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Generalized preview and delayed H^∞ control: output feedback case

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***Abstract*— A generalized H^∞ control problem, which covers preview and delayed control strategies, is discussed in the output feedback setting. By introducing analytic solutions to the corresponding control/filtering operator Riccati equations, the solvability is clarified based on the fundamental solutions to ordinary differential equations. The solutions obtained here enable us to deal with important class of H^∞ control problems, which include multiple input/output delays and the preview tracking strategies.**

I. INTRODUCTION

H^∞ control problem has been studied for a broader class of infinite-dimensional systems [1], [11] and, especially for a class of time delay systems, explicit solutions are clarified based on various approaches [2], [6], [10]. Recently, by applying the approaches for delay systems, the effect of preview action is further investigated in terms of the H^∞ performance [3], [4], [5], [7].

In the state-space approach for infinite-dimensional systems, the abstract system theory has been discussed for a class of systems (Pritchard-Salamon systems, e.g. [8], [9]) and, if the plant is in this class, the typical control problems such as H_2 (LQ) and H^∞ control/filtering problems are characterized via corresponding operator representations [11]. These fundamental frameworks enable us to deal with the broader class of control problems beyond the apparent system representation and, further, have a potential to provide an insight on the underlying property of resulting systems.

In this paper, we focus on a generalized H^∞ control problem, which covers preview and delayed control strategies, and derive explicit formulas in the output feedback setting. The state-space approach employed here also provides a base to solve typical control/filtering problems, which are characterized with the operator Riccati equations.

The paper is organized as follows. In Section 2, the generalized H^∞ control problem is defined and an abstract result with corresponding operator Riccati equations [11] is stated. In Section 3, the calculation method of control/filtering operator Riccati equations is established and the solution to the H^∞ control problem is provided. A note on the calculation of H^∞ control law is summarized based on the feature of analytic solutions to the operator Riccati equations.

II. FORMULATION AND PRELIMINARIES

Define a generalized plant with multiple delays in the control/disturbance and the measurement/regulated output:

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{d} \left(B_{1i}w_i(t-h_i) + B_{2i}u_i(t-h_i)\right)$$

$$\Sigma : z(t) = \begin{bmatrix} C_{10}x(t-\check{h}_0) \\ C_{11}x(t-\check{h}_1) \\ \vdots \\ C_{1\ell}x(t-\check{h}_\ell) \end{bmatrix} + D_{12}u(t) \qquad (1)$$

$$y(t) = \begin{bmatrix} C_{20}x(t-\check{h}_0) \\ C_{21}x(t-\check{h}_1) \\ \vdots \\ C_{2\ell}x(t-\check{h}_\ell) \end{bmatrix} + D_{21}w(t)$$

$$w(t) := \begin{bmatrix} w_0(t) \\ w_1(t) \\ \vdots \\ w_d(t) \end{bmatrix} \in \mathbb{R}^{m_1}, u(t) := \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_d(t) \end{bmatrix} \in \mathbb{R}^{m_2},$$

$$w_i(t) \in \mathbb{R}^{m_{1i}}, u_i(t) \in \mathbb{R}^{m_{2i}} \quad (i = 0, 1, \cdots, d)$$
$$x(t) \in \mathbb{R}^n, \ z(t) \in \mathbb{R}^{p_1}, \ y(t) \in \mathbb{R}^{p_2}$$
$$B := [\,B_{1\bullet} \ \ B_{2\bullet}]$$
$$B_{1\bullet} := [\,B_{10}\ B_{11} \ \cdots \ B_{1d}],\ B_{2\bullet} := [\,B_{20}\ B_{21} \ \cdots \ B_{2d}]$$
$$C^{\mathrm{T}} := [\,C_{1\bullet}^{\mathrm{T}} \ \ C_{2\bullet}^{\mathrm{T}}]$$
$$C_{1\bullet}^{\mathrm{T}} := [\,C_{10}^{\mathrm{T}}\ C_{11}^{\mathrm{T}} \ \cdots \ C_{1\ell}^{\mathrm{T}}],\ C_{2\bullet}^{\mathrm{T}} := [\,C_{20}^{\mathrm{T}}\ C_{21}^{\mathrm{T}} \ \cdots \ C_{2\ell}^{\mathrm{T}}]$$
$$C_{1i} \in \mathbb{R}^{p_{1i}\times n}, C_{2i} \in \mathbb{R}^{p_{2i}\times n}(i = 0, 1, \cdots, \ell)$$

where x,w,u,z,y are the state, the disturbance, the control input, the regulated output, and the measurement of the system respectively. The matrices A,B,C,D_{12},D_{21} are with appropriate dimensions and h_i $(i = 0, 1, \cdots, d)$, $\check{h}_j$ $(j = 0, 1, \cdots, \ell)$ denote time delays in the increasing order: $0 =: h_0 \le h_1 \le h_2 \le \cdots \le h_d =: L$, $0 =: \check{h}_0 \le \check{h}_1 \le \check{h}_2 \le \cdots \le \check{h}_\ell =: \check{L}$. We make following assumptions for the system Σ.

(H1) $(C_{2\bullet},\ A,\ B_{2\bullet})$ is detectable and stabilizable.

(H2) $D_{12}^{\mathrm{T}}[\,C_{1\bullet}\ D_{12}\,] = [\,0\ I\,]$, $\begin{bmatrix} B_{1\bullet} \\ D_{21} \end{bmatrix} D_{21}^{\mathrm{T}} = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

(H3) $\mathrm{rank} \begin{bmatrix} A - j\omega I & B_{2\bullet} \\ C_{1\bullet} & D_{12} \end{bmatrix} = n + m_1$,

$\mathrm{rank} \begin{bmatrix} A - j\omega I & B_{1\bullet} \\ C_{2\bullet} & D_{21} \end{bmatrix} = n + p_1, \quad \forall \omega \in \mathbb{R}$.

The H^∞ control problem is to design a feedback control law such that the resulting closed loop system satisfies the following conditions (C1),(C2):

(C1) The closed loop system is internally stable, and

(C2) the resulting closed loop system Σ_{zw} from the disturbance w to the regulated output z satisfies $\|\Sigma_{zw}\|_\infty < \gamma$ for a given constant $\gamma > 0$.

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In this paper, we will clarify the solvability of the H^∞ control problem Σ. The generalized plant (1) covers a broader class of H^∞ control problems in the output feedback setting and enables us to deal with the preview and delayed control action. The time delays in the disturbance w equivalently describe previewable reference signals and those in the control/measurement define H^∞ control problems for multiple input/output delay systems. Typical control problems are illustrated by Example 1-2.

