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Nonlinear Control Analysis on Nonholonomic Dynamic Systems with Affine Constraints

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Abstract— **This paper deals with nonholonomic control systems subject to affine constraints. We first derive several preliminary properties of nonholonomic dynamic systems with affine constraints (NDSAC). We then investigate local accessibility and local controllability of the NDSAC based on both Sussmann's theorem and linear approximation approaches. Conditions for local asymptotic stabilizability of the NDSAC by linear state feedback and nonlinear smooth state feedback are also derived. Finally, two physical examples are illustrated to confirm the results.**

I. INTRODUCTION

Many researchers have studied nonlinear control systems with nonholonomic constraints or nonholonomic control systems. Roughly speaking, researches of nonholonomic control systems can be classified into two areas: kinematic systems and dynamic systems. In both the areas, *linear constraints* which are linear in velocities have been mainly investigated. Kinematic systems are directly derived from nonholonomic constraints, and in particular linear constraints can be transformed into symmetrically affine control systems. On the contrary, dynamic systems are derived from Euler-Lagrange equations with constraints by D'Alembert's principle. Especially, Bloch et al. [1] have analyzed nonholonomic dynamic systems with linear constraints. There are two common characteristics between kinematic and dynamic systems: (i) Their linear approximated systems are uncontrollable. (ii) They are locally controllable, but not locally asymptotically stabilizable by any nonlinear smooth state feedback from Brockett's theorem [2]. Therefore, many control laws which avoid Brockett's condition have been proposed such as time-variant feedback, discontinuous feedback and switching control.

There is another class of constraints which are affine in velocities and called *affine constraints*. It is a larger class of constraints than that of linear constraints. A space robot with initial angular momentum, a coin and a ball on a rotating table [3], a pneumatic tire [4], under-actuated manipulators and underwater vehicles [5] are typical examples of affine constraints. Until now, there have been much less researches on affine constraints than those on linear constraints. In [6], [7], we have derived the integrability and nonintegrability conditions for affine constraints using vector fields on a manifold and investigated accessibility of kinematically asymmetrically affine control systems (KAACS) derived from affine constraints. Moreover, in [8] we have analyzed the KAACS with nonholonomic affine constraints and shown the following two facts: (i) There exists a class of systems whose linear approximations are controllable, and hence they are stabilizable by linear state feedback. (ii) There exists a class of systems such that Brockett's condition holds, i.e., they may be stabilized by nonlinear smooth state feedback. These are far beyond the well known facts for nonholonomic systems so far.

In this paper, we analyze nonholonomic dynamic systems with affine constraints more rigorously than [9], and the main focus is to investigate whether above two facts also hold in dynamic systems with affine constraints rather than linear constraints. The rest of the paper is organized as follows. Section II presents some definitions and concepts of affine constraints. Moreover, a complete nonholonomicity condition is derived. In Section III, we first give the problem setting and introduce a nonholonomic dynamic system with affine constraints (NDSAC). Section IV is devoted to nonlinear control analysis of the NDSAC, which includes local accessibility, local controllability and local asymptotic stabilizability, and the main results of this paper are provided. Finally in Section V, we illustrate two physical examples, namely a coin on a rotating table and a ball on a rotating table, to confirm our results obtained in the paper. Through this paper, manifolds, vector fields, functions and distributions are all assumed to be smooth.

II. AFFINE CONSTRAINTS

A. Preliminaries

In this subsection, we first define affine constraints that we treat through this paper. Let Q be an n-dimensional manifold and an n-dimensional column vector $q = [\, q_1 \; \cdots \; q_n \,]^T \in \boldsymbol{R}^n$ be a local coordinate of Q. In this paper, we consider $n - m \; (n > m)$ affine constraints:

$$A(q) + B(q)\dot{q} = 0, \tag{1}$$

where $A(q)$ is an $(n - m)$-dimensional column vector and $B(q)$ is an $(n-m)\times n$ matrix. We assume the independence of the affine constraints as follows.

Assumption 1: The matrix $B(q)$ in the affine constraints (1) has row full-rank at any point $q \in Q$, that is,

$$\operatorname{rank} B(q) = n - m, \quad \forall q \in Q \tag{2}$$

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holds. ∎

Next, we introduce new concepts to classify the affine constraints. A point $q \in Q$ such that $A(q) = 0$ is called *an affine equilibrium point* and the set of affine equilibrium points is defined by

$$U^e = \{q \in Q \mid A(q) = 0\}\,. \tag{3}$$

We then define *the affine index at* $q^e \in U^e$:

$$r(q^e) := \mathrm{rank}\frac{\partial A}{\partial q}(q^e) \tag{4}$$

and *the affine index of the affine constraints*:

$$r := \max_{q^e \in U^e} r(q^e). \tag{5}$$

Using them, we classify the affine constraints as follows.

