

Agreement in presence of noise: pseudogradients on random geometric networks

Yuko Hatano, Arindam K. Das, and Mehran Mesbahi

Abstract—We consider the agreement problem over realizations of a (Poisson) random geometric network with noisy interconnections. The vertices of random geometric networks are assumed to be uniformly distributed on the unit square; an edge exists between a pair of vertices if the distance between them is less than or equal to a given threshold. Our treatment of the agreement problem in such a setting relies upon notions from stochastic stability. In this venue, we show that the noisy agreement protocol has a guaranteed convergence with probability one, provided that an embedded step size parameter meets certain constraints. These constraints turn out to be closely related to the spectra of the underlying graph Laplacian. Moreover, we point out the ramifications of having noisy networks by establishing connections between rate of convergence of the protocol and the range threshold in random geometric graphs.

Index Terms—Agreement problem, random geometric graphs, stochastic stability

I. INTRODUCTION

In recent years coordination problems for distributed dynamic systems have attracted the attention of many researchers in the systems and control community. The array of disciplines with a vested interest in studying these systems not only includes system and control theory, but also biology, computer sciences, statistical physics, and natural and social sciences. One of the basic, yet fundamental, problem that resurfaces from time to time in such distributed settings is the agreement problem and its convergence properties. In the agreement problem setup, all vertices of a network are eventually required to assume a particular value of interest. This value assumes distinct interpretations depending on the context: it can represent the common attitude in multiple spacecraft alignment, heading direction in flocking behavior, rendezvous of multiple vehicles, averaging in distributed computation, or the steady state of Markov chains. An agreement *protocol* provides means by which such agreement can be achieved [13], [19]. When the protocol is built on the available network for inter-element information exchange, the structure of the network—often abstracted in terms of graphs—naturally enters the convergence analysis [3], [6], [11], [19]. In this venue, even the theory of random graphs [2] has entered the analysis of the agreement protocol in situations where the underlying network can be modeled as

random dynamic networks [10]. The setup employed in [10] necessitated a probabilistic approach for the analysis of uncertain networked dynamic systems that was based on notions from stochastic stability. The present paper contributes to another facet of the research on the distributed agreement problem in a stochastic setting— that being the convergence properties of the protocol over noisy networks. We pursue such an analysis for networks that are realizations of random geometric graphs [15]. In this direction, we make several observations on the rate of convergence of the corresponding protocol that uses elementary notions from algebraic graph theory. In fact, one of the main contributions of the present paper is to highlight the role of the graph Laplacian in the convergence properties of the noisy agreement protocols.

Random geometric graphs have deep and interesting connections with geometric probability [15]. Such networks have found a variety of applications in wireless ad-hoc and sensor networks, biological networks, and social networks. The most studied class of random geometric graphs are referred to as proximity graphs, where vertices of the graph are randomly placed in d -dimensional space with some probability distribution, and an edge is added between two vertices if their Euclidean norm is within a given threshold. In the context of coordination over sensor networks, proximity graphs provide a viable model for the underlying relative sensing/communication topology. This model for example allows one to consider the required transmitters' power for reaching the corresponding receivers in order to maintain connectivity for the entire network [9].

The outline of the paper is as follows. In §II notions from graph theory, algebraic graph theory, and random geometric graphs are reviewed. The basic problem setup is then delineated upon in §III. In §IV, we provide an analytic framework for studying agreement protocols operating on noisy networks for fixed and random geometric networks. This will be accomplished via the general approach that is based on pseudogradients and stochastic convergence— this framework has a rich history in the convergence analysis of noisy numerical methods [4], [17]. A representative set of simulations results along with the insights that they provide conclude our presentation in §V.

