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Near Optimal H_∞ Tracking for a Compact Set of Plants

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Abstract— **Here we consider the problem of providing near optimal performance for a large set of possible models. We adopt the H_∞ framework in the single-input single-output (SISO) setting with structured uncertainty: a compact set of controllable and observable plant models of a fixed order; we consider the control problem of designing a controller to minimize the worst case performance. We consider two different feedback configurations, and under a mild assumption we prove that a linear periodic controller (LPC) exists which achieves the objective.**

I. Introduction

An important control problem is that of providing good performance in the face of plant uncertainty. The standard approaches to the problem are robust control techniques and adaptive control techniques. While the two approaches are similar, an implicit goal of the latter is to deal with (possibly rapidly) changing parameters. Here we consider an approach which is very much at the boundary between the two: we design a LPC which provides near optimal H_∞ performance for a large class of LTI plants; its ability to tolerate time-variations looks promising.

Here we adopt the H_∞ framework. Rather than dealing with the common problem of disturbance rejection, here we consider the equally important problem of (wideband) tracking; of course, they both fit equally well into this framework, and in most realistic situations one has to deal with reference signals, disturbance signals, and noisy measurements. Rather than dealing with the usual unstructured uncertainty model, here we consider structured uncertainty: this is appropriate in cases in which the model is well known but a few key parameters (such as mass, length, inductance, etc.) vary. The difficulty in this scenario is that if we restrict ourselves to LTI controllers, then there are limits on the amount of uncertainty that can be tolerated, e.g. there is no LTI controller which can simultaneously stabilize $\{\frac{1}{s-1}, \frac{-1}{s-1}\}$. Therefore, in order to handle a general class of structured uncertainty, we must use either a nonlinear or time-varying controller. While adaptive control can be used to attack the problem of handling a general compact (possibly convex) set of plant parameters, e.g. classical pole placement [3], as well as logic-based switching, e.g. see [2], [7], [8] and [4], these approaches often provide poor transient behaviour, or, in the case of

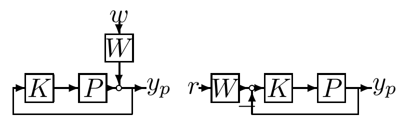

Fig. 1. One degree of freedom feedback for disturbance rejection and for tracking.

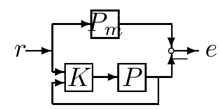

Fig. 2. Two degree of freedom configuration.

high-gain approaches (restricted to minimum phase systems), a large control signal. Here the goal is to obtain near optimal H_∞ performance immediately while using a modest control signal. Recent work by the first author has shown how to solve this problem in the LQR context [5], [6], and in this paper the approach is extended to deal with a class of H_∞ robust control problems.

We start with a siso LTI plant with parameters in a compact set. We can set up a tracking problem by modifying the classical setup of [10] to replace the weight on the disturbance to a weight on the reference input: see Figure 1. Alternatively, we can adopt a common approach used in the adaptive control: we choose a reference model P_m, which embodies the desired behaviour of the closed loop system to the exogenous input, and then adopt a general controller as illustrated in Figure 2. In both cases, the outputs to be controlled would be the weighted tracking error signal and the weighted control signal; the control objective is the standard one of minimizing the worst case performance (optimized over all plants). We provide an approach which provides near optimal performance under a condition on the weight W.

We use standard notation throughout the paper. We let $\mathbf{N}$, $\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{Z}$, and $\mathbf{Z}^+$ denote the natural numbers, real numbers, non-negative real numbers, integers, and non-negative integers, respectively. We use the Holder 2-norm for vectors and the corresponding induced norm for matrices, and denote the norm of a vector or matrix by $\|\cdot\|$. We let $L_2(\mathbf{R}^n)$ denote the set of square integrable Lebesgue measureable signals, and

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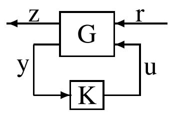

Fig. 3. The Standard Feedback Configuration.

use $\|f\|$ to denote the norm of an element f; occasionally we will be computing the norm of a truncated signal: with $0 \le t_1 < t_2 < \infty$, we define

$$\|f\|_{[t_1,t_2]} := [\int_{t_1}^{t_2} \|f(t)\|^2 \, dt]^{1/2}.$$

In our work it is convenient to adopt the standard plant-configuration framework illustrated in Figure 3. Here G is the generalized plant, K is the controller, and the standard definition of stability is that if we insert noise signals at the two plant controller interfaces, then the map from the exogenous signals to the internal signals is boundeded. We define $T_{r,z}(G,K)$ to be the closed loop map from r to z; $\|T_{r,z}(G,K)\|$ denotes its induced norm.