Definition 1: The affine constraints (1) are categorized into three types by affine indices (5) as follows.

(a) $r = 0$: *completely linear constraints*

(b) $1 \leq r \leq n-m-1$: *r-th order partially affine constraints*

(c) $r = n - m$: *completely affine constraints* ∎

The completely linear constraints correspond with linear constraints which have been mainly studied so far. The partially affine constraints includes both linear and affine constraints such as a coin on a rotating table which will be shown in Subsection V-A. The completely affine constraints are constraints that consist of only affine constraints such as a ball on a rotating table which will appear in Subsection V-B.

Finally, we explain a geometric representation of the affine constraints, which plays important roles through this paper. Since $n - m$ row vectors of $B(q)$ are all independent from (2), we can find m vector fields $Y_1, \cdots, Y_m$ which are all independent and annihilate $n-m$ row vectors of $B(q)$. Let us denote a space spanned by $Y_1, \cdots, Y_m$, that is, a distribution on Q as

$$D = \mathrm{span}\{Y_1, \cdots, Y_m\}. \tag{6}$$

A curve $q : I \to Q$ is said to satisfy the affine constraints (1) for a time interval I if and only if there exists a vector field X satisfying

$$A(q) + B(q)X(q) = 0, \ \forall q \in Q \tag{7}$$

and the curve satisfies

$$\dot{q}(t) - X(q(t)) \in D(q(t)) \qquad \forall\, t \in I. \tag{8}$$

Therefore, a geometric representation of the affine constraints is defined as follows.

Definition 2 [3], [10], [6]: The affine constraints (1) are geometrically represented by a pair (D, X), where D is a distribution defined by (6), and X is a vector field which satisfies (7) and called *an affine vector field*. ∎

For the geometric representation of the affine constraints, the following proposition holds.

Proposition 1 [6], [8]: For the geometric representation of the affine constraints (D, X), $X(q) \in D(q)$ holds at a point $q \in Q$ if and only if the point is an affine equilibrium point. Conversely, $X(q) \notin D(q)$ holds at a point $q \in Q$ if and only if the point is an affine regular point. ∎

B. Complete Nonholonomicity

In this subsection, we discuss completely nonholonomicity of the affine constraints. If all the $n-m$ affine constraints (1) are nonintegrable, that is, there do not exist any independent first integrals, they are said to be *completely nonholonomic* or *completely nonintegrable*. Now we define a smallest and involutive distribution C_0 which contains $Y_1, \cdots, Y_m$ and satisfies $[X, W] \in C_0$, $\forall W \in C_0$. A necessary and sufficient condition of complete nonholonomicity for the affine constraints is derived as follows (see also [9], [8]).

Theorem 1: The affine constraints (1) are completely nonholonomic if and only if

$$\dim C_0 = n \tag{9}$$

holds.

Proof. Consider the product space $\bar{Q} := \boldsymbol{R} \times Q$ with $(n+1)$-dimension, where $\boldsymbol{R}$ is the space of the time variable. On $\bar{Q}$, the affine constraints (1) are represented by Pfaffian equations of $n - m$ differential forms:

$$A(q)dt + B(q)dq = 0. \tag{10}$$

Since an vector field X of the geometric representation satisfies (7), $m + 1$ vector fields on $\bar{Q}$ which annihilate (10) are given by

$$\bar{X} := \frac{\partial}{\partial t} \oplus X, \ \ \bar{Y}_i := 0 \oplus Y_i \ \ (i = 1, \cdots, m). \tag{11}$$

Now we define an involutive distribution $\bar{C}$ defined on $\bar{Q}$, which contains $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$ and iterated Lie brackets that consist of $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$. Therefore, a necessary and sufficient condition for complete nonintegrability is given by

$$\dim \bar{C} = n + 1 \tag{12}$$

(cf. Frobenius' theorem [11], [12]). Calculate the iterated Lie brackets which consist of $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$, then we have

$$\begin{aligned}
&[\bar{X}, \bar{Y}_i] = 0 \oplus [X, Y_i]\,, \\
&[\bar{X}, [\bar{X}, \bar{Y}_i]] = 0 \oplus [X, [X, Y_i]], \ \cdots \\
&[\bar{Y}_j, \bar{Y}_i] = 0 \oplus [Y_j, Y_i]\,, \\
&[\bar{Y}_k, [\bar{Y}_j, \bar{Y}_i]] = 0 \oplus [Y_k, [Y_j, Y_i]], \ \cdots .
\end{aligned} \tag{13}$$