II. PRELIMINARIES: GRAPHS, RANDOM GEOMETRIC GRAPHS, AND RANDOM NUMERICAL SEQUENCES

We consider graphs $G = (V, E)$ consisting of a set of vertices (or nodes) V , in our case $[n] := \{1, 2, \dots, n\}$, and a set of edges, E , representing an incidence relation between pairs of nodes. Graphs admit convenient algebraic representations

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in terms of matrices. For example, the adjacency matrix of a simple graph G , denoted by $A(G)$, is a $n \times n$ binary matrix such that $[A(G)]_{ij} = 1$ if there is an edge between nodes i and j in G , and $[A(G)]_{ij} = 0$, otherwise. The adjacency matrix for simple undirected graphs is symmetric and its diagonal elements are all equal to 0. The degree matrix of G , $D(G) = [D_{ij}]$, is a diagonal matrix such that $D_{ii} = \delta_i$, where δ_i is the degree of node i . Finally, the Laplacian matrix of a graph G , denoted by $L(G)$ is an $n \times n$ matrix defined as $L(G) = D(G) - A(G)$. We note that for simple connected undirected graphs, $L(G)$ is positive semi-definite with row and column sums equal to 0; moreover, $\text{rank}(L(G)) \leq n - 1$. In fact, when G has r connected components, the rank of its Laplacian is equal to $n - r$ [8]. The ordered (real) spectrum of a graph Laplacian is then of the form $0 = \lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G)$, with $\lambda_2(G) > 0$ if and only if the graph is connected; for a complete graph K_n , $\lambda_2(G) = \lambda_n(G) = n$.

A. Random geometric graphs in the plane

Consider n independent and identically distributed points in $[0, 1]^2$, and let r be a positive real number which will be referred to as the range or radius of a node. Let z_i denote the position of node i in the plane with respect to some given reference frame. A random geometric graph on $[0, 1]^2$, denoted by $G(n, r)$, is defined by the node set $V = [n]$ and the edge set $E = \{ij : i, j \in [n], i \neq j\}$ such that $ij \in E$ if $\|z_i - z_j\|_2 \leq r$, with $\|\cdot\|_2$ denoting the Euclidean norm. A random geometric graph on a Poisson point processes is known as a Poisson geometric graph. By definition, the following properties hold for a Poisson geometric graph: (1) for every region S in the unit square $[0, 1]^2$, the number of nodes in S , $N(S)$, is Poisson distributed with parameter $n|S|$, where n is the total number of nodes in $[0, 1]^2$ and $|S|$ is the area of the region S , i.e., $\Pr\{N(S) = k\} = ((n|S|)^k / k!) e^{-n|S|}$, and (2) for every finite collection of disjoint regions in the unit square $[0, 1]^2$, $\{S_1, S_2, \dots, S_m\}$, the random variables $N(S_1), N(S_2), \dots, N(S_m)$ are independent. The above properties can be used to derive the distribution of node degrees in a Poisson geometric graph, as expressed by the following lemma [15].

Lemma 2.1: Consider a Poisson random geometric graph $G(n, r)$ on $[0, 1]^2$. Assume that $n \gg 1$ and $\pi r^2 \ll 1$. Let δ_i denote the degree of node i , located in a subregion S of area πr^2 . Then, δ_i is Poisson distributed with parameter $n\pi r^2$, i.e., for all i

$$\Pr\{\delta_i = \delta\} = \frac{(n\pi r^2)^\delta}{\delta!} e^{-n\pi r^2}, \quad (1)$$

and therefore, for all i , $\mathbf{E}\{\delta_i\} = n\pi r^2$, where \mathbf{E} denotes the expected value operator.

For a random geometric graph the expected value of the vertex degree proves to be useful in bounding the noise variance effecting the node updates in the agreement protocol. We denote this expected value by $\Delta(G)$. However, once a random geometric graph is realized, we adopt the convention

of setting $\Delta(G)$ to be equal to the maximum vertex degree in G instead.

Our next observation pertains to probabilistic connectivity of Poisson geometric graphs as a function of the node range r . Connectivity of random graphs has been studied by several researchers; see for example [1], [9], [15], [16], and [21]. The following lemma has been adopted from [22].

Lemma 2.2: Consider a Poisson random geometric graph $G(n, r)$ on the plane. For any real number α , one has

$$\Pr\{G(n, r) \text{ is connected}\} \geq e^{-e^{-\alpha}} \quad (2)$$

when $n\pi r^2 > \ln n + \alpha$.

Lemma 2.2 provides a guideline on choosing the node range threshold r to meet a desired probability of connectivity. For example, if we want the probability of connectivity to be at least 0.99, we can use the above results to set $\alpha = 4.7$. For a 10-node network, $r > 0.473$ would suffice; for 20 node, one must have $r > 0.35$.

