymptotically stabilizing feedbacks; see e.g. [15]. It is fair to conclude that the stabilization problem is well understood from a mathematical point of view, and that a rich array of design tools is available for applications.

In contrast, the robust stabilization problem is not yet completely solved, and is under active investigation. Many designs of controllers have been given to treat this problem: discontinuous sampling feedbacks [5], [29], [4], time varying control laws [7], [6], [21], [23], or patchy feedbacks [1], [2]. They enjoy different robustness properties depending on the class of considered errors and on the design structure.

Out of different errors that come into play, measurement errors are probably the most troublesome. For example, in [29], [4], where discontinuous feedbacks were used, the sampling rate of the π-solutions had to be adjusted based on the class of measurement errors allowed. In the work most important to our paper, [1] showed that an asymptotically controllable system can be stabilized with a patchy feedback, i.e. appropriately defined feedback from a family of open sets and constant feedbacks on each on them, and such a feedback is robust to external disturbances. In [2], impulsive perturbations were also considered. When a hybrid strategy is used, one may consider unknown parameters or unmodeled dynamics, see *e.g.* [18], [14], [22], [25], [24], where several types of hybrid stabilizing control laws have been considered. In [27], [28], quasi-optimal robust stabilization has been achieved by means of hybrid feedbacks.

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In this paper, we show that any asymptotically controllable to a compact set nonlinear system can be stabilized with a hybrid feedback that is robust to small measurement noise, actuator error, and external disturbances. We achieve this by recasting the patchy feedback of [1] as a hybrid feedback, and altering the latter slightly to eliminate some delicate issues of the existence of solutions under time-varying perturbations. This part of our work is conceptually similar to [24]. However, we do take advantage of some general results on hybrid systems made possible by working in the framework that was motivated in [9] by the pursuit of robust stability for hybrid systems and developed in [10] (see also [11]). A particular advantage of this framework, besides its generality, is that it guarantees good compactness and upper semicontinuity properties of the set of solutions to hybrid systems, and also that Zeno solutions do not require special treatment. This features are not always present in the numerous other approaches to hybrid systems, see for example [30], [20], [19], [31], [24].

The hybrid feedback we build, and the resulting closed loop hybrid system, has favorable growth and closedness structure, similar to that used by [10]. Thus, robustness of the constructed feedback is shown as a general property of hybrid systems possessing the said structure, rather than as a consequence of the particular construction. This generalizes the results of [24] and parallels those outlined in [9] and developed in [10] for a slightly different setting.

The link between notion of solution and robust asymptotic stability for nonlinear (but not hybrid) systems is well-understood. Consider for example the generalized solutions a la Filippov [8], or a la Krasovskii [16], or limits of solutions under vanishing noise. [13], [12]. We note that the notion of solutions for hybrid systems defined in [9], [10] can been considered in the context of the generalized solutions for nonlinear systems with a discontinuous right-hand side defined in [8], [16], [13], [12].

The paper is organized as follows. Section II states the robust stabilization via hybrid feedback results, and gives several definitions that make the result precise. Section III recalls the patchy feedback concepts of [1] and describes how a patchy vector field can be recast in the hybrid framework. Section IV deals with the robustness of stability issue: Subsection IV-A states an interesting on its own result on the generic robustness to autonomous perturbations of stability in hybrid systems, Subsection IV-B deals with the "robust" existence of solutions to hybrid systems under time-varying perturbations, finally Subsection IV-C collects various facts from the paper and proves the main result.

