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On Global Output Feedback Tracking Control of Planar Robot Manipulators

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Abstract— **In this paper a simple solution to the global output feedback tracking control problem for planar robot manipulators is presented. The proposed tracking controller renders the origin of the error dynamics uniformly globally asymptotically stable. The novelty of our approach is that our control design is based on a new model for robot manipulators. This model described in the Cartesian coordinate space gives a redundant (i.e., nonreduced-order) dynamics of the system. The main stream for stabilization is that the nonredundant dynamics part of this model, for planar manipulators, is linear in the unmeasurable velocities.**

I. INTRODUCTION

Motivated by its technical complexity and the fact that velocity measurements are often contaminated by noise, the problem of global output feedback tracking control of manipulators has been studied by many authors throughout the last decade. Due to space constraints, only some important references are cited instead of detailed review. For more details the reader is invited to see [5].

In chronological order, we will start with [3] where a global convergence of the tracking errors was proved, using singular perturbation techniques hence, no explicit control gains can be established. In [4], the author presented the first smooth controller which renders the one-degree-of-freedom (dof) Euler-Lagrange system in closed loop, *uniformly* globally asymptotically stable (UGAS). In order to automatically enlarge the domain of attraction in despite of keeping the controller and observer gains constant, Loría introduced hyperbolic trigonometric functions in both controller and observer. A drawback of this approach is the complexity of the obtained stabilizing control law. In addition, this control law contains terms which exponentially increase hence, yield high control input signals to the system. In [2], an elegant result for 1-dof systems was reported. The controller is based upon a global nonlinear change of coordinates which makes the system *linear* in the unmeasured velocities and proved UGAS of the closed loop. In [9], the result of [4] was recently extended in one direction, for n degrees of freedom systems. Global (in the tracking initial errors) convergence to zero of the position and velocity tracking errors was proven, provided that the initial conditions of the dynamic extension are sufficiently small.

In this paper, a simple dynamic controller which makes the overall closed loop system UGAS is proposed. Specifically, we utilize a linear filter to compensate for the need for velocity measurements. The proposed approach is based on a new model for robot manipulators [7]. This model described in the Cartesian coordinate space gives a redundant (i.e., nonreduced-order) dynamics of the manipulator system where the latter can be regarded as a set of rigid bodies (that is the links) subjected to holonomically mechanical constraints. The main tool for stability analysis and control design is that, for planar manipulators, the dynamics of this set of *free* unconstrained rigid bodies is linear in the unmeasurable velocities. We then show that Lyapunov stability and control structure for the manipulator system are deducted by projection in the submanifold of movement from appropriate Lyapunov stability and stabilizing control for the corresponding unconstrained rigid body system.

II. SYSTEM MODEL

Here, our dynamic model for planar robot manipulators is briefly presented. A detailed development of this latter can be found in [7].

Only manipulators driven by DC motors through rigid transmissions will be considered. We consider the manipulator as an open kinematic chain of $(n+1)$ rigid bodies, that is the base (0-th link) plus n other links, interconnected by n rigid joints.

We start by defining a notational system to describe the geometry of the manipulator. Following the notations of [8], let us introduce the following variables. Let $\dot{p}_{\ell_i} \in \mathbb{R}^3$ and $\omega_i \in \mathbb{R}^3$ denote respectively the vectors of linear and angular velocities of the i-th link's center of mass expressed in the base frame. Similarly, $\dot{p}_{m_i} \in \mathbb{R}^3$ and $\omega_{m_i} \in \mathbb{R}^3$ are respectively the vectors of linear and angular velocities of the i-th rotor's center of mass expressed in the base frame. Also, $\omega_{i-1,m_i} \in \mathbb{R}^3$ is the angular velocity of the i-th rotor with respect to the $(i-1)$-th link on which such motor is located. The *constant* m_{ℓ_i} denotes the mass of the i-th link (including the mass of the $(i+1)$-th stator) and m_{m_i} is the mass of the i-th rotor. $I^i_{\ell_i} \in \mathbb{R}^{3\times3}$ corresponds to the *constant* inertia tensor of the i-th link relative to its center of mass expressed in a frame attached to the link itself (as in the Denavit-Hartenberg convention). $I^{m_i}_{m_i} \in \mathbb{R}^{3\times3}$ is the *constant* inertia tensor of the i-th rotor relative to its center of mass expressed in a frame attached to the rotor by the center of mass and whose axis z parallel to its axis of rotation.