B. Convergence of random sequences

The convergence of the agreement protocol can naturally be studied in the general framework of convergence of numerical sequences. One of the fundamental observations on such sequences- one that has been utilized in many different contexts- relates to the convergence of decreasing yet bounded sequences. This observation has a stochastic analogue in the theory of supermartingales [7], which is deceptively simple to state. First, recall that a random sequence $\{V(k)\}_{k \geq 0}$ converges to another random variable V^* (w.p.1) (i.e., with probability one), if for every $\epsilon > 0$,

$$\Pr\{\sup_{k \geq N} \|V(k) - V^*\| \geq \epsilon\} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

On the other hand $\{V(k)\}_{k \geq 0}$ converges in the mean if the sequence $\{\mathbf{E}\{V(k)\}\}_{k \geq 0}$ converges to (a constant) V^* . Suppose now that the sequence of nonnegative random variables $\{V(k)\}_{k \geq 0}$ is such that $\mathbf{E}\{V(k) | V(0), \dots, V(k-1)\} \leq V(k)$ and $\mathbf{E}\{V(0)\} < \infty$; such a sequence is called a nonnegative supermartingale. If $\{V(k)\}_{k \geq 0}$ is a nonnegative supermartingale, then there exists a random variable $V^* \geq 0$ such that $V(k) \rightarrow V^*$ (w.p.1). The power and utility of this result has been recognized by many researchers in control and optimization. One of the useful consequences of this supermartingale convergence theorem is the following result (see Lemma 10 in §2.2 of [17]).

Lemma 2.3: Consider the sequence of nonnegative random variables $\{V(k)\}_{k \geq 0}$ with $\mathbf{E}\{V(0)\} < \infty$. Let

$$\mathbf{E}\{V(k+1) | V(0), \dots, V(k)\} \leq (1 - c_1(k)) V(k) + c_2(k),$$

where

$$0 \leq c_1(k) \leq 1, \quad c_2(k) \geq 0, \quad \sum_{k=0}^{\infty} c_2(k) < \infty,$$

and $\sum_{k=0}^{\infty} c_1(k) = \infty$ and $c_2(k)/c_1(k) \rightarrow 0$. Then $V(k) \rightarrow 0$ (w.p.1).

Proof: We just provide a sketch of the proof; the reader is referred to [17] for details. First, construct the auxiliary

nonnegative sequence $U(k) := V(k) + \sum_{i=k}^{\infty} c_2(k)$. Then using the first three conditions on $c_1(k)$ and $c_2(k)$, it can be shown that $\{U(k)\}_{k \geq 0}$ forms a nonnegative supermartingale and thus its convergence to the some nonnegative random variable (w.p.1). However since the sequence $\{c_2(k)\}_{k \geq 0}$ is summable, convergence (w.p.1) of the sequence $\{V(k)\}_{k \geq 0}$ also follows. The final step involves employing the last two conditions on $c_1(k)$ and $c_2(k)$ to ensure that the limiting value of the sequence $\{V(k)\}_{k \geq 0}$ is the origin (w.p.1). ■

III. PROBLEM SETUP

Our setup involves a distributed dynamic system consisting of n elements, labeled $1, 2, \dots, n$, interconnected via a noisy information network. Let $x_i(k)$ denote the state of node i at time step k . We assume that initial states, $\{x_i(1), i \in V\}$, are deterministic but unknown. It is also possible for the initial states to be drawn from some probability distribution with finite mean and variance. We now study the evolution of the system in discrete time as,

$$x_i(k+1) = x_i(k) - \gamma(k) u_i(k), \quad k = 1, 2, \dots, \quad (3)$$

where $\gamma(k) > 0$ is a time-varying step size parameter and $u_i(k)$ is the control input or update direction at time k . We assume that at time k , node i receives a noisy measurement of its relative state with respect to its neighbors, and computes its control input as,

$$u_i(k) = \sum_{j \in N(i)} (x_i(k) - x_j(k)) + \eta_{ji}(k), \quad (4)$$

where $N(i)$ denotes the set of neighbors to node i , and $\eta_{ji}(k)$ is the random noise on the link $j \rightarrow i$ at time k . We assume that all η_{ij} 's are independent, uncorrelated and Gaussian distributed with zero mean and variance σ^2 . The control input (4) can be written in terms of the Laplacian and adjacency matrices of the network graph as,

$$u_i(k) = \sum_{j=1}^n [L(G)]_{ij} x_j(k) + \sum_{j=1}^n [A(G)]_{ji} \eta_{ji}(k). \quad (5)$$