II. Main Result

Let the set $\widetilde{O} \subset \mathbb{R}^n$ be open, the set $\mathcal{A} \subset \widetilde{O}$ and the set of feasible controls $U \subset \mathbb{R}^m$ be compact, and the function $f : \widetilde{O} \times U \to \mathbb{R}^n$ be smooth. Consider the system

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U, \quad \text{for all } t \geq 0. \tag{1}$$

It is called *asymptotically controllable* on $\widetilde{O}$ to $\mathcal{A}$ if

(a) for each $\xi \in \widetilde{O}$ there exists a measurable $u_\xi : [0, \infty) \to \mathbb{R}^{n_u}$ with $u(t) \in U$ for almost all t such that the maximal trajectory x to (1) with u replaced by u_ξ is complete and such that $x(t) \to \mathcal{A}$ as $t \to \infty$;

(b) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\xi \in \widetilde{O}$ with $\|\xi\|_{\mathcal{A}} < \delta$ one can find u_ξ as in (a) so that the resulting x is such that $\|x(t)\|_{\mathcal{A}} < \varepsilon$ for all $t \geq 0$.

Here, $\|x\|_{\mathcal{A}}$ denotes the distance from the set $\mathcal{A}$ to x.

Theorem 2.1: *If* (1) *is asymptotically controllable on* $\widetilde{O}$ *to* $\mathcal{A}$*, then there exists a hybrid feedback on* $\widetilde{O} \setminus \mathcal{A}$ *which renders* $\mathcal{A}$ *asymptotically stable on* $\widetilde{O}$ *for the system* (1)*, robustly to measurement noise, actuator errors, and external disturbances.*

Below, we explain the meaning of each term in the result above. For simplicity, we set $O := \widetilde{O} \setminus \mathcal{A}$.

Definition 2.2: *A hybrid feedback consists of*

- *a totally ordered set* $Q \subset \mathbb{Z}^2$*,*
- *for each* $q \in Q$*,*
  - *sets* $C_q \subset O$ *and* $D_q \subset O$*,*
  - *a function* $k_q : C_q \to U$*,*
  - *a set-valued mapping* $g_q : D_q \rightrightarrows Q$*.*

The set Q is the set where the discrete variable of the hybrid system resulting from the application of the hybrid feedback evolves. The sets C_q, respectively D_q, describe the set in which the continuous variable of the hybrid system can flow, respectively, the set which enables the jump of the discrete variable. The function k_q, in closed loop with (1) determines how the continuous variable flows, while g_q describes how the discrete variable jumps.

The nonlinear system (1), with the state augmented to include the discrete variable, and "in closed loop with the hybrid feedback", will result in the hybrid system which can be informally described as

$$\begin{array}{rl} \dot{x} \in F_q(x),\, \dot{q} = 0 & \text{if} \quad x \in C_q, \\ q^+ \in G_q(x),\, x^+ = x & \text{if} \quad x \in D_q. \end{array} \tag{2}$$

In particular, the continuous variable x only changes continuously and does not jump, while the discrete variable q only changes value via jumps. To make the definition of a solution to such a system precise, we recall some concepts from [10]. A subset $S \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. S is a *hybrid time domain* if for all $(T, J) \in S$, $S \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid domain; equivalently, if S is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$. In what follows, we will write $\sup_t(S)$ for the supremum of all t such that $(t, j) \in S$ for some j.

A *solution to the hybrid system* (2) consists of: a nonempty hybrid time domain S, a function $x : S \to O$ such that $x(t, j)$ is locally absolutely continuous in t for a fixed j and $(t, j) \in S$, and a function $q : S \to Q$ such that $q(t, j)$ is constant in t for a fixed j and $(t, j) \in S$ meeting the following conditions: $x(0, 0) \in C_{q(0,0)} \cup D_{q(0,0)}$ and

(S1) For all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in S$,

$$\dot{x}(t, j) \in F_{q(t,j)}(x(t, j)), \qquad x(t, j) \in C_{q(t,j)}.$$

(S2) For all $(t, j) \in S$ such that $(t, j + 1) \in S$,

$$q(t, j + 1) \in G_{q(t,j)}(x(t, j)), \qquad x(t, j) \in D_{q(t,j)}.$$

Given a solution to (2) we will usually not mention the hybrid time domain explicitly, but will identify the solution by (x, q), and when needed, refer to the associated hybrid time domain by $\mathrm{dom}(x, q)$.