II. The Setup

Our plant model is given by

$$\begin{array}{rcl} \dot{x}_p &=& A_p x_p + B_p u_p, \;\; x(t_0) = x_0 \\ y_p &=& C_p x_p, \end{array} \tag{1}$$

with $x_p(t) \in \mathbf{R}^{n_p}$ representing the state, $u_p(t) \in \mathbf{R}$ the control signal, and $y_p(t) \in \mathbf{R}$ the measured output. We associate the plant with the triple (A_p, B_p, C_p). With n_p fixed throughout the paper, we let Γ denote the subset of (A_p, B_p, C_p) triples for which (A_p, B_p) is controllable and (C_p, A_p) is observable. For each such triple, we let $\mathcal{C}(A_p, B_p)$ denote the controllability matrix and $\mathcal{O}(C_p, A_p)$ denote the observability matrix. In this paper our goal is to control the plant when the model is uncertain: we assume that it lies in a compact subset of Γ, which we label $\mathcal{P}$.

At this point we consider two different feedback configurations.

A. 1-DOF Configuration

As in [10], which is focussed in large part on disturbance rejection, we assume that the reference signal energy is concentrated in specific frequency ranges, typically low frequency. To this end, with W a stable transfer function, the set of reference signals for which the controller is to be optimized is

$$\{Wr : r \in L_2, \; \|r\| \le 1\};$$

we label the filter output as y_{ref}. Here we adopt the common 1-DOF configuration, as illustrated in Figure 1: the tracking error is $e := Wr - y_p$. With W_1 and W_2 stable transfer functions, the output to be controlled is of the form

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_1 e \\ W_2 u_p \end{bmatrix}.$$

B. 2-DOF Configuration

Here we adopt a 2-DOF configuration as illustrated in Figure 2; the goal is to make the map from $r \to y$ as close as desired to a desired stable "reference model" P_m, which embodies the desired closed loop behaviour, while maintaining a desirable control signal. This is an archetypical problem in the adaptive control literature, although there it is typically focussed on minimum phase plant models for the simple reason that then exact (or near exact) matching is possible. With W_1 and W_2 stable weights, the output to be controlled is

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_1 e \\ W_2 u \end{bmatrix} = \begin{bmatrix} W_1(P_m r - y) \\ W_2 u \end{bmatrix}.$$

C. A Canonical Plant Model

It is convenient at this point to put the plant into observable canonical form. To this end, we define

$$\bar{x}_p := \mathcal{O}(C_p, A_p) x.$$

which yields a corresponding state-space model of

$$\left.\begin{array}{rcl} \dot{\bar{x}}_p &=& \bar{A}_p \bar{x}_p + \bar{B}_p \bar{u}_p \\ y &=& \bar{C}_p \bar{x}_p; \end{array}\right\} \tag{2}$$

it is easy to see that $\bar{A}_p$ is in controllable canonical form, and that

$$\bar{B}_p^T = \begin{bmatrix} C_p B_p & C_p A_p B_p & \cdots & C_p A_p^{n_p - 1} B_p \end{bmatrix},$$

$$\bar{C}_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

While $\bar{B}_p$ and $\bar{C}_p$ are clearly polynomials in the first n_p plant Markov parameters, it turns out that we can prove something almost as nice for $\bar{A}_p$.

Lemma 1: (Parametrization Lemma) [6] $\bar{A}_p$, $\bar{B}_p$, and $\bar{C}_p$ are analytic functions of the first $2n_p$ Markov parameters

$$\{C_p B_p, C_p A_p B_p, ..., C_p A_p^{2n_p - 1} B_p\}$$

for all $(A_p, B_p, C_p) \in \Gamma$.

Remark 1: If there is additional structure in $\mathcal{P}$, then it could very well be that $\bar{A}_p$, $\bar{B}_p$, and $\bar{C}_p$ are analytic functions of a subset of the first $2n_p$ parameters. For example, if we have a gain margin problem, the gain is embedded in the first non-zero Markov parameter.