Note that the second term on the right hand side of (5) is the total noise input at node i at time k ; denote this term by $w_i(k)$. We observe that for all i and k , $w_i(k)$'s are zero mean, independent, uncorrelated Gaussian distributed random variables. Let M be a positive integer-valued random variable. For independent, identically, distributed (i.i.d.) random variables Y_1, Y_2, \dots , with mean μ and variance σ^2 , the mean and variance of $Y := Y_1 + Y_2 + \dots + Y_M$ are $\mathbf{E}[Y] = \mu \mathbf{E}\{M\}$ and

$$\mathbf{var}\{Y\} = \mu^2 \mathbf{var}\{M\} + \sigma^2 \mathbf{E}\{M\}. \quad (6)$$

For the Poisson random geometric model of §II.A, since all η_{ji} 's are Gaussian distributed with zero mean and variance σ^2 , it follows from (6) that for all k , $\mathbf{var}[w_i(k)] = \sigma^2 \Delta(G)$. For a realization of random geometric graphs, one can bound the total variance on each node as

$$\mathbf{E}[|w_i(k)|^2] \leq \sigma^2 \Delta(G) \quad \text{for all } i \text{ and } k,$$

where in this case $\Delta(G)$ is the maximum node degree in G . The protocol dynamics can now be expressed as

$$x(k+1) = x(k) - \gamma(k) u(k) \quad (7)$$

with

$$u(k) := L(G)x(k) + w(k)$$

and

$$w(k) := [w_1(k), \dots, w_n(k)]^T.$$

In this paper we consider the convergence properties of the protocol (7) with respect to agreement set.

Definition 3.1: The agreement set $\mathcal{A} \subseteq \mathbb{R}^n$ is the subspace $\text{span}\{\mathbf{1}\}$. We will write $\{x(k)\} \rightarrow \mathcal{A}$ (w.p.1) if for every $\epsilon > 0$,

$$\Pr \left\{ \sup_{k \geq N} \mathbf{dist}(x(k), \mathcal{A}) \geq \epsilon \right\} \rightarrow 0, \quad \text{as } N \rightarrow \infty,$$

where $\mathbf{dist}(x, \mathcal{A}) := \inf_{z \in \mathcal{A}} \|x - z\|$.

It is apparent from (7) that convergence (or lack thereof) of the agreement protocol is dependent on the choice of the step size, $\gamma(k)$. For fixed $\gamma(k) = \gamma$ the variance of the state for any time step can not be less than $\gamma \sigma^2 \Delta(G)$ for the fixed/random geometric graph G . In this case, the agreement protocol would fail to converge (w.p.1). For a given $\gamma(k)$, it is interesting to note that a higher average node degree implies higher variance of $u_i(k)$'s, and thereby, slower convergence of the agreement protocol. This is in sharp contrast to the noise-free case where a higher node degree implies greater information sharing, leading to faster convergence. In fact, only one iteration is all that is needed to achieve agreement if all node degrees are equal to $n-1$; this node degree corresponds to a complete graph. In presence of noise, higher node degrees have both a beneficial effect (faster mixing of state information) and a detrimental effect (higher noise variance). Since the average node degree is proportional to the range r in a random geometric graph (see Lemma 2.1), we postulate the existence of a threshold range, r_T , beyond which the detrimental effect of further increasing r negates its beneficial effect, and no further improvement in the convergence rate is achieved. While we will not provide an analytical expression for r_T in the present paper, we do provide empirical evidence that validates this statement to the effect that a higher r does not necessarily imply faster convergence in noisy information networks. This issue will be discussed further in §V.

One way via which the variance of the state vector can be driven to zero is to adopt a time-varying step size parameter in the agreement protocol, satisfying

$$\lim_{k \rightarrow \infty} \gamma(k) = 0, \quad \sum_{k=1}^{\infty} \gamma(k) = \infty, \quad \sum_{k=1}^{\infty} \gamma^2(k) < \infty. \quad (8)$$

Note that the second condition is necessary to allow for a sufficient number of updates for the protocol to converge [4]. These conditions are necessary yet not sufficient for convergence. As we will see, an additional condition on $\gamma(k)$ is required to ensure convergence in noisy networks. It turns out that this constraint is intimately related to the spectra of the underlying graph Laplacian.