Example 1 (H^∞ preview control): Define Σ with $B_{1\bullet} = [B_{10}, B_{11}]$, $B_{2\bullet} = [B_{20}, 0]$, $0 = h_0 < h_1 = L$, $C_{1\bullet} = C_{10}$, $C_{2\bullet} = C_{20}$ ($\check{L} = 0$), and describe the uncertainty and the previewable reference signals by $B_{10}w_0(t)$, $B_{11}w_1(t-L)$ respectively. Rewriting the reference signal by $w_1(t) =: r_1(t+L)$, the problem is equivalently given as follows.

$$\Sigma^{\text{prev}}: \begin{array}{rcl} \dot{x}(t) &=& Ax(t) + B_{10}w_0(t) + B_{11}r_1(t) + B_{20}u(t) \\ z(t) &=& C_{1\bullet}x(t) + D_{12}u(t) \\ y(t) &=& C_{2\bullet}x(t) + D_{21}\begin{bmatrix} w_0(t) \\ r_1(t+L) \end{bmatrix} \end{array} \quad (2)$$

As the future reference signal $r_1(t+L)$ is included in the measurement $y(t)$, the preview control problem is formulated in (1). Preliminary results are reported by [3], which solve the full-information problem. ■

Example 2 (H^∞ control with input/output delays): The H^∞ control problem for multiple input/output delay systems is defined with $B_{1\bullet} = [B_{10}, 0, \cdots, 0]$, $B_{2\bullet} = [B_{20}, B_{21}, \cdots, B_{2d}]$, $C_{1\bullet} = [C_{10}^{\mathrm{T}}, 0, \cdots, 0]^{\mathrm{T}}$, $C_{2\bullet} = [C_{20}^{\mathrm{T}}, C_{21}^{\mathrm{T}}, \cdots, C_{2d}^{\mathrm{T}}]^{\mathrm{T}}$. It broadens the class of problems, where analytic solutions are clarified. ■

The relation to filtering problems or LQ (H_2) control problems are mentioned after Theorem 1, which states the abstract state-space approach for H^∞ control problems.

In order to solve the problem Σ, we will prepare an abstract system description on appropriate function space. Introduce a Hilbert space $\mathcal{X} := \mathbb{R}^n \times L_2(-L, 0; \mathbb{R}^m) \times L_2(-\check{L}, 0; \mathbb{R}^p)$ endowed with the inner product:

$$\langle \psi, \phi \rangle := \psi^{0\mathrm{T}}\phi^0 + \int_{-L}^{0} \psi^{1\mathrm{T}}(\beta)\phi^1(\beta)\,d\beta + \int_{-\check{L}}^{0} \psi^{2\mathrm{T}}(\beta)\phi^2(\beta)\,d\beta \quad (3)$$

$$\psi = (\psi^0, \psi^1, \psi^2) \in \mathcal{X}, \quad \phi = (\phi^0, \phi^1, \phi^2) \in \mathcal{X},$$

and describe the generalized plant Σ by the following evolution equation [8].

$$\hat{\Sigma}: \begin{array}{rcl} \dot{\hat{x}}(t) &=& \mathcal{A}\hat{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t) \\ z(t) &=& \mathcal{C}_1\hat{x}(t) + D_{12}u(t) \\ \hat{y}(t) &=& \mathcal{C}_2\hat{x}(t) + D_{21}w(t) \end{array} \quad (4)$$

The operator $\mathcal{A}$ is an infinitesimal generator defined by

$$\mathcal{A}\phi = \begin{bmatrix} A\phi^0 + B\phi^1(-L) \\ \phi^{1\prime} \\ \phi^{2\prime} \end{bmatrix}, \quad (5)$$

$$\mathrm{D}(\mathcal{A}) = \{\phi \in \mathcal{X} : \phi^1 \in W^{1,2}(-L, 0; \mathbb{R}^m), \phi^2 \in W^{1,2}(-\check{L}, 0; \mathbb{R}^p), \phi^1(0) = 0, \phi^2(0) = C\phi^0\}$$

where $W^{1,2}(-L, 0; \mathbb{R}^m)$ denotes the Sobolev space of $\mathbb{R}^m$-valued, absolutely continuous functions with square integrable derivatives on $[-L, 0]$. For the subspaces $\mathcal{W} := \{\phi \in \mathcal{X} : \phi^2 \in W^{1,2}(-\check{L}, 0; \mathbb{R}^p), \phi^2(0) = C\phi^0\}$, $\mathcal{V}^* := \{\psi \in \mathcal{X} : \psi^1 \in W^{1,2}(-L, 0; \mathbb{R}^m), \psi^1(-L) = B^{\mathrm{T}}\psi^0\}$, it is shown that $\mathcal{W} = \mathrm{D}_{\mathcal{V}}(\mathcal{A})$ holds and the subspaces $\mathcal{W}$, $\mathcal{X}$, $\mathcal{V}$ are with continuous, dense injections satisfying $\mathcal{W} \subset \mathcal{X} \subset \mathcal{V}$ ([8] Remark 2.6).

Denoting the components of the product space $\phi \in \mathcal{X}$ by the following manner:

$$\begin{array}{rcl} \phi &=& (\phi^0, \phi^1, \phi^2) \in \mathcal{X} \\ \phi^0 &\in& \mathbb{R}^n \\ \phi^1 &=& (\phi^{11}, \phi^{12}) \in L_2(-L, 0; \mathbb{R}^m) \\ && \phi^{11} = (\phi_0^{11}, \phi_1^{11}, \cdots, \phi_d^{11}) \\ && \phi^{12} = (\phi_0^{12}, \phi_1^{12}, \cdots, \phi_d^{12}) \\ && \phi_i^{11} \in L_2(-L, 0; \mathbb{R}^{m_{1i}}) \\ && \phi_i^{12} \in L_2(-L, 0; \mathbb{R}^{m_{2i}}) \ \ (i = 0, 1, \cdots, d) \\ \phi^2 &=& (\phi^{21}, \phi^{22}) \in L_2(-\check{L}, 0; \mathbb{R}^p) \\ && \phi^{21} = (\phi_0^{21}, \phi_1^{21}, \cdots, \phi_\ell^{21}) \\ && \phi^{22} = (\phi_0^{22}, \phi_1^{22}, \cdots, \phi_\ell^{22}) \\ && \phi_i^{21} \in L_2(-\check{L}, 0; \mathbb{R}^{p_{1i}}) \\ && \phi_i^{22} \in L_2(-\check{L}, 0; \mathbb{R}^{p_{2i}}) \ \ (i = 0, 1, \cdots, \ell) \end{array} \quad (6)$$