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We will introduce the vector of *Cartesian* velocities

$$\nu = \mathrm{col}[\nu_1, \nu_2, \nu_3] \ \in \mathbb{R}^{12n} \tag{1}$$

with

$$\begin{aligned} \nu_1 &= \mathrm{col}[\dot{p}_{\ell_1}, \cdots, \dot{p}_{\ell_n}, \dot{p}_{m_1}, \cdots, \dot{p}_{m_n}] \ \in \mathbb{R}^{6n} \\ \nu_2 &= \mathrm{col}[\omega_1, \cdots, \omega_n] \ \in \mathbb{R}^{3n} \\ \nu_3 &= \mathrm{col}[\omega_{0,m_1}, \omega_{1,m_2}, \cdots, \omega_{n-1,m_n}] \ \in \mathbb{R}^{3n} \end{aligned}$$

which collects the linear and angular velocities of the links and rotors. We also introduce the *constant* "Cartesian inertia matrix"

$$\mathcal{M} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & M_3 \\ 0 & M_3^\top & M_4 \end{bmatrix} \in \mathbb{R}^{12n \times 12n} \tag{2}$$

where the symmetric positive definite matrices $M_1 \in \mathbb{R}^{6n\times 6n}, M_2 \in \mathbb{R}^{3n\times 3n}, M_4 \in \mathbb{R}^{3n\times 3n}$ and the strict upper triangular matrix $M_3 \in \mathbb{R}^{3n\times 3n}$ are given by

$$\begin{aligned} M_1 &= \mathrm{diag}\{m_{\ell_1} I_3, \cdots, m_{\ell_n} I_3, m_{m_1} I_3, \cdots, m_{m_n} I_3\} \\ M_2 &= \text{block-diag}\left\{ I_{\ell_1}^1 + I_{m_2}^{m_2}, \cdots, I_{\ell_{n-1}}^{n-1} + I_{m_n}^{m_n}, I_{\ell_n}^n \right\} \\ M_3 &= \begin{bmatrix} 0 & I_{m_2}^{m_2} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & I_{m_3}^{m_3} & & \cdots & 0 \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ & \cdots & & & 0 & I_{m_n}^{m_n} \\ 0 & & & \cdots & & 0 \end{bmatrix} \\ M_4 &= \mathrm{diag}\{I_{m_1}^{m_1}, \cdots, I_{m_n}^{m_n}\}, \end{aligned}$$

hence, $\mathcal{M}$ is symmetric and positive definite.

Then, our dynamic model for planar robot manipulators is

$$\mathcal{M}\,\dot{\nu} + v = \tau + \tau_c \tag{3}$$

with

$$\begin{aligned} v = \mathrm{col}\,[&-m_{\ell_1} g_o, \cdots, -m_{\ell_n} g_o, -m_{m_1} g_o, \\ &\cdots, -m_{m_n} g_o, 0_{6n\times 1}] \end{aligned}$$

where $\tau \in \mathbb{R}^{12n}$ is the vector of Cartesian forces and torques, $\tau_c \in \mathbb{R}^{12n}$ is the vector of forces and torques corresponding to the Cartesian mechanical constraints between the different links of the chain, and $g_o \in \mathbb{R}^3$ is the gravity acceleration vector in the base frame.

In words, (3) without the term of constraint τ_c gives the dynamics of a set of $2n$ *free* rigid bodies, that is the n links and the n rotors, whose elements can reach any position in two-dimensional space. By taking into account the term of constraint τ_c we obtain a redundant dynamics for planar robot manipulators.

On the other hand, the (holonomic) constraints between the links allow eliminating $11n$ out of $12n =: m$ coordinates of the redundant dynamics (3). With the remaining n coordinates, it is possible to determine the *minimal* configuration of this manipulator. Such coordinates that will be defined as the vector q are the *nonredundant* generalized coordinates and n is the number of degrees of freedom of this manipulator. Consequently, the Cartesian kinematic motion of the system that gives the Cartesian positions $\pi := \int \nu dt$ for a given value of generalized coordinates $q(t)$ can be described by equation of the form

$$\pi = \pi(q(t))\,. \tag{4}$$

By differentiating the equation above with respect to time, we obtain the Cartesian kinematics equation of the system

$$\nu = \left(\frac{\partial \pi(q)}{\partial q}\right)^\top \dot{q} =: \mathcal{J}(q)\,\dot{q} \tag{5}$$

where the "Jacobian matrix" $\mathcal{J}(q)$ of dimension $(m \times n)$ has full-column rank, globally with respect to q. The computation of the Jacobian matrix above follows by using the Denavit-Hartenberg convention. See [7] for more details.