As the state of the original nonlinear system (1) evolves in $\widetilde{O}$, while the continuous variable of the hybrid system under discussion evolves in $O = \widetilde{O} \setminus \mathcal{A}$, in the definition of asymptotic stability of the latter we need to allow for solutions reaching $\mathcal{A}$ in finite time.

Definition 2.3: *The set* $\mathcal{A}$ *is asymptotically stable on* O *for the hybrid system* (2) *if:*

- *for any* $(x_0, q_0) \in O \times Q$ *there exists a solution to* (2) *with* $x(0, 0) = x_0$, $q(0, 0) = q_0$*;*
- *for any maximal solution* (x, q) *to* (2) *we have* $x(t, j) \to \mathcal{A}$ *as* $t \to \sup_t(\mathrm{dom}(x, q))$*;*
- *for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any solution* (x, q) *to* (2) *with* $\|x(0, 0)\|_{\mathcal{A}} \leq \delta$ *satisfies* $\|x(t, j)\|_{\mathcal{A}} \leq \varepsilon$ *for all* $(t, j) \in \mathrm{dom}(x, q)$.

Consequently, we say that the hybrid feedback as in Definition 2.2 renders $\mathcal{A}$ asymptotically stable on $\widetilde{O}$ for (1) if setting $F_q(x) = f(x, k_q(x))$ and $G_q(x) = g_q(x)$ yields a hybrid system for which $\mathcal{A}$ is asymptotically stable on O.

In what follows, *admissible measurement noise* and *admissible external disturbance* are functions ξ and ζ in $\mathcal{L}^\infty_{loc}(O \times \mathbb{R}_{\geq 0}; \mathbb{R}^n)$ that are continuous in $x \in O$ for each $t \in \mathbb{R}_{\geq 0}$. As noted in [17, Remark 1.4], with the presence of ζ and the continuity of f in u, we can omit any explicit reference to actuator errors. Furthermore, due to the discrete nature of the set Q, we do not consider noise or disturbances "for the discrete variable". The presence of measurement noise and external disturbances leads to a time-varying hybrid system, which we will denote by $\mathcal{H}_{\xi,\zeta}$.

Definition 2.4: *A solution to* $\mathcal{H}_{\xi,\zeta}$ *consists of: a nonempty hybrid time domain* S*, a function* $x : S \to O$ *such that* $x(t, j)$ *is locally absolutely continuous in* t *for a fixed* j *and* $(t, j) \in S$*, and a function* $q : S \to Q$ *such that* $q(t, j)$ *is constant in* t *for a fixed* j *and* $(t, j) \in S$ *meeting the following conditions:* $x(0, 0) \in C_{q(0,0)} \cup D_{q(0,0)}$ *and*

(S_p1) For all $j \in \mathbb{N}$ *and almost all* t *with* $(t, j) \in S$*,*

$$\dot{x}(t, j) \in f\left(x(t), k_{q(t,j)}(x(t, j) + \xi(t, j), t\right) + \zeta(x(t), t),$$

$$x(t, j) + \xi(x(t, j), t) \in C_{q(t,j)}.$$

(S_p2) For all $(t,j) \in S$ such that $(t,j+1) \in S$,

$$q(t,j+1) \in g_{q(t,j)}\left(x(t,j)+\xi(x(t,j),t)\right),$$

$$x(t,j)+\xi(x(t,j),t) \in D_{q(t,j)}.$$

Let us compare this definition with the solution notion from [24] (which, essentially, is taken from [3]). The definition of the hybrid time domain implies that, in the standard terminology, the set of jump times is of measure zero; this was also the case in [24]. When a jump occurs, the continuous variable must be in D_q. This was the case of [24], where the discrete-component of a trajectory was right-continuous on a subset of $O \times Q$ (this set is denoted $\mathcal{RC}$ in [24]). The main difference between the notions is the interpretation of the memory in a hybrid system. In [24] it is seen as the left-limit of the discrete component of the solution. Here it is seen as $k_{q(t,j)}\left(x(t,j)+\xi(x(t,j),t)\right)$ only, which requires *a priori* less information to be computed. This discrete evolution and the notion of hybrid time domain imply that Zeno-solutions are included in the class of solution under study, which is not the case in [24].