input/output operators $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{C}_1$, $\mathcal{C}_2$ are defined by

$$\mathcal{B}_k : \mathbb{R}^{m_k} \to \mathcal{V}, \ \mathcal{B}_k^*\phi = \begin{bmatrix} \phi_0^{1k}(-L + h_0) \\ \phi_1^{1k}(-L + h_1) \\ \vdots \\ \phi_d^{1k}(-L + h_d) \end{bmatrix}, \ \phi \in \mathcal{V}^* \quad (7)$$

$$\mathcal{C}_k : \mathcal{W} \to \mathbb{R}^{p_k}, \ \mathcal{C}_k\phi = \begin{bmatrix} \phi_0^{2k}(-\check{h}_0) \\ \phi_1^{2k}(-\check{h}_1) \\ \vdots \\ \phi_\ell^{2k}(-\check{h}_\ell) \end{bmatrix}, \ \phi \in \mathcal{W} \quad (8)$$

where $k = 1, 2$.

For the system $\hat{\Sigma}$, the solvability and the solution (control law) to the H^∞ control problem are formally characterized based on abstract operator Riccati equations.

Theorem 1 ([11], Theorem 5.4): The H^∞ control problem $\hat{\Sigma}$ is solvable iff (A),(B),(C) are satisfied.

(A) The operator Riccati equation

$$\mathcal{S}\mathcal{A}\phi + \mathcal{A}^*\mathcal{S}\phi + \tfrac{1}{\gamma^2}\cdot\mathcal{S}\mathcal{B}_1\mathcal{B}_1^*\mathcal{S}\phi - \mathcal{S}\mathcal{B}_2\mathcal{B}_2^*\mathcal{S}\phi + \mathcal{C}_1^*\mathcal{C}_1\phi = 0, \quad \phi \in \mathcal{W} \quad (9)$$

has a stabilizing solution $\mathcal{S} \geq 0$ ($\mathcal{S} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$) such that $\mathcal{A} + \frac{1}{\gamma^2}\cdot\mathcal{B}_1\mathcal{B}_1^*\mathcal{S} - \mathcal{B}_2\mathcal{B}_2^*\mathcal{S}$ generates exponentially stable semigroups on $\mathcal{W}$ and $\mathcal{V}$.

(B) The operator Riccati equation

$$\mathcal{A}\mathcal{P}\psi + \mathcal{P}\mathcal{A}^*\psi + \tfrac{1}{\gamma^2}\cdot\mathcal{P}\mathcal{C}_1^*\mathcal{C}_1\mathcal{S}\psi - \mathcal{P}\mathcal{C}_2^*\mathcal{C}_2\mathcal{P}\psi + \mathcal{B}_1\mathcal{B}_1^*\psi = 0, \quad \psi \in \mathcal{V}^* \quad (10)$$

has a stabilizing solution $\mathcal{P} \geq 0$ ($\mathcal{P} \in \mathcal{L}(\mathcal{W}^*, \mathcal{W})$) such that $\mathcal{A} + \frac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{C}_1^*\mathcal{C}_1 - \mathcal{P}\mathcal{C}_2^*\mathcal{C}_2$ generates exponentially stable semigroups on $\mathcal{W}$ and $\mathcal{V}$.

(C) The stabilizing solutions defined by (11),(12) satisfy

$$\lambda_{\max}(\mathcal{P}\mathcal{S}) < \gamma^2. \tag{11}$$

If (A),(B),(C) hold, an H^∞ control law is given as follows.

$$\dot{\hat{\underline{x}}}(t) = \mathcal{A}\hat{\underline{x}} + \mathcal{B}_2 u(t) + \frac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{C}_1^*\mathcal{C}_1\hat{\underline{x}}, + \mathcal{P}\mathcal{C}_2^*(y(t) - \mathcal{C}_2\hat{\underline{x}})$$

$$u(t) = -\mathcal{B}_2^*\mathcal{Q}\hat{\underline{x}}, \quad \mathcal{Q} := \mathcal{S}\left(\mathcal{I} - \frac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{S}\right)^{-1} \tag{12}$$

■

Remark 2: The condition (A) characterizes the solvability of full-information (FI) problem $(y(t) = [\, x^{\mathrm{T}}(t), w^{\mathrm{T}}(t)\,]^{\mathrm{T}})$ and provides interpretations on the related H_2 (LQ) control problem. It is also noted that the solution to (B) enables us to solve filtering problems, which clarify the effect of preview/delayed estimation of the measurement. A related problem is discussed in [7]. ■

In the sequel, we explore the analytic solution to (9),(10) and clarify the solvability of H^∞ control problem Σ in the framework of finite-dimensional operations. A note for the calculation of the control law (12) is also provided based on the property of analytic solutions. For the preliminary to solve the filtering operator Riccati equation (10), we define a transposed system of (1):

$$\dot{\tilde{x}}(t) = A^{\mathrm{T}}\tilde{x}(t) + \sum_{i=0}^{\ell} \left(C_{1i}^{\mathrm{T}}\tilde{w}_i(t - \check{h}_i) + C_{2i}^{\mathrm{T}}\tilde{u}_i(t - h_i)\right)$$

$$\Sigma^{\mathrm{T}}: \tilde{z}(t) = \begin{bmatrix} B_{10}^{\mathrm{T}}\tilde{x}(t - h_0) \\ B_{11}^{\mathrm{T}}\tilde{x}(t - h_1) \\ \vdots \\ B_{1d}\tilde{x}(t - h_d) \end{bmatrix} + D_{21}^{\mathrm{T}}\tilde{u}(t) \tag{13}$$

$$\tilde{y}(t) = \begin{bmatrix} B_{20}^{\mathrm{T}}\tilde{x}(t - h_0) \\ B_{21}^{\mathrm{T}}\tilde{x}(t - h_1) \\ \vdots \\ B_{2d}^{\mathrm{T}}\tilde{x}(t - h_d) \end{bmatrix} + D_{12}^{\mathrm{T}}\tilde{w}(t)$$

which is employed to explore the duality of $\hat{\Sigma}$.