Note that from the system Cartesian kinematics equation (5) we also have

$$\dot{q} = \mathcal{J}^\dagger(q)\,\nu \tag{6}$$

where $\mathcal{J}^\dagger(q)$ is any left pseudo-inverse of the Jacobian matrix $\mathcal{J}(q)$.

A. Generalized Coordinate Model

By substituting (6) in the system Cartesian kinematics equation (5), the holonomic constraints between the rigid bodies of the planar manipulator can be *explicitly* defined by the following velocity-level form

$$\left(I_m - \mathcal{J}(q)\,\mathcal{J}^\dagger(q)\right)\,\nu = 0_{m\times 1} \tag{7}$$

where the functional dependence of the Jacobian matrix as well as its left pseudo-inverse is still on q and not on π, owing to the fact that this substitution is not essential and that, from a computational point of view, it is more advantageous to keep the explicit dependence on generalized coordinates.

Following the principle of virtual work, the vector of Cartesian constraint forces τ_c is [7]

$$\tau_c = \left(I - \mathcal{J}_{\mathcal{M}}^\dagger(q)^\top \mathcal{J}(q)^\top\right)\left[\mathcal{M}\,\dot{\mathcal{J}}(q,\dot{q})\,\dot{q} + v - \tau\right] \tag{8}$$

where $\dot{\mathcal{J}}(q,\dot{q}) := d\mathcal{J}(q)/dt$ and $\mathcal{J}_{\mathcal{M}}^\dagger(q)$ is the left pseudo-inverse of the Jacobian matrix $\mathcal{J}(q)$ weighted by $\mathcal{M}$, that is

$$\mathcal{J}_{\mathcal{M}}^\dagger(q) = \left(\mathcal{J}(q)^\top \mathcal{M} \mathcal{J}(q)\right)^{-1} \mathcal{J}(q)^\top \mathcal{M}\,.$$

Because only $m - n$ equations among the m constraint equations (7) are independent (see [7] for more details), the vector of Cartesian constraint forces can be rewritten in an advantageous manner as [6]

$$\tau_c = F(q)^\top\, Y(q)\left(\mathcal{M}\,\dot{\mathcal{J}}(q,\dot{q})\,\dot{q} + v - \tau\right) \tag{9}$$

with

$$Y(q) = \left(F(q)\mathcal{M}^{-1}F(q)^\top\right)^{-1} F(q)\,\mathcal{M}^{-1} \tag{10}$$

where $F(q) \in \mathbb{R}^{(m-n)\times m}$ is the reduced constraint matrix which satisfies that $F(q)\mathcal{J}(q) = 0$.

From (3), (9) and (10), the "weighting matrix" $\mathcal{K}(q) := (I - F(q)^\top Y(q))$ of the Cartesian forces vector τ in the dynamics (3) is so that $\mathcal{K}(q)F(q)^\top = 0$. That is $\mathcal{K}$ is a projection operator that filters out all Cartesian forces lying in the range of the transpose of the reduced constraint matrix $F(q)$. These correspond to Cartesian forces that tend to violate the imposed Cartesian space constraints. The obtained Cartesian positions π are supposed to be in a subset Ω_π of $\mathbb{R}^m$. This means that $\Omega_\pi = \{\pi \in \mathbb{R}^m / \exists\, q \in \mathbb{R}^n, \pi = \pi(q)\}$.

To eliminate the Cartesian constraint forces τ_c and therefore reduce the dimension of the manipulator redundant dynamics (3), it suffices to use the Cartesian kinematics equation (5) in the dynamics (3) and premultiply on both sides of (3) by $\mathcal{J}(q)^\top$. Hence, the Cartesian constraint forces τ_c are eliminated owing to the fact that $\mathcal{J}(q)^\top \tau_c = 0$, and the generalized coordinate model of the planar manipulator system is then given by the following equation

$$\begin{aligned} \underbrace{\mathcal{J}(q)^\top \mathcal{M}\, \mathcal{J}(q)}_{D(q)}\, \ddot{q} \quad + \quad \underbrace{\mathcal{J}(q)^\top \mathcal{M}\, \dot{\mathcal{J}}(q,\dot{q})}_{C(q,\dot{q})}\, \dot{q} \\ + \quad \underbrace{\mathcal{J}(q)^\top v}_{g(q)} \quad = \quad \underbrace{\mathcal{J}(q)^\top \tau}_{e} \end{aligned} \qquad (11)$$

where the generalized matrices and vectors $D(q)$, $C(q,\dot{q})$ and $g(q)$ are now given by jacobian-type expressions. Also, $\mathcal{J}(q)^\top \tau = e$ gives the relationship between the Cartesian forces vector τ and the generalized forces vector e.