Naturally, given ξ and ζ, we will say that $\mathcal{A}$ is asymptotically stable on O for the hybrid system $\mathcal{H}_{\xi,\zeta}$ if Definition 2.3 is satisfied, when solutions to $\mathcal{H}_{\xi,\zeta}$ are considered in place of those to (2). By an *admissible perturbation radius* we will understand any continuous function $\rho : O \to \mathbb{R}_{>0}$ such that $x+\rho(x)\mathbb{B} \subset O$ for all $x \in O$. (Here and in what follows, $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$.)

Definition 2.5: *A hybrid feedback on $\widetilde{O} \setminus \mathcal{A}$ renders $\mathcal{A}$ asymptotically stable on $\widetilde{O}$ for* (1)*, robustly to measurement noise, actuator errors, and external disturbances if there exists an admissible perturbation radius $\delta : O \to \mathbb{R}_{>0}$ such that for all admissible measurement noise ξ and admissible external disturbance ζ such that*

$$\begin{array}{ll} \|\xi(x,t)\| \le \delta(x), & \text{for all } x \in O, t \in \mathbb{R}_{\ge 0}, \\ \|\zeta(x,t)\| \le \delta(x) & \end{array} \qquad (3)$$

$\mathcal{A}$ is asymptotically stable on O for the hybrid system $\mathcal{H}_{\xi,\zeta}$.

III. Patchy vector fields and hybrid systems

Definition 3.1: *([1]) A mapping $\phi : \Omega \to \mathbb{R}^n$ is a patchy vector field on Ω if there exist a set $\mathcal{Q}$, and for each $\alpha \in \mathcal{Q}$, sets $\Omega_\alpha \subset \Omega$, $O_\alpha \subset \Omega$ and a function f_α such that*

(a) *for each $\alpha \in \mathcal{Q}$, the triple Ω_α, O_α, f_α forms a patch, that is:*
   - (a1) *Ω_α, O_α are open, $\overline{\Omega_\alpha} \subset O_\alpha$, and the boundary of Ω_α is smooth;*
   - (a2) *$f_\alpha : O_\alpha \to \mathbb{R}^n$ is smooth;*
   - (a3) *for any point $\xi \in \operatorname{bdry}\Omega_\alpha$*

$$\langle f_\alpha(\xi), n_\alpha(\xi)\rangle < 0,$$

where $n_\alpha(\xi)$ is the outer normal to $\overline{\Omega_\alpha}$ at ξ;

(b) *$\mathcal{Q}$ is a totally ordered set;*

(c) *the sets Ω_α form a locally finite covering of Ω;*

and ϕ can be written in the form

$$\phi(\xi) = f_\alpha(\xi) \quad \text{if} \quad \xi \in \Omega_\alpha \setminus \bigcup_{\beta \succ \alpha} \Omega_\beta,$$

where $\succ$ is the ordering of $\mathcal{Q}$.

Solutions to a patchy vector field are understood in the Caratheodory sense, i.e. as locally absolutely continuous functions that satisfy $\dot{x}(t) = \phi(x(t))$ almost everywhere. In Lemma 3.6, we will relate the solutions of the patchy vector field to those of a corresponding hybrid system.

Definition 3.2: *([1]) A mapping $u : \Omega \to U$ is a patchy feedback if there exists a patchy vector field $\phi : \Omega \to \mathbb{R}^n$ on Ω (given by the index set $\mathcal{Q}$, and sets Ω_α, O_α and a function f_α for each $\alpha \in \mathcal{Q}$) and control values $u_\alpha \in U$ such that*

$$u(\xi) = u_\alpha \quad \text{if} \quad \xi \in \Omega_\alpha \setminus \bigcup_{\beta \succ \alpha} \Omega_\beta$$

and ϕ can be written in the form

$$\phi(\xi) = f(\xi, u(\xi)) \quad \text{if} \quad \xi \in \Omega_\alpha \setminus \bigcup_{\beta \succ \alpha} \Omega_\beta.$$