We can see that $\bar{X}$ is independent of $\bar{Y}_i, \cdots, \bar{Y}_m$ and the iterated Lie brackets (13). Then, the necessary and sufficient condition (12) is changed into a condition that $\bar{Y}_1, \cdots, \bar{Y}_m$ and the iterated Lie brackets which consist of $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$ span an n-dimensional space. From (11) and (13), we can only consider $Y_1, \cdots, Y_m$ on Q instead of $\bar{Y}_1, \cdots, \bar{Y}_m$ on $\bar{Q}$, and iterated Lie brackets which consist of $X, Y_1, \cdots, Y_m$ on Q instead of those which consist of $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$ on $\bar{Q}$. Therefore, a necessary and sufficient condition for complete nonholonomicity is that $Y_1, \cdots, Y_m$ and the iterated Lie brackets which consist of $X, Y_1, \cdots, Y_m$ span an n-dimensional space, that is, (9) holds. ∎

We then assume the following, that is, we deal with nonholonomic affine constraints.

Assumption 2: The affine constraints (1) are completely nonholonomic, that is, (9) holds. ∎

III. NONHOLONOMIC DYNAMIC SYSTEMS WITH AFFINE CONSTRAINTS

A. Problem Setting

This subsection is devoted to the problem setting of this paper. Let Q be an n-dimensional configuration manifold and a column vector $q = [\, q_1 \, \cdots \, q_n \,]^T \in \boldsymbol{R}^n$ be generalized configuration variables. We then denote generalized velocity and acceleration variables as $\dot{q} = [\, \dot{q}_1 \cdots \dot{q}_n \,]^T \in \boldsymbol{R}^n$ and $\ddot{q} = [\, \ddot{q}_1 \cdots \ddot{q}_n \,]^T \in \boldsymbol{R}^n$, respectively. We consider the Lagrangian of general form, that is, the kinetic energy minus the potential energy:

$$L(q, \dot{q}) := \frac{1}{2} \dot{q}^T G(q) \dot{q} - U(q), \tag{14}$$

where $G(q)$ is an $n \times n$ inertia matrix which is symmetric and positive definite, and $U(q)$ is a potential function.

The system is assumed to be subject to $n - m$ affine constraints (1) under Assumption 1. From (2), $B(q)$ can be partitioned as $B(q) = [\, B_1(q) \, B_2(q) \,]$ by changing the order of the generalized configuration variable q, where $B_1(q)$ is an $(n-m) \times m$ matrix and $B_2(q)$ is an $(n-m) \times (n-m)$ matrix which is non-singular for any point $q \in Q$. Depending on this partition, the generalized configuration variables q is also partitioned into $q = [\, q_1^T \;\; q_2^T \,]^T$ where q_1 and q_2 are an m-dimensional and $(n-m)$-dimensional column vectors, respectively. Consequently, we have $\dot{q}_2 = -B_2(q)^{-1} A(q) - B_2(q)^{-1} B_1(q) \dot{q}_1$ and we then rewrite the affine constraints (1) as

$$\dot{q} = \hat{X}(q) + \hat{Y}(q) \dot{q}_1, \tag{15}$$

where $X(q)$ is an n-dimensional column vector and $Y(q)$ is an $n \times m$ matrix, which are defined by

$$\hat{X}(q) := \begin{bmatrix} 0 \\ -B_2(q)^{-1} A(q) \end{bmatrix}, \; \hat{Y}(q) := \begin{bmatrix} I_m \\ -B_2(q)^{-1} B_1(q) \end{bmatrix}.$$

Note that $\hat{X}$ and $\hat{Y}$'s column vectors $\hat{Y}_1, \cdots, \hat{Y}_m$ satisfy the properties of the geometric representation of the affine constraints.