Introduce a Hilbert space $\mathcal{X}^{\mathrm{T}} := \mathbb{R}^n \times L_2(-\check{L}, 0;\, \mathbb{R}^p) \times L_2(-L, 0;\, \mathbb{R}^m)$ for Σ^{T}, the evolution equation is given by

$$\hat{\Sigma}^{\mathrm{T}}: \begin{array}{rcl} \dot{\hat{x}}(t) &=& \mathcal{A}^{\mathrm{T}}\hat{x}(t) + \mathcal{C}_1^{\mathrm{T}}w(t) + \mathcal{C}_2^{\mathrm{T}}u(t) \\ z(t) &=& \mathcal{B}_1^{\mathrm{T}}\hat{x}(t) \qquad\qquad + D_{21}^{\mathrm{T}}u(t) \\ \hat{y}(t) &=& \mathcal{B}_2^{\mathrm{T}}\hat{x}(t) + D_{12}^{\mathrm{T}}w(t) \end{array} \tag{14}$$

where $\mathcal{A}^{\mathrm{T}}$ is an infinitesimal generator defined by

$$\mathcal{A}^{\mathrm{T}}\phi = \begin{bmatrix} A^{\mathrm{T}}\phi^0 + C^{\mathrm{T}}\phi^1(-\check{L}) \\ \phi^{1\prime} \\ \phi^{2\prime} \end{bmatrix}, \tag{15}$$

$$\mathrm{D}(\mathcal{A}^{\mathrm{T}}) = \{\phi \in \mathcal{X}^{\mathrm{T}} : \phi^1 \in W^{1,2}(-\check{L}, 0;\, \mathbb{R}^p), \phi^2 \in W^{1,2}(-L, 0;\, \mathbb{R}^m),\ \phi^1(0) = 0, \phi^2(0) = B^{\mathrm{T}}\phi^0\}.$$

For the subspaces $\mathcal{W}^{\mathrm{T}} := \{\phi \in \mathcal{X}^{\mathrm{T}} : \phi^2 \in W^{1,2}(-L, 0;\, \mathbb{R}^m), \phi^2(0) = B^{\mathrm{T}}\phi^0\}$, $\mathcal{V}^{\mathrm{T}*} := \{\psi \in \mathcal{X} : \psi^1 \in W^{1,2}(-\check{L}, 0;\, \mathbb{R}^p), \psi^1(-\check{L}) = C\psi^0\}$, $\mathcal{W}^{\mathrm{T}} = \mathrm{D}_{\mathcal{V}^{\mathrm{T}}}(\mathcal{A}^{\mathrm{T}})$ holds and the Hilbert spaces $\mathcal{W}^{\mathrm{T}}$, $\mathcal{X}^{\mathrm{T}}$, $\mathcal{V}^{\mathrm{T}}$ are with continuous, dense injections satisfying $\mathcal{W}^{\mathrm{T}} \subset \mathcal{X}^{\mathrm{T}} \subset \mathcal{V}^{\mathrm{T}}$ [8]. Along (7),(8), the input/output operators $\mathcal{B}_1^{\mathrm{T}}$, $\mathcal{B}_2^{\mathrm{T}}$, $\mathcal{C}_1^{\mathrm{T}}$, $\mathcal{C}_2^{\mathrm{T}}$ are defined by

$$\mathcal{C}_k^{\mathrm{T}} : \mathbb{R}^{p_k} \to \mathcal{V}^{\mathrm{T}},\ \mathcal{C}_k^{\mathrm{T}*}\phi = \begin{bmatrix} \phi_0^{1k}(-\check{L} + \check{h}_0) \\ \phi_1^{1k}(-\check{L} + \check{h}_1) \\ \vdots \\ \phi_\ell^{1k}(-\check{L} + \check{h}_\ell) \end{bmatrix}, \phi \in \mathcal{V}^{\mathrm{T}*} \tag{16}$$

$$\mathcal{B}_k^{\mathrm{T}} : \mathcal{W}^{\mathrm{T}} \to \mathbb{R}^{m_k},\ \mathcal{B}_k^{\mathrm{T}}\phi = \begin{bmatrix} \phi_0^{2k}(-h_0) \\ \phi_1^{2k}(-h_1) \\ \vdots \\ \phi_d^{2k}(-h_d) \end{bmatrix}, \phi \in \mathcal{W}^{\mathrm{T}} \tag{17}$$

where $k = 1, 2$ and the subscripts of the component $\mathcal{X}^{\mathrm{T}}$ is similarly denoted along (6).

III. Main Result

In order to solve the H^∞ control problem $\hat{\Sigma}$, we will provide analytic solutions to (9),(10) and characterize the spectral radius condition (11) by the root of transcendental equation. Employing the preliminary result [4], which solves the FI-problem for $\check{h}_0 = \check{h}_1 = \cdots = \check{h}_\ell = 0$, the analytic solution to the operator Riccati equation (9) is obtained as follows.

Theorem 3: For a given $\gamma > 0$, the condition (A) holds iff (a1),(a2) are satisfied.

(a1) The Hamiltonian matrix

$$J := \begin{bmatrix} A & \frac{1}{\gamma^2} \cdot B_{1\bullet}B_{1\bullet}^{\mathrm{T}} - B_{2\bullet}B_{2\bullet}^{\mathrm{T}} \\ -C_{1\bullet}^{\mathrm{T}}C_{1\bullet} & -A^{\mathrm{T}} \end{bmatrix} \tag{18}$$

does not have eigenvalues on the imaginary axis.

(a2) Under the condition (a1), define a column full-rank matrix $V \in \mathbb{R}^{2n \times n}$ as follows.

$$JV = V\Lambda_c,\ \Lambda_c \in \mathbb{R}^{n \times n} : \text{stable matrix}$$
$$V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad V_1, V_2 \in \mathbb{R}^{n \times n} \tag{19}$$

Then the minimal root of the transcendental equation

$$\det[\tilde{V}(\mu)] = 0,$$
$$\tilde{V}(\mu) := V^{\mathrm{T}} \begin{bmatrix} I & 0 \\ -\mu \cdot I & I \end{bmatrix} \tilde{\Psi}_d(-L) \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{20}$$

is not negative where $\tilde{\Psi}_d$ is a fundamental solution to the differential equation:

$$\begin{cases} \tilde{\Psi}_d(0) = I \\ \frac{d}{dt}\Psi_d(t) = \Psi_d(t)\tilde{J}_i^{\mathrm{T}}, \quad -L + h_i \leq t \leq -L + h_{i+1} \end{cases},$$

$$\tilde{J}_i := \begin{bmatrix} A & \sum_{j=0}^{i}\left(\frac{1}{\gamma^2} \cdot B_{1j}B_{1j}^{\mathrm{T}} - B_{2j}B_{2j}^{\mathrm{T}}\right) \\ -\frac{1}{1-\mu} \cdot C_{1\bullet}^{\mathrm{T}}C_{1\bullet} & -A^{\mathrm{T}} \end{bmatrix}$$
$$(i = 0, 1, \cdots, d) \tag{21}$$

with a scalar parameter $\mu \neq 1$.