We will be interested in patchy feedbacks that, for the system (1), render the set $\mathcal{A}$ globally asymptotically stable on O (for a precise definition of this see [1]). The result below extends Theorem 1 of [1] to compact attractors $\mathcal{A}$ and a general open domain O. In contrast to [1], we do not require the patchy feedback to have any robustness properties, those will come from reformulating the patchy feedback as hybrid feedback. By a *proper indicator* of $\mathcal{A}$ with respect to $\widetilde{O}$ we understand a continuous function $\omega : \widetilde{O} \to \mathbb{R}_{\ge 0}$ such that $\omega(\xi) = 0$ if and only if $\xi \in \mathcal{A}$, and $\omega(\xi) \to \infty$ if $\xi \to (\operatorname{bdry}\widetilde{O}) \setminus \mathcal{A}$ or $\|\xi\| \to \infty$.

Theorem 3.3: *For any asymptotically controllable on $\widetilde{O}$ to $\mathcal{A}$ nonlinear system* (1) *there exists a patchy feedback on O that renders $\mathcal{A}$ asymptotically stable on O for* (1)*. The patchy feedback can be chosen so that*

(a) *the index set $\mathcal{Q}$ is a subset of $\mathbb{N}^2$;*

(b) *for each $\alpha \in \mathcal{A}$, $\overline{\Omega_\alpha}$ is a compact subset of O;*

(c) *for some proper indicator ω of $\mathcal{A}$ with respect to $\widetilde{O}$, we have the following: for each $\alpha \in \mathcal{Q}$ there exist $\delta, \Delta > 0$ so that $\sup\omega(\Omega_\beta) \le \delta$ implies $\beta \succ \alpha$, and $\beta \succ \alpha$ implies $\sup\omega(\Omega_\beta)$.*

From now on, let $\mathcal{PVF}$ denote a patchy vector field resulting from the application of a stabilizing patchy feedback as in Theorem 3.3 to (1).

Lemma 3.4: *Let Ω_α, O_α, and f_α form a patch, and suppose that Ω_α is bounded. Then, there exist an open set Ω'_α so that $\overline{\Omega_\alpha} \subset \Omega'_\alpha \subset \overline{\Omega'_\alpha} \subset O_\alpha$ and a constant $T_\alpha > 0$ so that any maximal solution to $\dot{x}(t) = f_\alpha(x(t))$ with $x(0) \in \overline{\Omega'_\alpha}$ is complete and such that $x(t) \in \Omega_\alpha$ for all $t > T_\alpha$.*

We add that, in Lemma 3.4, the set Ω'_α and the constant T_α can be picked arbitrarily small. For the former, this means that Ω'_α can be picked as a subset of an arbitrary neighborhood of Ω_α.

Given a $\mathcal{PVF}$, let $Q := \mathcal{Q}$, and for each $q \in Q$, use Lemma 3.4 to find Ω'_q and $T_q > 0$ so that, for some proper indicator ω of $\mathcal{A}$ with respect to $\widetilde{O}$ and all $q \in Q$,

$$\inf\omega(\Omega_q)/2 \le \omega(x) \le 2\sup\omega(\Omega_q)$$

for all $x \in \overline{\Omega'_q}$. Note that this entails $\{\Omega'_q\}_{q\in Q}$ being a locally finite covering of O. Now, let $\mathcal{H}_{\mathcal{PVF}}$ be the hybrid system defined on $O \times Q$ as follows: for each $q \in Q$, let

$$
\begin{aligned}
C_q &= \overline{\Omega'_q} \setminus \bigcup_{\beta \succ q} \Omega_\beta \\
F_q(x) &= f_q(x) \\
D_q &= \bigcup_{\beta \succ q} \overline{\Omega_\beta} \cup \left(O \setminus \Omega'_q\right)
\end{aligned} \tag{4}
$$

$$
G_q(x) = \begin{cases} \{\beta \in Q \,|\, x \in \overline{\Omega_\beta}, \beta \succ q\} & x \in \bigcup_{\beta \succ q} \overline{\Omega_\beta} \\ \{\beta \in Q \,|\, x \in \overline{\Omega_\beta}\} & x \in O \setminus \Omega'_q \end{cases}
$$

Lemma 3.5: *The sets and mappings defined in* (4) *are such that, for all* $q \in Q$,

(A_q1) C_q and D_q are relatively closed subsets of O.