At this point we freeze $m \in N$ with the property that all plant uncertainty is embedded in the first m plant Markov parameters. To this end, we define

$$S := \{ \underbrace{\begin{bmatrix} C_p B_p \\ C_p A_p B_p \\ \vdots \\ C_p A_p^{m-1} B_p \end{bmatrix}}_{=:p} : (A_p, B_p, C_p) \in \mathcal{P}\};$$

since $\mathcal{P}$ is compact so is S. Henceforth, all plant uncertainty will be expressed in terms of p.

D. A State Space Model

Using minimal models of W, W_1, W_2, and P_m together with (2), we can obtain a state-space representation of the generalized plant $G(p)$:

$$\begin{aligned} \dot{x} &= A(p)x + B_1 r + B_2(p)u && (3)\\ z &= C_1 x + D_{11} r + D_{12} u && (4)\\ y &= C_2 x + D_{21} r; && (5) \end{aligned}$$

we associate this system with the 8−tuple $(A(p), B_1, B_2(p), C_1, C_2, D_{11}, D_{12}, D_{21})$. We denote the dimension of x by n.

E. The Control Problem

The goal here is the one most common in the robust control literature: minimize the worst-case performance. To make this precise, we need two definitions:

$$\alpha_{opt}(G(p)) := \inf_{K \text{ is LTI and stabilizes G(p)}} \|T_{r,z}(G(p), K)\|$$

defined for all $p \in S$, and

$$\alpha_{opt} := \sup_{p\in S} \alpha_{opt}(G(p)).$$

The Control Problem: Given $\rho > 0$, find a controller K which, for every $p \in S$, provides closed loop stability and satisfies

$$\|T_{r,z}(G(p), K)\| \leq \alpha_{opt} + \rho.$$

Remark 2: Given that only the magnitudes of W, W_1 and W_2 play a role in the optimization problem, henceforth we shall assume that all three weights are minimum phase, i.e. there are no zeros in the open right-half of the complex plant.

F. The Structure of the Controller

The controller will consist of three parts:
(i) an anti-aliasing filter at the input;
(ii) a continuous-time compensator of order equal to that of the (near optimal) LTI controller, and
(iii) a linear periodic discrete-time controller.

To this end, with $\sigma > 0$, we define the anti-aliasing filter by

$$\dot{y}_f = -\sigma y_f + \sigma y. \quad (6)$$

We know that there always exists a controller of order n_P which stabilizes P; to obtain near optimal performance we may need one of higher order - the approach of [1] produces one of order n. Furthermore, it is not hard to prove that this controller works almost as well if it measures the filtered version of y (namely y_f) rather than y. Hence, we incorporate this into the controller structure:

$$\begin{aligned} \dot{v} &= Fv + Hy_f, \quad v(t_0) = v_0 \in \mathbf{R}^n \\ u &= Jv + Ly_f; \end{aligned} \quad (7)$$

we associate this controller with the 4−tuple (F, H, J, L). Before proceeding with the third part, it is convenient to group the first two parts together with the generalized plant; if we define

$$\tilde{x} := \begin{bmatrix} x \\ y_f \\ v \end{bmatrix}, \quad \tilde{u} := \begin{bmatrix} u \\ \dot{v} \end{bmatrix}, \quad \tilde{y} := \begin{bmatrix} y_f \\ v \end{bmatrix},$$

then it is easy to construct the matrices $\tilde{A}(p)$, $\tilde{B}_1$, $\tilde{B}_2(p)$, $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{D}_{11}$ and $\tilde{D}_{12}$ so that

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}(p)\tilde{x} + \tilde{B}_1 r + \tilde{B}_2(p)\tilde{u} && (8)\\ z &= \tilde{C}_1\tilde{x} + \tilde{D}_{11} r + \tilde{D}_{12}\tilde{u} && (9)\\ \tilde{y} &= \tilde{C}_2\tilde{x}, && (10) \end{aligned}$$

which we label $\tilde{G}(p)$. We denote the dimensions of $\tilde{x}$, $\tilde{y}$, and $\tilde{u}$ by $\tilde{n}$, $\tilde{m}$ and $\tilde{r}$, respectively. It turns out that the first element of $\tilde{y}$ is a filtered version of y_p, which is an important quantity: we label the first row of $\tilde{C}_2$ by $\tilde{C}_{21}$,