For further development, we give the following lemma.

Lemma 1 *For a planar manipulator, the mapping $\pi(q) : \mathbb{R}^n \to \mathbb{R}^m$ is injective.*

The proof of Lemma 1 is straightforward and follows by observing that, for planar manipulators, the angular velocities of the links, belonging to the Cartesian velocities vector ν of (1), are given by the joint velocities $\dot{q}$. For example, the angular velocity of the first link in the base frame is given by $\omega_1 = \mathrm{col}[0\ \ 0\ \ \dot{q}_1]$.

III. CONTROL DESIGN AND STABILITY

The control problem can be stated as follows. Let $q_d : \mathbb{R}_+ \to \mathbb{R}^n$ be a given twice continuously differentiable reference trajectory and assume that there exists $\beta_d > 0$ such that $\max\{\|q_d\|_\infty, \|\dot{q}_d\|_\infty, \|\ddot{q}_d\|_\infty\} \leq \beta_d$. Consider the system (11) and assume that only q is measurable. Under these conditions, find a dynamic controller $\tau(t, q, \pi_c)$, $\dot{\pi}_c = \phi(t, \pi_c, q)$ such that, defining the tracking errors $\tilde{q}(t) := q(t) - q_d(t)$ and $\dot{\tilde{q}}(t) := \dot{q}(t) - \dot{q}_d(t)$, the origin $(\tilde{q}, \dot{\tilde{q}}) = (0, 0)$ be uniformly globally asymptotically stable for all initial conditions $(t_o, \tilde{q}(0), \dot{\tilde{q}}(0)) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$.

The control strategy is based on determining a stabilizing control law of the unconstrained dynamics of manipulator system. We then prove that the projection of this control law in the submanifold of movement achieves the control objective above for the manipulator system.

A. Unconstrained System

As pointed out before, the unconstrained dynamics of manipulator system is given by the following equation

$$\mathcal{M}\, \dot{\nu} + v = \tau\,. \qquad (12)$$

Under the condition that the Cartesian velocities vector ν is not available for measurement, an output feedback tracking controller which renders the closed loop system asymptotically stable is derived. We will utilize a linear filter to remove the need for velocity measurements. To accomplish this goal, we first define the position tracking error as $\tilde{\pi} = \pi - \pi_d$, where $\pi_d(t)$ represents the desired trajectory for π. Also, we define the velocity tracking error as $\tilde{\nu} = \nu - \nu_d$, where $\nu_d(t)$ stands for the desired trajectory of ν.

Then, the solution of the above control problem is stated by the following proposition.

Proposition 1 *Consider the unconstrained system (12) in closed loop with the control law*

$$\tau = \mathcal{M}\dot{\nu}_d + v - K_P\tilde{\pi} - K_D\eta \qquad (13)$$

$$\dot{\pi}_c = -A(\pi_c + B\tilde{\pi}) \qquad (14)$$

$$\eta = \pi_c + B\tilde{\pi} \qquad (15)$$

where[1] K_P, K_D, A *and* $B = \mathrm{diag}\{b_i\}$ *are diagonal positive definite matrices and*

$$b_i \geq \frac{\overline{\lambda}(\mathcal{M})}{\beta_2 \underline{\lambda}(\mathcal{M})} \qquad (16)$$

while β_2 is a constant such that $0 < \beta_2 < 1$. Then, the closed loop system is uniformly globally asymptotically stable for any tracking initial conditions.

Proof: The proof relies on classical Lyapunov theory. Indeed, we propose a Lyapunov function candidate for the closed loop system and then we prove that under the conditions of the proposition above the proposed function qualifies as a Lyapunov function. Stability is established by invoking the Lyapunov second method.

We start by defining a suitable error equation for the closed loop system. Indeed, substituting (13) into (12) yields

$$\mathcal{M}\, \dot{\tilde{\nu}} + K_P\, \tilde{\pi} + K_D\, \eta = 0. \qquad (17)$$

On the other hand, differentiating (15) and using (14) we get

$$\dot{\eta} = -A\, \eta + B\, \tilde{\nu}. \qquad (18)$$

Then, stability of the error system (17) and (18) will be studied.