(A_q2) $F_q : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded, and $F_q(x)$ is nonempty and convex for all $x \in C_q$.

(A_q3) $G_q : O \rightrightarrows Q$ is outer semicontinuous and locally bounded, and $G_q(x)$ is nonempty for all $x \in D_q$.

This means that $\mathcal{H}_{\mathcal{PVF}}$ has similar continuity and closedness properties to the systems analyzed by [10]; such properties are the key to showing the robustness result we give in subsection IV-A. Furthermore, for each $q \in Q$, $C_q \cup D_q = O$. This implies that solutions to $\mathcal{H}_{\mathcal{PVF}}$ exist for any initial point in $O \times Q$.

We now compare solutions of $\mathcal{PVF}$ to those of $\mathcal{H}_{\mathcal{PVF}}$.

Lemma 3.6:

(a) *Let $x : [0, T] \to O$ be a solution to $\mathcal{PVF}$. Let $t_1 < t_2 < \cdots < t_{J-1}$ be the sequence of discontinuities of $t \to \alpha^*(x(t))$ where*

$$q^*(\xi) = \max\{\beta \in Q \mid \xi \in C_\beta\},$$

let $t_0 = 0$, $t_J = T$. Then (x, q) given on the hybrid time domain

$$\bigcup_{j=1}^{J} [t_{j-1}, t_j] \times \{j\}$$

by $(x, q)(t, j) = (x(t), q^*(x(t_j)))$ for $t \in [t_{j-1}, t_j]$ is a solution to $\mathcal{H}_{\mathcal{PVF}}$.

(b) *Let (x, q) be a solution to $\mathcal{H}_{\mathcal{PVF}}$ with compact $\operatorname{dom}(x, q)$. Let $[t_{j-1}, t_j]$, $j = 1, 2, \ldots, J$ be the sequence of all nontrivial (i.e. with $t_{j-1} < t_j$) intervals such that for all j, $[t_{j-1}, t_j] \times \{i_j\} \in \operatorname{dom} y$ for some i_j. (It may happen that this sequence is empty.) Then the function $x' : [t_1, t_J] \to O$ given by $x'(t) = x(t, i_j)$ for $t \in [t_{j-1}, t_j]$ is a solution to $\mathcal{PVF}$.*

In particular, (b) above says that after the first jump, the continuous part of the solution to $\mathcal{H}_{\mathcal{PVF}}$ is (essentially) also a solution to $\mathcal{PVF}$. Note also that by the construction, and by Lemma 3.4, each maximal solution to $\mathcal{H}_{\mathcal{PVF}}$ does jump at least once (in fact infinitely many times). This, and the properties of sets C_q, yield the following.

Corollary 3.7: *For the hybrid system $\mathcal{H}_{\mathcal{PVF}}$ defined on $Q \times O$ by* (4), *$\mathcal{A}$ is asymptotically stable on O.*

IV. Proof of the main result

This section proves Theorem 2.1, by exploiting the generic robustness of stability in hybrid systems possessing some basic growth and closedness properties. We want to point out that a more direct approach is possible. Through a more careful and more explicit, in comparison to that following Lemma 3.4, construction of the hybrid system from a patchy vector field, one can build a hybrid feedback stabilizing (1), for which robustness can be verified by directly calculating the admissible error bounds. Details will be given in the forthcoming work [26].

A. Generic robustness of hybrid systems

As outlined in [9], much of the motivation for insisting on some closedness and outer semicontinuity properties of the data of hybrid systems (similar to those in Lemma 3.5) is that in presence of such properties, asymptotic stability of compact attractors is robust. Results outlined in [9] and shown in [10], [11] do not apply directly to the hybrid systems discussed in the current paper, partly because we consider convergence of the continuous variable only (which in a bigger framework corresponds to a noncompact attractor), and partly because the attractor is not in the state space. However, for a broad class of systems including $\mathcal{H}_{\mathcal{PVF}}$, generic robustness of stability can be shown.