Hence, in this context our control law (7) can be rewritten as

$$\tilde{u} = \begin{bmatrix} u \\ \dot{v} \end{bmatrix} = \begin{bmatrix} L & J \\ H & F \end{bmatrix} \tilde{y} = \underbrace{\begin{bmatrix} L & J \\ H & F \end{bmatrix}}_{=:\tilde{K}} \begin{bmatrix} y_f \\ v \end{bmatrix}. \quad (11)$$

A word on notation: $T_{r,z}(\tilde{G}(p), \tilde{K})$ simply means the closed loop map from $r \to z$ when (11) is applied to (8)-(10).

Of course, we do not know P so we do not know which $\tilde{K}$ to choose; hence the need for probing and estimation, which is carried out by the l^{th} order sampled-data compensator given by

$$\begin{aligned} \eta[k+1] &= F_d(k)\eta[k] + H_d(k)\tilde{y}(kh), \\ \tilde{u}(kh+\tau) &= J_d(k)\eta[k] + L_d(k)\tilde{y}(kh), \\ & \quad \tau \in [0, h) \end{aligned} \quad (12)$$

with the controller gains F_d, H_d, J_d, and L_d periodic of period $\ell \in \mathbf{N}$; the period of the controller is $T := \ell h$, and we associate this system with the 6-tuple $(F_d, H_d, J_d, L_d, h, \ell)$. Note that (12) can be implemented with a sampler, a zero-order-hold, and an l^{th} order periodically time-varying discrete-time system of period ℓ. Hence, the proposed controller consists of the filter (6), the sampled-data compensator (12), and the continuous-time compensator given by

$$\left.\begin{aligned} \begin{bmatrix} u \\ \dot{v} \end{bmatrix} &= \tilde{u} \\ \tilde{y} &= \begin{bmatrix} y_f \\ v \end{bmatrix}. \end{aligned}\right\} \quad (13)$$

At this point we provide a high level motivation of our approach. First of all, under suitable assumptions, it is proven that one can obtain a static output feedback law, labelled $\tilde{K}(p)$, with parameters which are a polynomial in the elements of $p \in S$, which stabilizes $\tilde{G}(p)$ and which is near optimal. We divide the period $[kT, (k+1)T)$ into two phases: the Estimation Phase and the Control Phase. In the Estimation Phase, we estimate $\tilde{K}(p)\tilde{y}(kT)$ by applying a sequence of test signals which are constructed on the fly. In the Control Phase we apply a suitably weighted estimate of $\tilde{K}(p)\tilde{y}(kT)$ (see Figure 4). We provide a **linear** approach

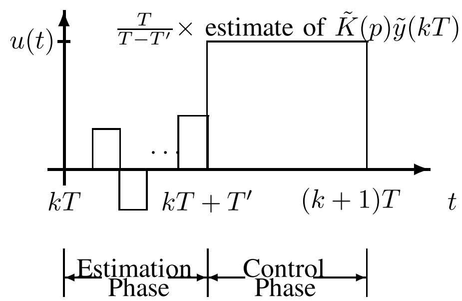

Fig. 4. Estimation Phase and Control Phase

to implemementing the Estimation Phase, and we end up with a linear time-varying controller which will achieve the objective.

The following steps are used to design the proposed controller:

- First, we present a result which allows us to do an experiment on the plant to obtain a certain number of the Markov parameters times a test signal.
- Second, we prove a technical result on the existence of a near optimal controller whose parameters are a polynomial in the elements of p.
- We introduce a data structure which allows a clean use of the aforementioned experimentation.
- Finally, we bring these three parts together to propose a controller.

III. Preliminary Technical Results

As discussed above, the goal is to iteratively estimate the desired control signal and then to apply this estimate. The first step is to determine what items can be easily estimated by experimenting on the plant **in a linear fashion**, and then to demonstrate that we can write our output feedback laws in terms of these items.