If (a1),(a2) hold, the stabilizing solution $\mathcal{S} \geq 0$ to (9) is given by

$$\mathcal{S} = \mathcal{G}^* \mathcal{V}_2 (\mathcal{V}_1 + \mathcal{G} \Pi \mathcal{G}^* \mathcal{V}_2)^{-1} \mathcal{G} \tag{22}$$

$$\mathcal{G} \in \mathcal{L}(\mathcal{X}, \mathcal{X}_r),$$
$$\mathcal{X}_r := \mathbb{R}^n \times L_2(-L, 0; \mathbb{R}^{p_1}) \times L_2(-\check{L}, 0; \mathbb{R}^{p_1})$$
$$\mathcal{G}\phi := \begin{bmatrix} (\mathcal{G}\phi)^0 \\ (\mathcal{G}\phi)^1 \\ (\mathcal{G}\phi)^2 \end{bmatrix} \tag{23}$$
$$(\mathcal{G}\phi)^0 := e^{AL}\phi^0 + \int_{-L}^{0} e^{-A\beta} B \phi^1(\beta)\, d\beta$$
$$(\mathcal{G}\phi)^1(\xi) := C_{1\bullet}\left(e^{A(\xi+L)}\phi^0 + \int_{-L}^{\xi} e^{A(\xi-\beta)} B\phi^1(\beta)\, d\beta\right)$$
$$(\mathcal{G}\phi)^2(\xi) := \begin{bmatrix} \chi_{[-\check{h}_0,0]}(\xi)\phi_0^{21}(\xi) \\ \chi_{[-\check{h}_1,0]}(\xi)\phi_1^{21}(\xi) \\ \vdots \\ \chi_{[-\check{h}_\ell,0]}(\xi)\phi_\ell^{21}(\xi) \end{bmatrix} \ (-\check{L} \leq \xi \leq 0)$$

$$\mathcal{V}_k := \begin{bmatrix} V_k & & \\ & \mathcal{I} & \\ & & \mathcal{I} \end{bmatrix} \in \mathcal{L}(\mathcal{X}_r) \quad (k = 1, 2) \tag{24}$$

$$\Pi := \begin{bmatrix} 0 & & \\ & \begin{bmatrix} -\gamma^{-2} \cdot \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} & \\ & & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{X}) \tag{25}$$
$$(\Pi_k \phi^{1k})(\beta) := \begin{bmatrix} \chi_{[-L+h_0,0]}(\beta) \cdot \phi_0^{1k}(\beta) \\ \chi_{[-L+h_1,0]}(\beta) \cdot \phi_1^{1k}(\beta) \\ \vdots \\ \chi_{[-L+h_d,0]}(\beta) \cdot \phi_d^{1k}(\beta) \end{bmatrix}$$
$$(k = 1, 2,\ -L \leq \beta \leq 0)$$

where χ is a characteristic function defined by $\chi_A(\beta) = \begin{cases} 1 & (\beta \in A) \\ 0 & (\beta \notin A) \end{cases}$. ■

Proof: (Sketch) By Remark 2, the FI-problem $(y(t) = [x^{\mathrm{T}}(t), w^{\mathrm{T}}(t)]^{\mathrm{T}})$ is solvable iff (A) holds. Furthermore the time delays in the regulated output z do not affect the solvability of the problem as the orthogonal condition (H2) is imposed. Hence, the solvability of (A) is characterized by [4]. The representation (22) is verified by substitution. ■

By Theorem 3, the solvability and the solution to the control operator Riccati equation is clarified with the integral operator (23), which describes the state transition of system Σ. Next, we will solve the filtering operator Riccati equation (10) by exploring the duality between $\hat{\Sigma}$ and $\hat{\Sigma}^{\mathrm{T}}$.

Introduce an auxiliary operator Riccati equation for $\hat{\Sigma}^{\mathrm{T}}$:

$$\tilde{\mathcal{P}}\mathcal{A}^{\mathrm{T}}\phi + \mathcal{A}^{\mathrm{T}*}\tilde{\mathcal{P}}\phi + \tfrac{1}{\gamma^2} \cdot \tilde{\mathcal{P}}\mathcal{C}_1^{\mathrm{T}}\mathcal{C}_1^{\mathrm{T}*}\tilde{\mathcal{P}}\phi$$
$$- \tilde{\mathcal{P}}\mathcal{C}_2^{\mathrm{T}}\mathcal{C}_2^{\mathrm{T}*}\tilde{\mathcal{P}}\phi + \mathcal{B}_1^{\mathrm{T}*}\mathcal{B}_1^{\mathrm{T}}\phi = 0, \quad \phi \in \mathcal{W}^{\mathrm{T}} \tag{26}$$

and an isomorphic operator $\mathcal{J} \in \mathcal{L}(\mathcal{X}, \mathcal{X}^{\mathrm{T}})$.

$$\mathcal{J}\phi = \begin{bmatrix} (\mathcal{J}\phi)^0 \\ (\mathcal{J}\phi)^1 \\ (\mathcal{J}\phi)^2 \end{bmatrix}, \quad \phi \in \mathcal{X} \tag{27}$$
$$(\mathcal{J}\phi)^0 = \phi^0$$
$$(\mathcal{J}\phi)^1(\alpha) = \phi^2(-\alpha - \check{L}), \quad -\check{L} \leq \alpha \leq 0$$
$$(\mathcal{J}\phi)^2(\beta) = \phi^1(-\beta - L), \quad -L \leq \beta \leq 0$$

As $\mathcal{J} \in \mathcal{L}(\mathcal{V}, \mathcal{W}^{\mathrm{T}*})$, $\mathcal{J} \in \mathcal{L}(\mathcal{W}, \mathcal{V}^{\mathrm{T}*})$ follows from the definition of $\mathcal{V}$,$\mathcal{W}$,$\mathcal{V}^{\mathrm{T}}$,$\mathcal{W}^{\mathrm{T}}$, the condition (B) is characterized as follows.

Theorem 4: The condition (B) holds iff the operator Riccati equation (26) has a stabilizing solution $\tilde{\mathcal{P}} \geq 0$ ($\tilde{\mathcal{P}} \in \mathcal{L}(\mathcal{V}^{\mathrm{T}}, \mathcal{V}^{\mathrm{T}*})$) such that $\mathcal{A}^{\mathrm{T}} + \frac{1}{\gamma^2} \cdot \mathcal{C}_1^{\mathrm{T}}\mathcal{C}_1^{\mathrm{T}*}\tilde{\mathcal{P}} - \mathcal{C}_2^{\mathrm{T}}\mathcal{C}_2^{\mathrm{T}*}\tilde{\mathcal{P}}$ generates exponentially stable semigroups on $\mathcal{W}^{\mathrm{T}}$ and $\mathcal{V}^{\mathrm{T}}$.