IV. AGREEMENT IN A STATIC AND A RANDOM GEOMETRIC INITIAL GRAPH WITH NOISY LINKS

In this section we consider the probabilistic convergence of the agreement protocol (7) in presence of noise. One of the crucial steps for such an analysis is to examine the notion of pseudogradient. Let

$$V(k) = V(x(k)) := \frac{1}{2} x(k)^T L(G) x(k). \quad (9)$$

We now note that the process (7) is a strong pseudogradient [17], [18] with respect to this choice of the quadratic function.

Proposition 4.1: For a connected network, $u(k) = L(G)x(k) + w(k)$ is a strong pseudogradient of $V(k)$ in (9), i.e.,

$$\nabla V(k)^T \mathbf{E} \{ u(k) | x(k) \} \geq \beta V(k), \quad (10)$$

where $\beta > 0$.

Proof: First observe that $\mathbf{E}\{u(k) | x(k)\} = \mathbf{E}\{L(G)x(k) + w(k) | x(k)\} = L(G)x(k)$, since $\mathbf{E}\{w(k) | x(k)\} = \mathbf{E}\{w(k)\} = 0$. For (10) to hold it suffices to ensure that

$$x(k)^T \left[L(G)^2 - \frac{\beta}{2} L(G) \right] x(k) \geq 0,$$

for all $x(k)$. The last inequality will certainly hold if $L(G)^2 - (\beta/2) L(G)$ is positive semi-definite for some $\beta > 0$. Since the eigenvalues of $L(G)^2$ are given by $\{\lambda_i^2 : 1 \leq i \leq n\}$, where λ_i is the i -th smallest eigenvalue of $L(G)$, and $\lambda_1(G) = 0$, it follows that $L(G)^2 - (\beta/2) L(G)$ is positive semidefinite when $\beta \leq 2\lambda_2(G)$. Since $\lambda_2(G) > 0$ for a connected graph, the strong pseudogradient condition (10) is satisfied for any $\beta \in (0, 2\lambda_2(G)]$. ■

The main result of this section is as follows.

Proposition 4.2: For a connected network, the trajectory of the system (7) converges to the agreement set \mathcal{A} (w.p.1) if conditions in (8) hold and for all k , $\gamma(k) \leq 2/\lambda_n(G)$.

Proof: Using the quadratic function (9), one has

$$\begin{aligned} V(k+1) &= \frac{1}{2} x(k+1)^T L(G) x(k+1) \\ &= \frac{1}{2} (x(k) - \gamma(k) u(k))^T L(G) (x(k) - \gamma(k) u(k)) \\ &= \frac{1}{2} x(k)^T L(G) x(k) - \gamma(k) x(k)^T L(G) u(k) \\ &\quad + \frac{\gamma^2(k)}{2} u^T(k) L(G) u(k) \\ &\leq V(k) - \gamma(k) \nabla V(k)^T u(k) + \frac{\gamma^2(k)}{2} \lambda_n(G) \|u(k)\|^2. \end{aligned}$$

Concurrently,

$$\begin{aligned} &\mathbf{E} \{ \|u(k)\|^2 | x(k) \} \\ &= \mathbf{E} \{ (L(G)x(k) + w(k))^T (L(G)x(k) + w(k)) | x(k) \} \\ &= x(k)^T L(G)^2 x(k) + \mathbf{E} \{ w^T(k) w(k) \} \\ &= x(k)^T L(G)^2 x(k) + \sum_i \mathbf{var} \{ w_i(k) \}, \end{aligned}$$

which is bounded by $\nabla V(k)^T \mathbf{E} \{ u(k) | x(k) \} + n\sigma^2 \Delta(G)$. Hence,

$$\begin{aligned} &\mathbf{E} \{ V(k+1) | x(k) \} \\ &\leq V(k) - \gamma(k) \nabla V(k)^T \mathbf{E} \{ u(k) | x(k) \} \\ &\quad + \frac{\lambda_n(G) \gamma^2(k)}{2} \mathbf{E} \{ \|u(k)\|^2 | x(k) \} \\ &= V(k) + c_2(k) \\ &\quad - \left(\gamma(k) - \frac{\lambda_n(G) \gamma^2(k)}{2} \right) \nabla V(k)^T \mathbf{E} \{ u(k) | x(k) \}, \end{aligned}$$