A. Linear Estimation

To proceed, we define an $(m+2)\times(m+2)$ matrix

$$V_m(h) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{m+1} \\ & & \vdots & & \\ 1 & m+1 & (m+1)^2 & \cdots & (m+1)^{m+1} \end{bmatrix} \times$$

$$\text{diag}\{1,\ h\ h^2/(2!),\ \cdots\ h^{m+1}/(m+1)!\} \qquad (14)$$

and an $m\times(m+2)$ matrix:

$$\bar{U}_m(\sigma) = \left[\begin{array}{cc} 0 & \begin{bmatrix} \sigma & & & \\ \sigma^2 & \sigma & & \\ \vdots & \ddots & \ddots & \\ \sigma^m & \cdots & \sigma^2 & \sigma \end{bmatrix}^{-1} \end{array}\right].$$

We will be sampling y to form our estimate: we define

$$\mathcal{Y}_m(t) = \begin{bmatrix} \tilde{y}_1(t) \\ \tilde{y}_1(t+h) \\ \vdots \\ \tilde{y}_1(t+(m+1)h) \end{bmatrix},$$

$$\tilde{\mathcal{Y}}_m(t) := \mathcal{Y}_m(t) - \mathcal{Y}_m(t+mh+h).$$

The relative degree of the transfer function from $r \to \tilde{y}_1$ plays a critical role - we label this m_{11}, which equals

$$m_{11} = \begin{cases} 1+ \text{ relative degree of } W & \text{1-DOF case} \\ \infty & \text{2-DOF case.} \end{cases}$$

Lemma 2: (Key Estimation Lemma) Suppose that $m < m_{11}-1$. Let $\bar{h}\in(0,1)$ and $\sigma>0$. Then there exists a constant $\gamma>0$ so that for all $t_0\in\mathbf{R}$, $\tilde{x}_0\in\mathbf{R}^{\tilde{n}}$, $h\in(0,\bar{h})$, $\bar{u}\in\mathbf{R}$, and $p\in S$ the solution of (8)-(10) with $\tilde{x}(t_0)=\tilde{x}_0$ and

$$\tilde{u}(t) = \begin{cases} \begin{bmatrix} \bar{u} \\ 0 \end{bmatrix}, & t\in[t_0, t_0+(m+1)h), \\ \begin{bmatrix} -\bar{u} \\ 0 \end{bmatrix}, & t\in[t_0+(m+1)h, \\ & \quad t_0+2(m+1)h), \end{cases}$$

satisfies the following:

$$\|\bar{U}_m(\sigma)V_m(h)^{-1}\tilde{\mathcal{Y}}_m(t_0) - 2p\bar{u}\| \le \gamma h(\|\tilde{x}_0\|+\|\bar{u}\|) + \gamma h^{1/2}\|r\|_{[t_0,t_0+2(m+1)h]},$$

$$\|\tilde{x}(t)-\tilde{x}_0\| \le \gamma h(\|\tilde{x}_0\|+\|\bar{u}\|) + \gamma h^{1/2}\|r\|_{[t_0,t_0+2(m+1)h]}, \quad t\in[t_0,t_0+2(m+1)h],$$

$$\|\tilde{x}(t_0+2(m+1)h) - e^{2\tilde{A}(m+1)h}\tilde{x}_0 - \int_0^{2(m+1)h} e^{\tilde{A}(2(m+1)h-\tau)}\tilde{B}_1 r(t_0+\tau)\,d\tau\| \le \gamma h^2|\bar{u}|.$$

Remark 3: Notice that in the $2-$DOF case, we always have $m<m_{11}-1$. In the $1-$DOF case, we will have $m<m_{11}-1$ if, in particular, the relative degree of the weight W is larger than $2n_P$; since we are usually interested in tracking low frequency signals, we can achieve this by rolling off W at high frequency if need be.

At this point we briefly outline how the above lemma will prove useful. First, we measure

$$\tilde{y}(kT)\in\mathbf{R}^{\tilde{m}}.$$

With ρ a scaling constant, we first set

$$\tilde{u}_1(t) = \begin{cases} \rho\tilde{y}_1(kT) & t\in[kT, kT+(m+1)h), \\ -\rho\tilde{y}_1(kT) & t\in[kT+(m+1)h, \\ & \quad kT+2(m+1)h), \end{cases}$$

(we set the other elements of $\tilde{u}$ to zero). Using the above lemma, at the end of $2(m+1)h$ time units we have a good estimate of

$$p\tilde{y}_1(kT).$$

We can repeat the procedure for each input, so at $t = kT + \tilde{m}2(m+1)h$ we have a good estimate of

$$p\tilde{y}(kT),$$

which contains $m\tilde{m}$ pieces of data. Of course, we can repeat this procedure as many times as we wish: we can create an estimate of the form

$$\phi(p)\tilde{y}(kT),$$

with ϕ a polynomial in its arguments.