If the Riccati equation (26) is solvable, the stabilizing solution $\mathcal{P} \geq 0$ to (10) is given by

$$\mathcal{P} = \mathcal{J}^{-1}\tilde{\mathcal{P}}(\mathcal{J}^*)^{-1} \geq 0 \tag{28}$$

where $\tilde{\mathcal{P}} \geq 0$ is the stabilizing solution to (26). ■

Proof: The following relations are obtained for $\hat{\Sigma}$, $\hat{\Sigma}^{\mathrm{T}}$.

$$\mathcal{A}^{\mathrm{T}*}\mathcal{J}\phi = \mathcal{J}\mathcal{A}\phi, \ \phi \in \mathcal{W} \tag{29}$$
$$\mathcal{C}_1^{\mathrm{T}*}\mathcal{J}\phi = \mathcal{C}_1\phi, \ \phi \in \mathcal{W} \tag{30}$$
$$\mathcal{C}_2^{\mathrm{T}*}\mathcal{J}\phi = \mathcal{C}_2\phi, \ \phi \in \mathcal{W} \tag{31}$$
$$\mathcal{B}_1^{\mathrm{T}}\psi = \mathcal{B}_1^*\mathcal{J}^*\psi, \ \psi \in \mathcal{W}^{\mathrm{T}} \tag{32}$$
$$\mathcal{B}_2^{\mathrm{T}}\psi = \mathcal{B}_2^*\mathcal{J}^*\psi, \ \psi \in \mathcal{W}^{\mathrm{T}} \tag{33}$$

Hence, if $\tilde{\mathcal{P}} \geq 0$ is a solution to (26), a solution $\mathcal{P} \geq 0$ to (10) is given by (28). By (28)-(31), the equality

$$\mathcal{J}\left(\mathcal{A} + \frac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{C}_1^*\mathcal{C}_1 - \mathcal{P}\mathcal{C}_2^*\mathcal{C}_2\right)$$
$$= \left(\mathcal{A}^{\mathrm{T}*} + \frac{1}{\gamma^2} \cdot \tilde{\mathcal{C}}_1^{\mathrm{T}}\tilde{\mathcal{C}}_1^{\mathrm{T}*}\tilde{\mathcal{P}} - \tilde{\mathcal{C}}_2^{\mathrm{T}}\tilde{\mathcal{C}}_2^{\mathrm{T}*}\tilde{\mathcal{P}}\right)^* \mathcal{J} \ \text{on} \ \mathcal{W} \tag{34}$$

is obtained for the solutions $\mathcal{P}$ and $\tilde{\mathcal{P}}$. Hence, if $\tilde{\mathcal{P}} \geq 0$ is a stabilizing solution to (26), the stabilizing solution $\mathcal{P} \geq 0$ to (10) is given by (28). Conversely, it follows from (28),(34) that a stabilizing solution $\mathcal{P} \geq 0$ to (10) provides the stabilizing solution to (26) by $\tilde{\mathcal{P}} = \mathcal{J}\mathcal{P}\mathcal{J}^*$. ■

Applying Theorem 3,4 to the transposed system Σ^{T}, the condition (B) is clarified as follows.

Lemma 5: For a give $\gamma > 0$, the condition (B) holds iff (b1),(b2) are satisfied.

(b1) The Hamiltonian matrix

$$K := \begin{bmatrix} A^{\mathrm{T}} & \frac{1}{\gamma^2} \cdot C_{1\bullet}^{\mathrm{T}} C_{1\bullet} - C_{2\bullet}^{\mathrm{T}} C_{2\bullet} \\ -B_{1\bullet}B_{1\bullet}^{\mathrm{T}} & -A \end{bmatrix} \tag{35}$$

does not have eigenvalues on the imaginary axis.

(b2) Under the condition (b1), define a column full-rank matrix $U \in \mathbb{R}^{2n \times n}$ as follows.

$$KU = U\Lambda_f, \ \Lambda_f \in \mathbb{R}^{n \times n} : \text{stable matrix}$$
$$U := \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \ U_1, U_2 \in \mathbb{R}^{n \times n} \tag{36}$$

Then the minimal root of the transcendental equation

$$\det[\tilde{U}(\mu)] = 0,$$
$$\tilde{U}(\mu) := U^{\mathrm{T}} \begin{bmatrix} I & 0 \\ -\mu \cdot I & I \end{bmatrix} \tilde{\Phi}_\ell(-\check{L}) \begin{bmatrix} I \\ 0 \end{bmatrix} \quad (37)$$

is not negative where $\tilde{\Phi}_\ell$ is a fundamental solution to the differential equation:

$$\begin{cases} \tilde{\Phi}_\ell(0) = I \\ \frac{d}{dt}\Phi_\ell(t) = \Phi_\ell(t)\tilde{K}_i^{\mathrm{T}}, \ -\check{L}+\check{h}_i \le t \le -\check{L}+\check{h}_{i+1} \end{cases},$$

$$\tilde{K}_i := \begin{bmatrix} A^{\mathrm{T}} & \sum_{j=0}^{i}(\frac{1}{\gamma^2}\cdot C_{1j}^{\mathrm{T}}C_{1j} - C_{2j}^{\mathrm{T}}C_{2j}) \\ -\frac{1}{1-\mu}\cdot B_{1\bullet}B_{1\bullet}^{\mathrm{T}} & -A \end{bmatrix}$$
$$(i = 0, 1, \cdots, \ell) \quad (38)$$

with a scalar parameter $\mu \neq 1$.