Finally, we set control inputs. Let an r-dimensional ($r < n$) column vector $q = [\, u_1 \, \cdots \, u_r \,]^T \in \boldsymbol{R}^r$ be control input variables and an $n \times r$ matrix $E(q)$ be an input change matrix. Then control inputs to a system are denoted by $E(q)u$. We here impose the following assumption on the control inputs.
Assumption 3: The number of the control inputs is equal to the dimension of D, that is $r = m$. Furthermore, the constraint force and the control inputs are complimentary, that is, the $m \times m$ matrix $\hat{Y}(q)^T E(q)$ is non-singular at any point $q \in Q$. ∎

B. NDSAC and Normal Form

In this subsection, we derive *nonholonomic dynamic systems with affine constraints (NDSAC)* based on the problem setting in the previous subsection. Firstly, we substitute the Lagrangian (14) for the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \tag{16}$$

and the constraint force $B(q)^T \lambda$ and the control inputs $E(q)u$ are added in the right-hand side of (16), then we have

$$G(q)\ddot{q} + \Gamma(q, \dot{q}) + \nabla U(q) = B(q)^T \lambda + E(q)u, \tag{17}$$

where an $(n - m)$-dimensional column vector $\lambda = [\, \lambda_1 \, \cdots \, \lambda_{n-m} \,]^T \in \boldsymbol{R}^{n-m}$ is the Lagrange multiplier and an n-dimensional column vector $\Gamma(q, \dot{q})$ is the Coriolis' and centrifugal force term defined by $\Gamma(q, \dot{q}) := \dot{G}(q)\dot{q} - \frac{1}{2}\frac{\partial}{\partial q}\{\dot{q}^T G(q) \dot{q}\}$. Since the $m \times m$ matrix $\hat{Y}(q)^T G(q) \hat{Y}(q)$ is non-singular at any point $q \in Q$, then we can solve (17) for $\ddot{q}$ as follows

$$\begin{aligned} \ddot{q}_1 = &-\{\hat{Y}(q)^T G(q) \hat{Y}(q)\}^{-1} \hat{Y}(q)^T \{G(q)\dot{\hat{X}}(q) \\ &+ G(q)\dot{\hat{Y}}(q)\dot{q}_1 + \Gamma(q, \hat{X}(q) + \hat{Y}(q)\dot{q}_1) + \nabla U(q)\} \\ &+ \{\hat{Y}(q)^T G(q) \hat{Y}(q)\}^{-1} \hat{Y}(q)^T E(q)u. \end{aligned} \tag{18}$$

Now, we define a new vector and matrix as $\alpha(q) := \{\hat{Y}(q)^T G(q) \hat{Y}(q)\}^{-1} \hat{Y}(q)^T$, $\beta(q, \dot{q}_1) := G(q)\dot{\hat{X}}(q) + G(q)\dot{\hat{Y}}(q)\dot{q}_1 + \Gamma(q, \hat{X}(q) + \hat{Y}(q)\dot{q}_1) + \nabla U(q)$ and set new state variables as $z_1 := q_1 \in \boldsymbol{R}^m, z_2 := q_2 \in \boldsymbol{R}^{n-m}, z_3 := \dot{q}_1 \in \boldsymbol{R}^m$. From (15) and (18), we can obtain the NDSAC:

$$\begin{aligned} \dot{z}_1 &= z_3 \\ \dot{z}_2 &= -\bar{X}(z_1, z_2) - \bar{Y}(z_1, z_2) z_3 \\ \dot{z}_3 &= -\alpha(z_1, z_2)\beta(z_1, z_2, z_3) + \alpha(z_1, z_2) E(z_1, z_2) u, \end{aligned} \tag{19}$$

where $\bar{X}(z_1, z_2) := B_2(z_1, z_2)^{-1} A(z_1, z_2)$, $\bar{Y}(z_1, z_2) := B_2(z_1, z_2)^{-1} B_1(z_1, z_2)$.

We next transform the NDSAC (19) using a state feedback. Let an m-dimensional column vector $v = [\, v_1 \, \cdots \, v_m \,]^T \in \boldsymbol{R}^m$ be new control input variables and consider the following transformation of control input variables:

$$\begin{aligned} u = \{\alpha(z_1, z_2) E(z_1, z_2)\}^{-1} \alpha(z_1, z_2) \beta(z_1, z_2, z_3) \\ + \{\alpha(z_1, z_2) E(z_1, z_2)\}^{-1} v. \end{aligned} \tag{20}$$