In the next section we show that there is a near optimal controller of the form

$$\tilde{u} = \phi(p)\tilde{y}$$

with ϕ a polynomial.

B. Parametrization of Near Optimal Controllers

By "regularizing" the plant model and examining the associated Hamiltonian equation [1], we can prove the following:

Proposition 1: For every $\varepsilon > 0$, there exists a $\sigma > 0$ and a function $\tilde{K}$ from S to $\mathbf{R}^{\tilde{r}\times\tilde{m}}$ which is defined and continuous on S, and with the property that for every $p \in S$, it stabilizes $\tilde{G}(p)$ and yields the following level of performance:

$$\|T_{r,z}(\tilde{G}(p), \tilde{K}^{\varepsilon}(p)\| \leq \alpha_{opt} + \varepsilon.$$

By the Stone-Weierstrass Approximation Theorem, for every $\delta > 0$ there exists a polynomial $\tilde{K}^{\varepsilon}_{\delta}$ satisfying

$$\|\tilde{K}^{\varepsilon}_{\delta}(p) - \tilde{K}^{\varepsilon}(p)\| \leq \delta, \ \ p \in S.$$

Proposition 2: There exist constants $\bar{\delta} > 0$, $\gamma_0 > 0$, and $\lambda_0 < 0$ so that for every $\delta \in [0, \bar{\delta})$ and $p \in S$:

$$\|e^{(\tilde{A}+\tilde{B}_2\tilde{K}^{\varepsilon}_{\delta}(p)\tilde{C}_2)t}\| \leq \gamma_0 e^{\lambda_0 t}, \quad t \geq 0,$$
$$\|T_{r,z}(\tilde{G}(p), \tilde{K}^{\varepsilon}(p)) - T_{r,z}(\tilde{G}(p), \tilde{K}^{\varepsilon}_{\delta}(p))\| \leq \gamma_0\delta\|r\|_2.$$

Now freeze $\varepsilon > 0$ and $\sigma > 0$, choose $\bar{\delta} > 0$, $\gamma_0 > 0$, and $\lambda_0 < 0$ that Proposition 2 asserts to exist, and **freeze** $\delta \in (0, \bar{\delta})$ **so that** $\gamma_0\delta < \varepsilon$; clearly

$$\|T_{r,z}(\tilde{G}(p), \tilde{K}^{\varepsilon}_{\delta}(p))\| \leq \alpha_{opt} + 2\varepsilon,$$

$$sp[\underbrace{\tilde{A} + \tilde{B}_2\tilde{K}^{\varepsilon}_{\delta}(p)\tilde{C}_2}_{=:\tilde{A}^{\varepsilon}_{cl}(p)}] \subset \mathbf{C}^{-}, \quad p \in \mathcal{S}.$$

We finish out the section with a subsection on the notation needed to represent the associated polynomial in such a form that an associated LPC can be implemented.

C. Polynomial Notation

Following [5] and [6], it is convenient at this point to parametrize our polynomial approximation in such a way that we can estimate the various terms in a straight-forward and systematic fashion. With

$$\tilde{y}(t) \otimes^0 p := \tilde{y}(t),$$

$$\tilde{y}(t) \otimes^{i+1} p := (\tilde{y}(t) \otimes^i p) \otimes p, \ \ i \in \mathbf{N},$$

it turns out that there exists an integer q and row vectors d_i of length $\tilde{m}_i := \tilde{m}m^i$ so that we can rewrite the near optimal control law as

$$\tilde{u}^{\varepsilon}(t) = \tilde{K}^{\varepsilon}_{\delta}\tilde{y}(t) = \sum_{i=0}^{q} d_i(\tilde{y}(t) \otimes^i p). \tag{15}$$

We can use the KEL to estimate this quantity.