where $c_2(k)$ is defined as

$$c_2(k) := \frac{n\sigma^2 \gamma(k)^2}{2} \lambda_n(G) \Delta(G). \quad (11)$$

We now invoke the strong pseudogradient property (10) to observe that

$$\begin{aligned} &\mathbf{E} [V(k+1) | x(k)] \\ &\leq \left(1 - \beta\gamma(k) + \frac{\beta\gamma^2(k) \lambda_n(G)}{2} \right) V(k) + c_2(k) \\ &= (1 - c_1(k)) V(k) + c_2(k) \end{aligned} \quad (12)$$

where

$$c_1(k) := \beta\gamma(k) \left(1 - \frac{\gamma(k) \lambda_n(G)}{2} \right). \quad (13)$$

It is now straightforward to show that if $\gamma(k)$ satisfies the conditions in (8) and for all k , $\gamma(k) \leq 2/\lambda_n(G)$, $c_1(k)$ and $c_2(k)$ satisfy

$$0 \leq c_1(k) \leq 1, \quad c_2 \geq 0, \quad \sum_{k=1}^{\infty} c_1(k) = \infty, \quad \sum_{k=1}^{\infty} c_2(k) < \infty,$$

and $\lim_{k \rightarrow \infty} c_2(k)/c_1(k) = 0$. These conditions in turn allow us to invoke Lemma 2.3 to conclude that $V(k) \rightarrow 0$ (w.p.1) (as well as in the mean). ■

The utility of Proposition 4.2 in the context of convergence to the agreement subspace is facilitated by the following observations. First, we remark that since $\mathbf{E} \{ w(k) \} = 0$ the expected value of the limiting state is also the mean of the initial states of the elements in the network. In particular, for all $k \geq 1$

$$\begin{aligned} \mathbf{1}^T x(k+1) &= \mathbf{1}^T x(k) - \gamma(k) \mathbf{1}^T (Lx(k) + w(k)) \\ &= \mathbf{1}^T x(k) - \gamma(k) \mathbf{1}^T w(k), \end{aligned}$$

and hence for all $k \geq 0$,

$$\mathbf{E} \{ \mathbf{1}^T x(k+1) \} = \mathbf{E} \{ \mathbf{1}^T x(k) \}.$$

This remark can then be used to prove the following.

Proposition 4.3: Let G be a connected graph. If for the quadratic function $V(x) = x^T L(G) x$ the trajectories of the system satisfy $V(x(k)) \rightarrow 0$ (w.p.1), then $\mathbf{dist}(x(k), \mathcal{A}) \rightarrow 0$ (w.p.1). Thus, under the condition of Proposition 4.2 one has $x(k) \rightarrow \mathcal{A}$ (w.p.1).

We note that for the agreement protocol to converge, the step size $\gamma(k)$ not only needs to satisfy conditions in (8),

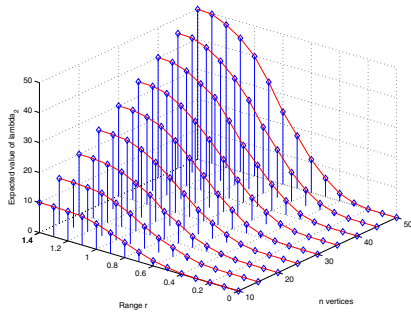


Fig. 1. Expected value of $\lambda_2(n, r)$ of 100 graphs for each n and r ; $10 \leq n \leq 50$ and $0 \leq r \leq 1.4$.

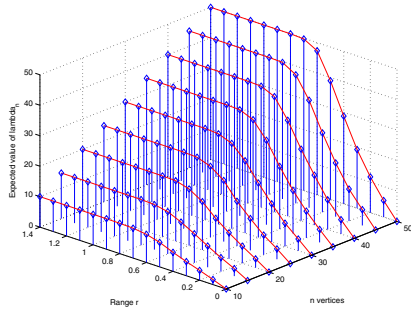


Fig. 2. Expected value of $\lambda_n(n, r)$ of 100 graphs for each n and r ; $10 \leq n \leq 50$ and $0 \leq r \leq 1.4$.