Theorem 4.1: *Consider the hybrid system* (2) *and assume that:*

(A0) *For all $q \in Q$, $C_q \cup D_q = O$;*

(•) *For all $q \in Q$, (A_q1), (A_q2), (A_q3) of Lemma 3.5 hold;*

(A4) *The family $\{C_q\}_{q\in Q}$ forms a locally finite covering of O;*

(A5) *The mappings $G_q : O \to Q$ are locally bounded in x uniformly in q.*

If $\mathcal{A}$ is asymptotically stable on O for (2), *then it is robustly asymptotically stable. That is, there exists an admissible perturbation radius ρ such that the system $\mathcal{H}^\rho$ given by*

$$
\begin{aligned}
F^\rho_q(\xi) &:= \operatorname{con} F_q(\xi + \rho(\xi)\mathbb{B}) + \rho(\xi)\mathbb{B}, \\
G^\rho_q(\xi) &:= G_q(\xi + \rho(\xi)\mathbb{B}), \\
C^\rho_q &:= \{\xi \in O \mid (\xi + \rho(\xi)\mathbb{B}) \cap C_q \neq \emptyset\}, \\
D^\rho_q &:= \{\xi \in O \mid (\xi + \rho(\xi)\mathbb{B}) \cap D_q \neq \emptyset\},
\end{aligned} \tag{5}
$$

is asymptotically stable on O to $\mathcal{A}$.

B. Robustness to time varying perturbations

Consider the hybrid system (2) under the assumptions of Theorem 4.1. Let ρ be an admissible perturbation radius so that the system $\mathcal{H}^\rho$ is asymptotically stable on O to $\mathcal{A}$, and suppose that ξ and ζ are admissible measurement noise and external disturbance, such that (3) hold. Then, any solution to the (time-varying) system $\mathcal{H}_{\xi,\zeta}$ is also a solution to $\mathcal{H}^\rho$. However, the very existence of solutions to $\mathcal{H}_{\xi,\zeta}$ can be problematic. For example, consider a hybrid system (2) on $O = \mathbb{R}$ and $Q = \{0\}$ given by $C = (-\infty, 0]$, $D = [0, \infty)$, $F(x) = 0$, $G(x) = 0$, measurement noise $\xi(x, t) = \xi(t)$

with $\xi(0) = -1$, $\xi(t) = 1$ for all $t > 0$ (similar, but arbitrarily small noise could be used here) and any external disturbance. Then, there does not exist a solution to $\mathcal{H}_{\xi,\zeta}$ with $x(0,0) = 0$. Indeed, the solution can not jump at the hybrid time $(0,0)$ (i.e. $(0,0),(0,1) \in \operatorname{dom}(x,q)$ can not happen) since $x(0,0) + \xi(x(0,0),0) = -1 \notin D$. Similarly, the solution can not flow at the hybrid time $(0,0)$ (i.e. $[0,\varepsilon) \times \{0\} \in \operatorname{dom}(x,q)$ can not happen for $\varepsilon > 0$) because $x(t,0) + \xi(x(t,0),t) = 1 \notin C$ for any $t > 0$.

It turns out that if the sets C_q and D_q "overlap", the existence can be guaranteed. Below, by $\operatorname{int} S$ we mean the interior of a set S.