IV. The Controller

Here we adopt the notation from the previous section and combine it with the KEL to design an algorithm to implement our proposed control law, the general operation of which was briefly discussed at the end of Section 2. This proposed control law is periodic of period T; we begin by describing its open loop behavior on a period of the form $[kT, (k+1)T)$. Here T is partitioned into $\ell >> m$ chunks of length h. Our near optimal LTI controller is (15); we would like to approximate this using a periodic sampled data controller. Following the discussion after the KEL, we measure $\tilde{y}(kT)$ and successively estimate terms of the form

$$\tilde{y}(kT) \otimes^i p \in \mathbf{R}^{m_i}, \ \ i = 1, ..., q;$$

it is easy to see that this will take $2\tilde{m}_{i-1}(m+1)h$ units of time. To this end, we define certain important points in time:

$$T_1 := 0,$$

$$T_{i+1} = T_i + 2\tilde{m}_{i-1}(m+1)h, \ \ i = 1, ..., q.$$

The idea is that on the interval $[kT + T_i, kT + T_{i+1})$ we estimate $\tilde{y}(kT) \otimes^i p$. Last of all, with $T := \ell h > T_{q+1}$ an integer multiple of h, on the interval $[kT + T_{q+1}, (k+1)T)$ we implement the Control Phase. With this in mind, we can now write down our proposed controller, presented in open loop form. To make this more transparent, we partition each interval $[T_i, T_{i+1})$, $i = 1, ..., q$, into $\tilde{m}_{i-1}$ consecutive subintervals of length $2(m+1)h$ on which probing takes place:

$$\begin{aligned}
[T_1, T_2) &= [T_{1,1}, T_{1,2}) \cup \cdots \cup [T_{1,\tilde{m}_0}, T_{1,\tilde{m}_0+1}) \\
[T_2, T_3) &= [T_{2,1}, T_{2,2}) \cup \cdots \cup [T_{2,\tilde{m}_1}, T_{2,\tilde{m}_1+1}) \\
&\vdots \\
[T_q, T_{q+1}) &= [T_{q,1}, T_{q,2}) \cup \cdots \\
&\qquad \cup [T_{q,\tilde{m}_{q-1}}, T_{q,\tilde{m}_{q-1}+1}).
\end{aligned}$$

Before proceeding, we fix $\rho > 0$.

THE PROPOSED CONTROLLER ($t_0 = 0$)

Initialization:
Define

$$\text{Est}[\tilde{y}(kT) \otimes^0 p] := \tilde{y}(kT). \quad (16)$$

Estimation Phase: $[kT + T_1, kT + T_{q+1})$
For $i = 1, ..., q$ and $j = 1, ..., \tilde{m}_{i-1}$, we set

$$u(t) = \begin{cases} \begin{bmatrix} \rho \text{Est}[\tilde{y}(kT) \otimes^{i-1} p]_j \\ 0 \end{bmatrix}, \\ \quad t \in [kT + T_{i,j}, kT + T_{i,j} + (m+1)h), \\ \begin{bmatrix} -\rho \text{Est}[\tilde{y}(kT) \otimes^{i-1} p]_j \\ 0 \end{bmatrix}, \\ \quad t \in [kT + T_{i,j} + (m+1)h, kT + T_{i,j+1}), \end{cases} \quad (17)$$

and we define

$$\text{Est}[\tilde{y}(kT) \otimes^i p] := \frac{1}{2\rho}\text{diag}\{\bar{U}_m(\sigma)V_m(h)^{-1}, ..., \bar{U}_m(\sigma)V_m(h)^{-1}\} \times \begin{bmatrix} \tilde{\mathcal{Y}}_m(kT + T_{i,1}) \\ \vdots \\ \tilde{\mathcal{Y}}_m(kT + T_{i,\tilde{m}_{i-1}}) \end{bmatrix}. \quad (18)$$

Control Phase: $[kT + T_{q+1}, (k+1)T)$.

$$u(t) = \frac{T}{T - T_{q+1}} \sum_{i=0}^{q} d_i \;\; \text{Est}[\tilde{y}(kT) \otimes^i p], \quad t \in [kT + T_{q+1}, (k+1)T). \quad (19)$$

It turns out that this controller can be written in a more compact form.