Using (20), the NDSAC (19) can be transformed into

$$\underbrace{\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} z_3 \\ -\bar{X}(z_1, z_2) - \bar{Y}(z_1, z_2) z_3 \\ 0 \end{bmatrix}}_{f(z)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ e_i \end{bmatrix}}_{g_i} v_i, \tag{21}$$

where $z = [\, z_1^T \, z_2^T \, z_3^T \,]^T \in \boldsymbol{R}^{n+m}$ and e_i is an m-dimensional column vector that the i-th element is 1 and the others are all 0. We shall call (21) *the normal form of NDSAC*. Here we define two subsets of $\boldsymbol{R}^{n+m}$ as follows. The set of points such that configuration are arbitrary and velocities are zero of (19) is denoted by $V^* := \{ z \in \boldsymbol{R}^{n+m} \mid z_3 = 0 \}$, and we call them *zero velocity points*. Next, the set of equilibria of (21) is denoted by $V^e := \{ z \in \boldsymbol{R}^{n+m} \mid A(z_1, z_2) = 0, \, z_3 = 0 \}$.

Finally, the linear approximation of the normal form of the NDSAC at an equilibrium $z^e = [\, q^{e\,T} \; 0^T \,]^T \in V^e$ is represented by

$$\dot{z} = \underbrace{\begin{bmatrix} \dfrac{\partial \hat{X}}{\partial q}(q^e) & \hat{Y}(q^e) \\ O_{m,n} & O_m \end{bmatrix}}_{\mathcal{A}} (z - z^e) + \underbrace{\begin{bmatrix} O_{n,m} \\ I_m \end{bmatrix}}_{\mathcal{B}} v. \tag{22}$$

IV. NONLINEAR CONTROL ANALYSIS

A. Accessibility

We investigate strong local accessibility of the NDSAC in this subsection. If $\Lambda^V(t, z^0)$, which is the accessible set from an initial point z_0 at any time t, contains a non-empty open set of $\boldsymbol{R}^{n+m}$ for all neighborhood V of z^0, then the NDSAC (19) is called to be *strongly locally accessible from* z^0. We can show strong local accessibility of the NDSAC (19) as the following theorem (See also [9]).

Theorem 2 : The NDSAC (19) is strongly locally accessible at any zero-velocity point $z^* \in V^*$.

Proof. By calculating iterated Lie brackets which consist of $f(z)$ and g_i in (21) in detail, we can prove this theorem. ∎

Bloch et al. [1] have shown that the nonholonomic dynamic system with linear constraints is strongly locally accessible at any equilibrium. Theorem 2 guarantees that strong local accessibility is conserved in expanding the class of constraints from linear to affine.

B. Controllability

This subsection is concerned with analysis of local controllability of the NDSAC. If a nonlinear system is locally accessible and $\Lambda^V(t, z^0)$ contains the initial point z^0, then the system is called *locally controllable at* z^0. In general, there are two methods to analyze local controllability of nonlinear control systems. One is based on Sussmann's theorem [13] by calculating iterated Lie brackets. The other is based on the linear approximation of nonlinear control systems. These methods provide us sufficient conditions for local controllability. We here consider above both methods. We first take Sussmann's theorem approach. Applying Sussmann's theorem to the normal form of the NDSAC (21), we derive a sufficient condition for local controllability as the following theorem.

Theorem 3: If iterated Lie brackets which consist of $\hat{X}, \hat{Y}_1, \cdots, \hat{Y}_m$: $\mathrm{ad}_{\hat{X}}^k \hat{Y}_i$ $(i = 1, \cdots, m;\ k = 0, 1, \cdots)$ span n-dimensional at an equilibrium point $q^e \in U^e$, then the NDSAC (19) is locally controllable at an equilibrium point $z^e \in V^e$.

Proof. Calculating iterated Lie brackets which consist of f and g_i of (21), then we obtain

$$[g_i, g_j] = 0$$
$$\mathrm{ad}_f^{k+1} g_i(z^e) = \begin{bmatrix} -\mathrm{ad}_{\hat{X}}^k \hat{Y}_i(q^e) \\ 0 \end{bmatrix} \ (k = 0, 1, \cdots).$$

Consequently, from Sussmann's theorem [13], the proof is completed. ∎

We next take linear approximation approach. It is known that if the linear approximation of a nonlinear system at an equilibrium is controllable, then the nonlinear system is locally controllable at the equilibrium. Now we show the following proposition for controllability of the linear approximation of the NDSAC.