but also that of $\gamma(k) \leq 2/\lambda_n(G)$. A function which satisfies both sets of conditions is,

$$\gamma_1(k) = k^{-1} \left(\frac{2}{\lambda_n(G)} - \epsilon \right), \quad 0 < \epsilon < \frac{2}{\lambda_n(G)}, \quad (14)$$

since $\sum_{k=1}^{\infty} k^{-1} = \infty$, and $\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6} < \infty$. Alternately, one can use

$$\gamma_2(k) = \begin{cases} \frac{2}{\lambda_n(G)} - \epsilon, & \text{if } k \leq \lceil \frac{\lambda_n(G)}{2} \rceil \\ \frac{1}{k}, & \text{otherwise,} \end{cases} \quad (15)$$

where $\lceil x \rceil$ denotes the least integer upper bound for $x \in \mathbb{R}$.

V. SIMULATION RESULTS AND RATE OF CONVERGENCE

The inequality (12) points to the fact that besides the initial state distribution, the rate of convergence of the agreement protocol is dictated by the quantities $c_1(k)$ and $c_2(k)$. These quantities depend on (see §IV): (1) graph parameters, n and r , (2) the Laplacian eigenvalues, $\lambda_2(G)$ and $\lambda_n(G)$ (see Figs. 1,2), (3) the noise variance σ^2 ; the higher the variance, the slower is the rate of convergence, and finally, (4) the step size $\gamma(k)$. If the function (15) is chosen as the step size, it is also clear from (13) that the quantity $\prod_{j=1}^k [1 - c_1(j)]$ decays as $O(1/k)$ for sufficiently large k and dominates the $O(1/k^2)$ decay rate of $c_2(k)$ (11)- all other parameters kept constant. Even though the rate of convergence of the protocol is predominately dictated by $\gamma(k)$, it is still instructive to study the effect of n , r , and the Laplacian eigenvalues, on the rapidity of its convergence.

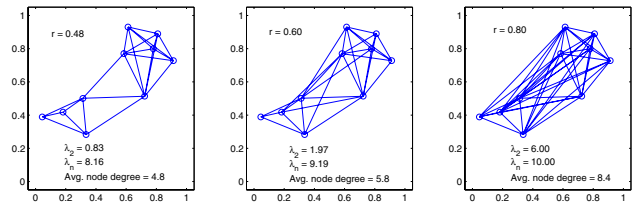


Fig. 3. Connectivity of a 10-node realization of random geometric graphs $G(10, r)$, for (from left to right) $r = 0.48$, $r = 0.65$ and $r = 0.80$.

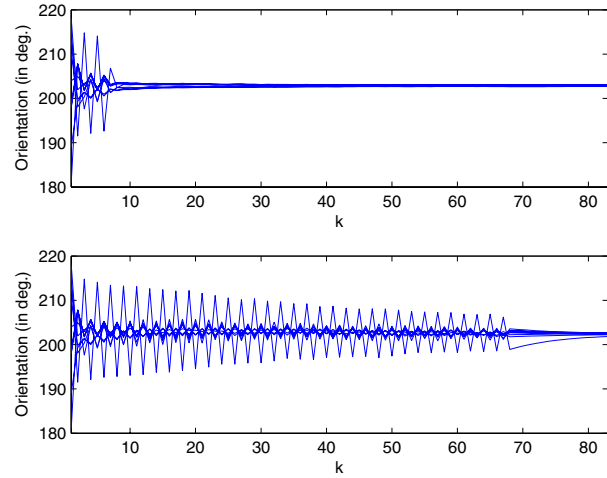


Fig. 4. Illustrating the dependence of the convergence rate on the choice of the function $\gamma(k)$.

Consider a group of ten dynamic units placed on the unit square according to a Poisson distribution. These units have adopted the agreement protocol to coordinate their orientation- however, the underlying information exchange network is noisy. Figure 3 depicts the corresponding sensor network with ranges $r = 0.48$, $r = 0.60$, and $r = 0.80$. For $G(10, 0.48)$, we first considered the dependency of the rate of convergence on the functional form of $\gamma(k)$. The initial states are uniformly distributed between 180 and 220 degrees with a mean of 202.6 degrees. The top plot in Fig. 4 shows the convergence behavior of the protocol when $\gamma(k) = 1.99/\lambda_n(G)$ for $k \leq \lceil \lambda_n(G)/2 \rceil = 5$, and $\gamma(k) = 1/k$, otherwise. The protocol converged in 16 steps; it was deemed to have converged when states of all units were within ± 1 degree of the average of the initial states. The bottom plot in Fig. 4 depicts the convergence behavior of the algorithm when $\gamma(k)$ was maintained at $1.99/\lambda_n(G)$, until the maximum difference in states of any two units was less than 5 degrees, which happened at $k = 69$. After that, $\gamma(k)$ was switched to be equal to $1/k$; the protocol converged in 83 steps in this case. We observe that maintaining a larger step-size for a longer duration may not translate to an improved convergence rate.