Lemma 3: (Representation Lemma) The control law described by (16)-(19) has a representation of the form (12) given by $(F_d, H_d, J_d, L_d, h, \ell)$ which is deadbeat; in fact, $F_d(0) = 0$.

V. The Main Result

Theorem 1: Suppose that $m < m_{11} - 1$. Then for every $\rho > 0$ there exists a controller K of the form (6), (12), and (13) which stabilizes $\{G(p) : p \in S\}$, and which, for every $p \in S$, yields

$$\|T_{r,z}(G(p), K)\| \leq \alpha_{opt} + \rho.$$

Proof:

The proof shows that the controller constructed above will achieve the objective if $\sigma > 0$ is chosen large enough and $T > 0$ and $\varepsilon > 0$ are chosen small enough.

QED

Notice that in the $2-$DOF case, we always have $m < m_{11} - 1$. In the $1-$DOF case, we will have $m < m_{11} - 1$ if, in particular, the relative degree of the weight W is larger than $2n_P$, which can be achieved by rolling off W at high frequency if need be.

Because of the linearity of the controller we automatically get some tolerance to unmodeled dynamics. Since we are computing the control signal from scratch on each period, we would also expect that the approach should tolerate a degree of slow time variations in the parameters.

VI. Summary and Conclusions

In this paper we have considered the problem of providing near optimal H_∞ performance in the presence of structured uncertainty in the plant model. We consider the case of the set of possible models lying in a compact subset of the set of all controllable and observable models of a fixed order; we consider two different feedback configurations. We consider the control problem of designing a controller to minimize the worst case performance. This is always solvable in the 2-DOF case, and is solvable in the 1-DOF case if the pre-filter has a high enough relative degree. In all cases the proposed controller is linear periodic, its action when applied to a particular admissible model is very close to that provided by an associated sub-optimal LTI controller, and the approach adopted naturally allows for slow time-variations in the plant model. We are presently trying to improve the controller performance so that pointwise near optimal performance is achieved.

References

[1] J.C. Doyle,, K. Glover, P.P. Khargonekr, and B.A. Francis, "State Space Solutions to Standard H_2 and H_∞ Control Problems", *IEEE Transactions on Automatic Control*, vol. AC – 34, pp. 831 – 847, 1989.

[2] M. Fu and B.R. Barmish, "Adaptive Stabilization of Linear Systems Via Switching Control", *IEEE Transactions on Automatic Control*, vol. AC-31, pp. 1097-1103, Dec. 1986.

[3] G.C. Goodwin and K.S. Sin, "Adaptive Filtering Prediction and Control, Prentice Hall, Englewood Cliffs, New Jersey, USA, 1984.

[4] J. Hocherman-Frommer, S.R. Kulkarni and P.J. Ramadge, "Controller Switching Based on Output Prediction Errors", *IEEE Transactions on Automatic Control*, **AC-43**, pp. 596 – 607, 1998.

[5] D.E. Miller, "Near Optimal LQR Performance for a Compact Set of Plants", *Proceedings of the IEEE 2003 Conference on Decision and Control*, Maui, Hawaii, pp. 1807 – 1812, Dec. 2003.

[6] D.E. Miller "Near Optimal LQR Performance for a Compact Set of Plants", Technical Report UW-E&CE#2004-22, Dept. of Electrical and computer Engineering, University of Waterloo, Ontario, Canada, Sept. 24, 2004. This can be found on the web-site http://kingcong.uwaterloo.ca/˜miller/

[7] A.S. Morse, "Supervisory Control of Families of Linear Set-Point Controllers - Part 1: Exact Matching", *IEEE Transactions on Automatic Control*, **AC-41**, pp. 1413 – 1431, 1996.

[8] K.S. Narendra and J. Balakrishnan, "Adaptive Control Using Multiple Models", *IEEE Transactions on Automatic Control*, **AC-42**, pp. 171 – 187, 1997.

[9] W. Rudin, "Functional Analysis", McGraw-Hill, New York, USA, 1973.

[10] G. Zames and B. Francis, "Feedback, minimax sensitivity, and optimal robustness", *IEEE Transactions on Automatic Control*, AC-28, pp. 585 – 601, 1983.

[11] K. Zhou, J.C. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1996.