Theorem 4: The linear approximation of the normal form of the NDSAC at an equilibrium $z^e \in V^e$ (22) is controllable if and only if the matrix defined by

$$\mathcal{V} := \Big[\frac{\partial A}{\partial q}(q^e)\hat{Y}(q^e) \ \ \frac{\partial A}{\partial q_2}(q^e)B_2(q^e)^{-1}\frac{\partial A}{\partial q}(q^e)\hat{Y}(q^e) \ \cdots \ \Big\{ \frac{\partial A}{\partial q_2}(q^e)B_2(q^e)^{-1} \Big\}^{n-3} \frac{\partial A}{\partial q}(q^e)\hat{Y}(q^e) \Big] \quad (23)$$

has row full-rank, that is, $\mathrm{rank}\, \mathcal{V} = n - m$ holds.

Proof. The necessary and sufficient condition of controllability of the linear approximation (22) is that rank of the controllability matrix $\mathcal{W} := [\, \mathcal{B}\ \mathcal{AB}\ \cdots\ \mathcal{A}^{n-1}\mathcal{B}\,]$ is equal to n. By calculating $\mathcal{W}$ in detail, we can complete the proof. ∎

From Theorem 4, the following can be derived in the cases of completely linear and partially affine constraints.

Corollary 1: In the case of completely linear and partially affine constraints, the linear approximation of the normal form of the NDSAC at any equilibrium $z^e \in V^e$ (22) is uncontrollable.

Proof. If the affine constraints are completely linear or partially affine, then $0 \le r \le n - m - 1$ holds for their affine index (5). Then, rank of $\mathcal{V}$ (23) is smaller than $n - m$ and then we can see from Theorem 4 that the linear approximation at any equilibrium z^e (22) is uncontrollable. ∎

In the cases of completely linear constraints and partially affine constraints, the linear approximation is uncontrollable, and then we have to adopt Theorem 3 to check the local controllability. The NDSAC is locally controllable in the case of completely linear constraints [1] . The following corollary can be derived in the case of completely affine constraints from Theorem 4.

Corollary 2: In the case of completely affine constraints and $n \le 2m$, if the affine index at an equilibrium $q^e \in U^e$ is $n - m$ and

$$\mathrm{rank}\, \frac{\partial A}{\partial q}(q^e)\hat{Y}(q^e) = n - m \quad (24)$$

holds, then the linear approximation of the normal form of the NDSAC at an equilibrium $z^e \in V^e$ (22) is controllable. Therefore, the NDSAC (19) is also locally controllable at the equilibrium z^e. ∎

If $n \le 2m$ or (24) does not hold in case of completely affine constraints, we cannot check the local controllability of the NDSAC by linear approximation approach, and then we have to rely on Sussmann's theorem approach of Theorem 3.

C. Stabilizability

In the previous subsection, we have considered local controllability of the NDSAC. Generally, there exists a gap between controllability and stabilizability in nonlinear control systems. In this subsection, we investigate local asymptotic stabilizability of the NDSAC to equilibria. We first consider stabilizability of the NDSAC by linear state feedback. It is known that if the linear approximation of a nonlinear system at an equilibrium is controllable or all its uncontrollable modes are stable, then the nonlinear system is locally asymptotically stabilizable to the equilibrium. by linear state feedback. In the cases of completely linear constraints and partially affine constraints, if uncontrollable modes of the linear approximation of the normal form of the NDSAC are stable, then the NDSAC is locally asymptotically stabilizable by linear state feedback. We can derive the following result from Corollary 2 in the case of completely affine constraints.

Corollary 3: In case of completely affine constraints and $n \le 2m$, if the affine index at an equilibrium $q^e \in U^e$ is $n-m$ and (24) holds, then the NDSAC (19) is locally asymptotically stabilizable to any equilibrium $z^e \in V^e$ by linear state feedback. ∎

We next consider stabilizability of the NDSAC by nonlinear smooth state feedback, which are a larger class than linear state feedback. The necessary condition of locally asymptotic stabilizability by nonlinear smooth state feedback can be derived as the following theorem.

Theorem 5: If the NDSAC (19) is locally asymptotically stabilizable to an equilibrium $z^e \in V^e$ by nonlinear smooth state feedback, then the affine index at q^e is $n-m$.