Next we considered the dependence of the rate of convergence on the noise variance σ^2 for a random geometric

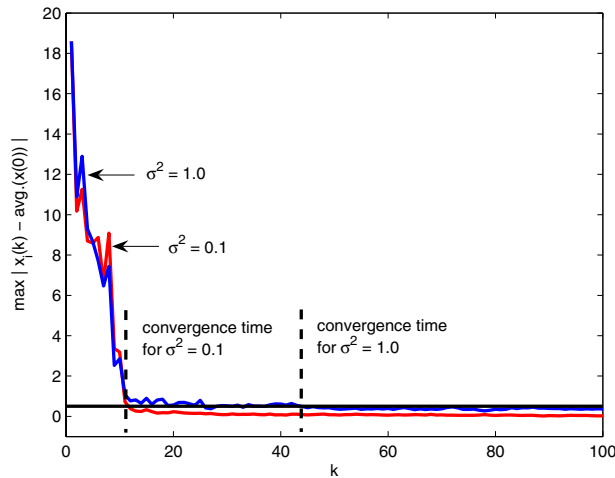


Fig. 5. Illustrating the dependence of the protocol's rate of convergence on the noise variance σ^2 for a random geometric graph realization $G(15, 0.8)$

graph realization $G(15, 0.80)$; these simulations are shown in Fig. 5. The initial states for generating this figure are uniformly distributed between 180 and 220 degrees; the step-size parameter is the same as in the top plot of Fig. 4. Fig. 5 depicts the influence of higher noise variance on the transient behavior of the protocol- at the same time- it highlights the dominant effect of the step size parameter on its rate of convergence.

We continued our studies by considering the dependence of the protocol's rapidity of convergence on the range of a 15-node random geometric graph realization $G(15, r)$; see Fig. 6. In this figure, for clarity, we have chosen to plot the maximum of the absolute differences between the units' states and the mean of the initial states, i.e.,

$$\max_i |x_i(k) - \frac{1}{n} \sum_i x_i(0)|,$$

as a function of k . The step-size parameter and convergence criterion are the same as in the top plot of Fig. 4. The same noise vector with a variance of $\sigma^2 = 1$ is chosen for all three cases. We note that in the noise-free case, the rate of convergence is improved for higher node range values. From Fig. 6, it is evident that the rate of convergence is improved when r is increased from 0.45 to 0.55, but not necessary when it is further increased to 0.65. This observation is in direct correspondence with our earlier assertion on the existence of a threshold, r_T , beyond which the detrimental effect of further increasing r negates its beneficial effect.

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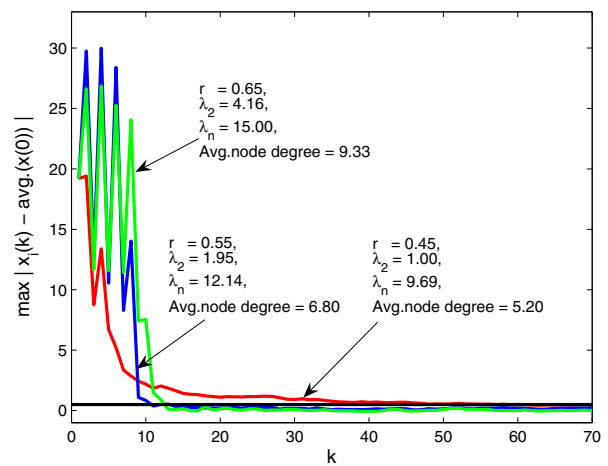


Fig. 6. Convergence properties of the agreement protocol on realizations of $G(15, r)$ for various values of r . The constant line represents the threshold of 0.5 degrees.

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