Lemma 4.2: *Consider the hybrid system* (2) *and assume that (A4) of Theorem 4.1, and for all $q \in Q$, A_q1, A_q2, A_q3 of Lemma 3.5 hold. Suppose that, for all $q \in Q$, each $x \in O$ is such that either $x \in \operatorname{int} C_q$ or $x \in \operatorname{int} D_q$. Then there exists an admissible perturbation radius ρ_e such that for any admissible measurement noise and external disturbance ξ and ζ for which* (3) *holds, solutions to $\mathcal{H}_{\xi,\zeta}$ exist for any initial point $(x_0,q_0) \in O \times Q$.*

C. Proof of the main result

Let $u : O \to U$ be a patchy feedback that asymptotically stabilizes (1), as guaranteed by Theorem 3.3. Let the associated patchy vector field $\mathcal{PVF}$ be given by the index set $\mathcal{Q}$, sets Ω_α, O_α and control values $u_\alpha \in U$ (leading to function f_α for each $\alpha \in \mathcal{A}$); recall Definition 3.2. Set $Q := \mathcal{Q}$, and for $q \in Q$, let Ω'_q be as constructed below Lemma 3.4. Then define the sets

$$\begin{aligned} C'_q &= \overline{\Omega'_q} \setminus \bigcup_{\beta \succ q} \Omega_\beta \\ D'_q &= \bigcup_{\beta \succ q} \overline{\Omega_\beta} \cup \left(O \setminus \Omega'_q\right) \end{aligned} \tag{6}$$

and mappings $k_q : O \to U$, $g_q : O \to Q$ by

$$\begin{aligned} k_q(x) &= u_q \\ g_q(x) &= \begin{cases} \{\beta \in Q \mid x \in \overline{\Omega_\beta}, \beta \succ q\} & x \in \bigcup_{\beta \succ q} \overline{\Omega_\beta} \\ \{\beta \in Q \mid x \in \overline{\Omega_\beta}\} & x \in O \setminus \Omega'_q \end{cases} \end{aligned} \tag{7}$$

Let $\mathcal{H}$ stand for the hybrid system (2) given by sets C'_q, D'_q, $F_q(x) := f(x, k_q(x))$, $G_q(x) = g_q(x)$. Let ρ be an admissible perturbation radius resulting from the application of Theorem 4.1 to $\mathcal{H}$; in particular, we thus have that $\mathcal{A}$ is asymptotically stable on O for $\mathcal{H}^\rho$.

Lemma 4.3: *Given sets C'_q, D'_q and the admissible perturbation ρ as above, there exists an admissible perturbation radius ρ' bounded above by ρ such that, for all $q \in Q$,*

$$\left(C'^{\rho'}_q\right)^{\rho'} \subset \left(C'_q\right)^\rho, \quad \left(D'^{\rho'}_q\right)^{\rho'} \subset \left(D'_q\right)^\rho.$$

Above, $\left(C'_q\right)^\rho$ is the "inflation" of the set C'_q as in (5), while $\left(C'^{\rho'}_q\right)^{\rho'}$ is the "inflation" of $\left(C'_q\right)^{\rho'}$. Similarly for D'_q. Now, for each $q \in Q$, define

$$C_q = \left(C'_q\right)^{\rho'}, \quad D_q = \left(D'_q\right)^{\rho'}. \tag{8}$$

These sets are such that for each $q \in Q$, each $x \in Q$ is either in $\operatorname{int} C_q$ or $\operatorname{int} D_q$ (or both). Note that as k_q and g_q are defined on O (and are nonempty etc. there), not just on C'_q and D'_q, they are certainly well-defined on C_q and D_q. Let ρ_e be the admissible perturbation radius as guaranteed by Lemma 4.2, and let $\delta(x) := \min\{\rho'(x), \rho_e(x)\}$.

Consider the hybrid feedback given Q, the sets C_q, D_q as in (8) and the mappings k_q, g_q as in (7). Let ξ and ζ be admissible measurement noise and admissible external disturbance such that (3) holds. By Lemma 4.2, solutions to $\mathcal{H}_{\xi,\zeta}$ exist for any initial point in $O \times Q$. By construction, and since $\delta(x) \le \rho'(x)$, any solution to $\mathcal{H}_{\xi,\zeta}$ is a solution to $\mathcal{H}^\rho$. As for the latter system, $\mathcal{A}$ is asymptotically stable on O, so is the case for $\mathcal{H}_{\xi,\zeta}$. This finishes the proof.

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