If (b1),(b2) hold, the stabilizing solution $\mathcal{P} \geq 0$ is given by

$$\mathcal{P} = \mathcal{J}^{-1}\tilde{\mathcal{P}}(\mathcal{J}^*)^{-1},$$
$$\tilde{\mathcal{P}} = \mathcal{F}^*\mathcal{U}_2(\mathcal{U}_1 + \mathcal{F}\Gamma\mathcal{F}^*\mathcal{U}_2)^{-1}\mathcal{F}. \quad (39)$$

$$\mathcal{F} \in \mathcal{L}(\mathcal{X}^{\mathrm{T}}, \mathcal{X}_r^{\mathrm{T}}),$$
$$\mathcal{X}_r^{\mathrm{T}} := \mathbb{R}^n \times L_2(-\check{L}, 0; \mathbb{R}^{m_1}) \times L_2(-L, 0; \mathbb{R}^{m_1})$$
$$\mathcal{F}\phi := \begin{bmatrix} (\mathcal{F}\phi)^0 \\ (\mathcal{F}\phi)^1 \\ (\mathcal{F}\phi)^2 \end{bmatrix} \quad (40)$$
$$(\mathcal{F}\phi)^0 := e^{A^{\mathrm{T}}\check{L}}\phi^0 + \int_{-\check{L}}^{0} e^{-A^{\mathrm{T}}\beta}C^{\mathrm{T}}\phi^1(\beta)\,d\beta$$
$$(\mathcal{F}\phi)^1(\xi) := B_1^{\mathrm{T}}\left(e^{A^{\mathrm{T}}(\xi+\check{L})}\phi^0 + \int_{-\check{L}}^{\xi} e^{A^{\mathrm{T}}(\xi-\beta)}C^{\mathrm{T}}\phi^1(\beta)\,d\beta\right)$$
$$(\mathcal{F}\phi)^2(\xi) := \begin{bmatrix} \chi_{[-h_0,0]}(\xi)\phi_0^{21}(\xi) \\ \chi_{[-h_1,0]}(\xi)\phi_1^{21}(\xi) \\ \vdots \\ \chi_{[-h_d,0]}(\xi)\phi_d^{21}(\xi) \end{bmatrix} \quad (-L \le \xi \le 0)$$

$$\mathcal{U}_k := \begin{bmatrix} U_k & & \\ & \mathcal{I} & \\ & & \mathcal{I} \end{bmatrix} \in \mathcal{L}(\mathcal{X}_r^{\mathrm{T}}) \quad (k = 1, 2) \quad (41)$$

$$\Gamma := \begin{bmatrix} 0 & & \\ & \begin{bmatrix} -\gamma^{-2}\cdot\Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} & \\ & & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{X}^{\mathrm{T}}) \quad (42)$$

$$(\Gamma_k\phi^{1k})(\beta) := \begin{bmatrix} \chi_{[-\check{L}+\check{h}_0,0]}(\beta)\cdot\phi_0^{1k}(\beta) \\ \chi_{[-\check{L}+\check{h}_1,0]}(\beta)\cdot\phi_1^{1k}(\beta) \\ \vdots \\ \chi_{[-\check{L}+\check{h}_\ell,0]}(\beta)\cdot\phi_\ell^{1k}(\beta) \end{bmatrix}$$
$$(k = 1, 2, \ -\check{L} \le \beta \le 0) \quad ■$$

Finally, we will clarify the spectral radius condition (C) based on Theorem 3, Lemma 5.

Theorem 6: The condition (C) holds iff (c) is satisfied.

(c) The maximal root $\sigma_{\max}$ of the following transcendental equation satisfies $\sigma_{\max} < \gamma$:

$$\det W(\sigma) = 0$$
$$W(\sigma) := U^{\mathrm{T}}\hat{\Phi}_\ell(\check{L})\begin{bmatrix} -\sigma\cdot I & 0 \\ 0 & \sigma^{-1}\cdot I \end{bmatrix}\hat{\Psi}_d(0)V \quad (43)$$

where $\hat{\Phi}_\ell$, $\hat{\Psi}_d$ are the fundamental solutions to the following differential equations.

$$\begin{cases} \hat{\Psi}_d(0) = I \\ \frac{d}{dt}\hat{\Psi}_d(t) = \hat{J}_i\hat{\Psi}_d(t), \ -L+h_i \le t \le -L+h_{i+1} \end{cases}$$

$$\hat{J}_i := \begin{bmatrix} A & \sum_{j=0}^{i}(\gamma^{-2}B_{1j}B_{1j}^{\mathrm{T}} - B_{2j}B_{2j}^{\mathrm{T}}) + \sum_{j=i+1}^{d}\sigma^{-2}B_{1j}B_{1j}^{\mathrm{T}} \\ -C_{1\bullet}^{\mathrm{T}}C_{1\bullet} & -A^{\mathrm{T}} \end{bmatrix}$$
$$(i = 0, 1, \cdots, d-1) \quad (44)$$

$$\begin{cases} \hat{\Phi}_\ell(-\check{L}) = I \\ \frac{d}{dt}\hat{\Phi}_\ell(t) = -\hat{K}_i\hat{\Phi}_\ell(t), \ -\check{L}+\check{h}_i \le t \le -\check{L}+\check{h}_{i+1} \end{cases}$$

$$\hat{K}_i := \begin{bmatrix} A & -B_{1\bullet}B_{1\bullet}^{\mathrm{T}} \\ \sum_{j=0}^{i}(\gamma^{-2}C_{1j}^{\mathrm{T}}C_{1j} - C_{2j}^{\mathrm{T}}C_{2j}) + \sum_{j=i+1}^{\ell}\sigma^{-2}C_{1j}^{\mathrm{T}}C_{1j} & -A^{\mathrm{T}} \end{bmatrix}$$
$$(i = 0, 1, \cdots, \ell-1) \quad (45)$$

■

Proof: (Sketch) We will prove that the roots of (43) meet the nonzero eigenvalues of $\mathcal{PS}$. Let $\lambda = \sigma^2 \neq 0$ be an eigenvalue of $\mathcal{PS}$ and $v \neq 0$ be the corresponding eigenvector such that

$$\sigma^2 \cdot v = \mathcal{PS}v \quad (46)$$

holds. By (22),(39), the equality (46) is described by

$$\mathcal{V}_1 g = \mathcal{G}(v - \sigma\cdot\Pi\mathcal{J}^*u) \quad (47)$$
$$\sigma\cdot\mathcal{J}^*u = \mathcal{G}^*\mathcal{V}_2 g \quad (48)$$
$$\mathcal{U}_1 f = \mathcal{F}(u - \sigma\cdot\Gamma\mathcal{J}v) \quad (49)$$
$$\sigma\cdot\mathcal{J}v = \mathcal{F}^*\mathcal{U}_2 f \quad (50)$$

and, further, $v \neq 0$ exists in (46) iff $(f, g, u, v) \neq 0$ exists in (47)-(50). In the following, we will characterize the condition such that $(f, g, u, v) \neq 0$ exists for (47)-(50).