Proof. Consider $A(q)$ as a map $A : U \to \boldsymbol{R}^{n-m}$, where U is an open set of Q. By the implicit function theorem, if the affine index at q^e is $n-m$, then there exists a diffeomorphism $\sigma : V \to W$ such that

$$\begin{aligned} &A \circ \sigma^{-1}(q_1, \cdots, q_m, q_{m+1}, \cdots, q_n) \\ &\qquad = (q_{m+1}, \cdots, q_n) + A(q^e), \ \ q \in W \end{aligned}$$

and $\sigma(q^e) = 0$, where V $(\subset U)$ is an open neighborhood of q^e in Q and W is an open neighborhood of 0 in $\boldsymbol{R}^n$. Now $A(q^e) = 0$, we have

$$\sigma \circ A^{-1}(q_{m+1}, \cdots, q_n) = (q_1, \cdots, q_m, q_{m+1}, \cdots, q_n).$$

Therefore, the subset of Q defined by

$$\begin{aligned} M :&= \sigma \circ A^{-1}(A(q^e)) = \sigma \circ A^{-1}(0) \\ &= (q_1, \cdots, q_m, 0, \cdots, 0) \end{aligned}$$

can be parameterized by m variables, and hence M is an m-dimensional submanifold of Q. On the other hand, it is known that if a nonlinear control system is locally asymptotically stabilizable, then the dimension of equilibria set is equal to the number of control inputs [2], [14]. Consequently, both the dimension of M and the number of inputs are m, this proves the theorem. ∎

In view of Theorem 5, the following can be derived in the cases of completely linear and partially affine constraints.

Corollary 4: In the cases of completely linear and partially affine constraints, the NDSAC (19) is not locally asymptotically stabilizable to any equilibrium $z^e \in V^e$ by any nonlinear smooth state feedback.

Proof. In this case, the affine index at any equilibrium q^e is smaller than $n-m$. Hence from Theorem 5, the proof is completed. ∎

We can see from Corollary 4 that the NDSAC is not locally asymptotically stabilizable by any nonlinear smooth state feedback in not only the completely linear constraints case but also the partially affine constraints case. On the other hand in case of completely affine constraints, the NDSAC has a possibility of locally asymptotic stabilizability by nonlinear smooth state feedback even though Corollary 3 does not hold.

V. PHYSICAL EXAMPLES

A. A Coin on a Rotating Table

We here consider *a coin on a rotating table* as shown in Fig. 1. Set the xy-coordinate whose origin corresponds to the center of rotation of the table with the angular rate Ω. (x, y) denotes the point that the coin contacts with the table and θ and ϕ denote the heading angle and self-rotation angle of the coin, respectively. Let R be the radius of the coin and M be the mass of the coin. Moreover, let be J_1 and J_2 be the moment of inertia of the coin in directions of θ and ϕ, respectively. The generalized configuration coordinate of the system is denoted by $q = [x \ \ y \ \ \theta \ \ \phi]^T \in SE(2) \times \boldsymbol{S}$ with $n = 4$.

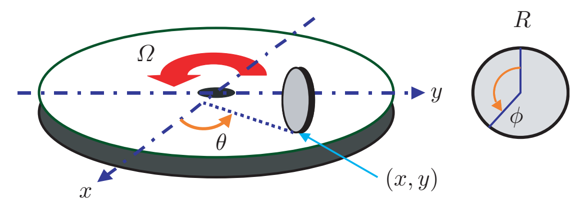

Fig. 1 : A Coin on a Rotating Table

Lagrangian of the system is given by

$$L(q, \dot{q}) = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J_1\dot{\theta}^2 + \frac{1}{2}J_2\dot{\phi}^2. \tag{25}$$

Considering equilibrium of the velocities in the heading and side directions of the coin, we have affine constraints of the system:

$$\underbrace{\begin{bmatrix} 0 \\ \Omega(y\cos\theta - x\sin\theta) \end{bmatrix}}_{A(q)} + \underbrace{\begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & R \end{bmatrix}}_{B(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = 0, \tag{26}$$

where $m = 2$. Therefore, the equilibria set is given by $U^e = \{q \in Q \mid y\cos\theta - x\sin\theta = 0\}$. We can find that the affine constraints (26) are completely nonholonomic by calculating C_0. Since the affine index at any equilibrium $q^e \in U^e$ is $r(q^e) = 1 < n - m = 2$, then affine constraints (26) are first order partially affine constraints. We now partition the generalized configuration coordinate of the system into $q_1 := [\theta \ \ \phi]^T$, $q_2 := [x \ \ y]^T$ and we have

$$\hat{X} := \begin{bmatrix} 0 \\ 0 \\ \Omega\cos\theta(x\sin\theta - y\cos\theta) \\ \Omega\sin\theta(x\sin\theta - y\cos\theta) \end{bmatrix}, \ \hat{Y} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -R\cos\theta \\ 0 & -R\sin\theta \end{bmatrix}.$$

We assume that we can control the torques in directions of θ and ϕ and denote them u_1 and u_2, respectively. Therefore, $E(q)$ and u are given by

$$E(q) := \begin{bmatrix} I_2 \\ O_2 \end{bmatrix}, \ u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Firstly, we can see from Theorem 2 that the NDSAC of the system is strongly locally accessible at any zero velocity point, since the affine constraints of the system (26) are completely nonholonomic. Next, it is seen from Corollary

1 that the linear approximated system at any equilibrium is uncontrollable due to partially affine constraints. However, we can prove that the NDSAC of the system is locally controllable at any equilibrium using Theorem 3. Finally, from Corollary 4, the NDSAC is not locally asymptotically stabilizable by any nonlinear smooth state feedback.

B. A Ball on a Rotating Table

We next consider *a ball on a rotating table* as depicted in Fig. 2. Set the xy-coordinate whose origin corresponds to the center of rotation of the table with the angular rate Ω. (x, y) denotes the point that the ball contacts with the table and (θ, ϕ, ψ) denotes the Eulerian angles of the ball. Let R be the radius of the ball and J be the moment of inertia of the ball. The generalized configuration coordinate of the system is denoted by $q = [x \ y \ \theta \ \phi \ \psi]^T \in \boldsymbol{R}^2 \times SO(3)$ with $n = 5$.

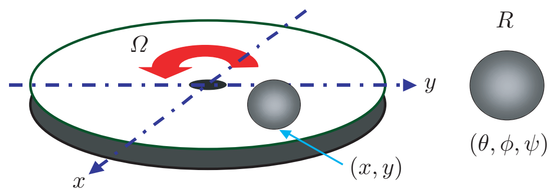

Fig. 2 : A Ball on a Rotating Table

Lagrangian of the system is given by

$$L(q, \dot{q}) = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2). \quad (27)$$

Considering equilibration of velocities in the x and y directions of the ball, we obtain affine constraints of the system as follows.

$$\underbrace{\begin{bmatrix} \Omega\, y \\ -\Omega\, x \end{bmatrix}}_{A(q)} + \underbrace{\begin{bmatrix} 1 & 0 & -R\sin\psi & R\sin\theta\cos\psi & 0 \\ 0 & 1 & R\cos\psi & R\sin\theta\sin\psi & 0 \end{bmatrix}}_{B(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = 0, \quad (28)$$

where $m = 2$. Therefore, the equilibria set is represented by $U^e = \{q \in Q \mid x = y = 0\}$. We can find that the affine constraints (28) are completely nonholonomic by calculating C_0. Since the affine index at any equilibrium $q^e \in U^e$ is $r(q^e) = 2 = n - m$, then the affine constraints (28) are completely affine constraints. We now partition the generalized configuration coordinate of the system into $q_1 := [\theta \ \phi \ \psi]^T$, $q_2 := [x \ y]^T$ and we have

$$\hat{X} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\Omega y \\ \Omega x \end{bmatrix}, \ \hat{Y} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ R\sin\psi & -R\sin\theta\cos\psi & 0 \\ -R\cos\psi & -R\sin\theta\sin\psi & 0 \end{bmatrix}.$$

We assume that we can control the torques in directions of θ, ϕ and ψ and denote them u_1, u_2 and u_3, respectively. Therefore, $E(q)$ and u are given by

$$E(q) := \begin{bmatrix} I_3 \\ O_{2,3} \end{bmatrix}, \ u := \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Firstly, we can confirm from Theorem 2 that the NDSAC of the system is strongly locally accessible at any zero velocity point, since the affine constraints of the system (28) are completely nonholonomic. Next, we can see from Corollary 1 that the condition for controllability of the linear approximated system (24) hold, and then the NDSAC of the system is locally controllable at any equilibrium. Finally, the NDSAC is locally asymptotically stabilizable by a linear state feedback from Corollary 3. This example has entirely different properties on locally asymptotic stabilizability from the example of a coin in Subsection V-A.

VI. Conclusion

In this paper, we have introduced and analyzed a class of nonholonomic dynamic systems with affine constraints (NDSAC) based on nonlinear control theory, which have never been discussed so far. As a result, we have shown that there exists a class of systems whose linear approximations are controllable and that are then locally asymptotically stabilizable by linear state feedback. Moreover, we have found that there exists a class of systems that are locally asymptotically stabilizable by nonlinear smooth state feedback. These properties are beyond well-known facts for nonholonomic dynamic systems with linear constraints till now as is the case with results of kinematic model case [8]. In this sense, a new class of control problems for affine constraints has been proposed in this paper.

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