Define $r = v - \sigma\cdot\Pi\mathcal{J}^*u$ and introduce auxiliary variables

$$p(\xi) = e^{A(\xi+L)}r^0 + \int_{-L}^{\xi} e^{A(\xi-\beta)}B_{1\bullet}r^1(\beta)\,d\beta \quad (51)$$
$$q(\beta) = -e^{-A^{\mathrm{T}}\beta}V_2 g^0 - \int_{\beta}^{0} e^{A^{\mathrm{T}}(\xi-\beta)}C_{1\bullet}^{\mathrm{T}}g^1(\xi)\,d\xi \quad (52)$$

to (47) and (48) respectively, then we have

$$\begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} = \hat{\Psi}_d(-L)\begin{bmatrix} p(0) \\ q(0) \end{bmatrix} \quad (53)$$

with boundary conditions.

$$v^0 = p(-L), \quad \sigma \cdot u^0 = q(-L) \tag{54}$$

$$\begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} g^0 = V g^0 \tag{55}$$

Similarly define $s = u - \sigma \cdot \Gamma \mathcal{J} v$ and introduce

$$\check{p}(\beta) = -e^{-A\beta} U_2 f^0 - \int_\beta^0 e^{A(\xi-\beta)} B_{1\bullet} f^1(\xi)\, d\xi \tag{56}$$

$$\check{q}(\xi) = e^{A^{\mathrm{T}}(\xi+\check{L})} s^0 + \int_{-\check{L}}^{\xi} e^{A^{\mathrm{T}}(\xi-\beta)} C_{1\bullet}^{\mathrm{T}} s^1(\beta)\, d\beta \tag{57}$$

to (49) and (50) respectively, then we have

$$\begin{bmatrix} \check{p}(0) \\ \check{q}(0) \end{bmatrix} = \hat{\Phi}_\ell(\check{L}) \begin{bmatrix} \check{p}(-\check{L}) \\ \check{q}(-\check{L}) \end{bmatrix} \tag{58}$$

with boundary conditions.

$$\check{p}(-\check{L}) = -\sigma \cdot v^0, \quad \check{q}(-\check{L}) = u^0 \tag{59}$$

$$\begin{bmatrix} U_1^{\mathrm{T}} & U_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \check{p}(0) \\ \check{q}(0) \end{bmatrix} = U^{\mathrm{T}} \begin{bmatrix} \check{p}(0) \\ \check{q}(0) \end{bmatrix} = 0 \tag{60}$$

Combining the conditions (54),(59) by

$$\begin{bmatrix} \check{p}(-\check{L}) \\ \check{q}(-\check{L}) \end{bmatrix} = \begin{bmatrix} -\sigma \cdot I & 0 \\ 0 & \sigma^{-1} \cdot I \end{bmatrix} \begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix}, \tag{61}$$

the condition $W(\sigma) g^0 = 0$ is obtained from (53),(55),(58),(60),(61).

Since $g^0 = 0$ implies $v = 0$, it is necessary for the eigenvalue $\lambda = \sigma^2$ that the matrix $W(\sigma)$ in (43) is singular. Conversely, if $g^0 \neq 0$ exists, the eigenvector $v \neq 0$ exists as $v = 0$ implies $(f, g, u, v) = 0$ in (47)-(50). ■

Based on Theorem 3, Lemma 5, and Theorem 6, the solvability of the H^∞ control problem Σ is summarized as follows.

Theorem 7: For a given $\gamma > 0$, the H^∞ control problem Σ is solvable iff the following conditions are satisfied.

(a) The Hamiltonian matrix (18) does not have eigenvalues on the imaginary axis and the minimal root of (20) is not negative.

(b) The Hamiltonian matrix (35) does not have eigenvalues on the imaginary axis and the minimal root of (37) is not negative.

(c) The maximal root $\sigma_{\max}$ of (43) satisfies $\sigma_{\max} < \gamma$. ■

Remark 8: In the general case, the conditions (a),(b) need rather complicated calculation to track the minimal roots. In the case $B_{2\bullet} = [B_{20}, 0, \cdots, 0]$, $C_{1\bullet} = [C_{10}^{\mathrm{T}}, 0, \cdots, 0]^{\mathrm{T}}$, $C_{2\bullet} = [C_{20}^{\mathrm{T}}, 0, \cdots, 0]^{\mathrm{T}}$ holds (preview control problem), the conditions (a),(b) are further reduced to a stability condition of some matrices [3]. ■

The calculation procedure of the control law (12), which employs analytic solutions to (9),(10), is summarized as follows.

Remark 9: For the calculation of the control law (12), we need the explicit representation of the operators $\mathcal{PC}_1^* \in \mathcal{L}(\mathbb{R}^{p_1}, \mathcal{X})$, $\mathcal{PC}_2^* \in \mathcal{L}(\mathbb{R}^{p_2}, \mathcal{X})$, and $\mathcal{B}_2^* \mathcal{Q} \in \mathcal{L}(\mathcal{X}, \mathbb{R}^{m_2})$.

By (30),(39), the operator $\mathcal{PC}_1^* \in \mathcal{L}(\mathbb{R}^{p_1}, \mathcal{X})$ or the relation $f = \mathcal{PC}_1^* v$ $(f \in \mathcal{X},\ v \in \mathbb{R}^{p_1})$ is equivalently described as follows.

$$\mathcal{J} f = \mathcal{F}^* \mathcal{U}_2 g, \quad \mathcal{U}_1 g = \mathcal{F}(\mathcal{C}_1^{\mathrm{T}*} v - \Gamma \mathcal{J} f) \tag{62}$$

In (62), the operators $\mathcal{F}$, $\mathcal{F}^*$ define forward/backward evolution of ordinary differential equations with boundary conditions. Based on this property, the direct relation between $f \in \mathcal{X}$ and $v \in \mathbb{R}^{p_1}$ is obtained. Similar approach is applicable to the representation of $\mathcal{PC}_2^* \in \mathcal{L}(\mathbb{R}^{p_2}, \mathcal{X})$.

By (12),(22), the relation $u = -\mathcal{B}_2^* \mathcal{Q} f$ $(u \in \mathbb{R}^{m_2},\ f \in \mathcal{X})$ is equivalently described as follows.

$$u = -\mathcal{B}_2 f, \quad f = \mathcal{S} e, \quad \gamma^2 \cdot (e - g) = \mathcal{P} f \tag{63}$$

Hence, by preparing the direct representations of (63) along the properties mentioned in (62), then eliminating e from (63), the relation $u = -\mathcal{B}_2^* \mathcal{Q} f$ is explicitly given. ■

4. Conclusion

A generalized H^∞ control problem, which covers preview and delayed control strategies, is discussed in the output feedback setting. Based on the analytic solutions to control/filtering operator Riccati equations, explicit formulas are derived for the H^∞ control problem. The calculation method on the solution to operator Riccati equations is also applicable to typical control/filtering problems, which are characterized with the Riccati or Lyapunov operator equations.

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