Lyapunov function candidate. Consider the function

$$\begin{aligned} V = \frac{1}{2}\tilde{\nu}^\top \mathcal{M}\tilde{\nu} + \frac{1}{2}\tilde{\pi}^\top K_P\tilde{\pi} + \frac{1}{2}\eta^\top K_D B^{-1}\eta \\ + \epsilon\tilde{\pi}^\top \mathcal{M}\tilde{\nu} - \epsilon\eta^\top \mathcal{M}\tilde{\nu}\,. \end{aligned} \qquad (19)$$

Now, in order to guarantee positive definiteness of V, sufficient conditions will be given. To easy the proof, let partition V as $V = V_1 + V_2$ where

$$\begin{aligned} V_1 = \frac{1}{4}\tilde{\nu}^\top \mathcal{M}\tilde{\nu} + \frac{1}{4}\tilde{\pi}^\top K_P\tilde{\pi} + \frac{1}{4}\eta^\top K_D B^{-1}\eta \\ + \epsilon\tilde{\pi}^\top \mathcal{M}\tilde{\nu} - \epsilon\eta^\top \mathcal{M}\tilde{\nu} \end{aligned} \qquad (20)$$

[1] The symbols $\underline{\lambda}(X)$ and $\overline{\lambda}(X)$ denote respectively the smallest and largest eigenvalues of the positive definite matrix X.

and

$$V_2 = \frac{1}{4}\tilde{\nu}^\top \mathcal{M}\tilde{\nu} + \frac{1}{4}\tilde{\pi}^\top K_P\tilde{\pi} + \frac{1}{4}\eta^\top K_D B^{-1}\eta\,. \tag{21}$$

Notice that (20) can be written in matrix form as

$$V_1 = \frac{1}{4}\begin{bmatrix}\tilde{\pi}\\ \tilde{\nu}\end{bmatrix}^\top \underbrace{\begin{bmatrix}K_P & 2\epsilon\mathcal{M}\\ 2\epsilon\mathcal{M} & \frac{1}{2}\mathcal{M}\end{bmatrix}}_{P_1}\begin{bmatrix}\tilde{\pi}\\ \tilde{\nu}\end{bmatrix}$$
$$+\frac{1}{4}\begin{bmatrix}\tilde{\nu}\\ \eta\end{bmatrix}^\top \underbrace{\begin{bmatrix}\frac{1}{2}\mathcal{M} & -2\epsilon\mathcal{M}\\ -2\epsilon\mathcal{M} & K_D B^{-1}\end{bmatrix}}_{P_2}\begin{bmatrix}\tilde{\nu}\\ \eta\end{bmatrix}.$$

From the definitions of K_P and $\mathcal{M}$, P_1 is positive definite if

$$\frac{1}{2}\sqrt{\frac{\overline{\lambda}(K_P)}{2\underline{\lambda}(\mathcal{M})}} > \epsilon\,. \tag{22}$$

Similarly, from the definitions of K_D, B and $\mathcal{M}$, P_2 is positive definite if

$$\frac{1}{2}\sqrt{\frac{\overline{\lambda}(K_D B^{-1})}{2\underline{\lambda}(\mathcal{M})}} > \epsilon\,. \tag{23}$$

While V_2 is trivially positive definite. Furthermore, by virtue of (22) and (23), we can prove that V_1 is strictly convex, hence radially unbounded by simply looking to the definite positivity of the obtained hessian matrix. In a similar way, V_1 is trivially strictly convex.

Global asymptotic stability. In this paragraph, we show that the time derivative of (19) along the trajectories of (17) and (18) is globally negative definite in the whole state $(\tilde{\pi}, \tilde{\nu}, \eta)$. Stability follows directly by invoking the Lyapunov second method. It has that

$$\begin{aligned}\dot{V} &= -\eta^\top K_D B^{-1}A\eta + \epsilon\tilde{\nu}^\top\mathcal{M}\tilde{\nu} - \epsilon\tilde{\pi}^\top K_P\tilde{\pi}\\ &-\epsilon\tilde{\pi}^\top K_D\eta + \epsilon\eta^\top A\mathcal{M}\tilde{\nu} - \epsilon\tilde{\nu}^\top B\mathcal{M}\tilde{\nu}\\ &+\epsilon\eta^\top K_P\tilde{\pi} + \epsilon\eta^\top K_D\eta\,. \end{aligned}\tag{24}$$

By virtue of the properties of K_D, K_P, A, B and $\mathcal{M}$, the following bounds can be established:

$$\begin{aligned}
-\eta^\top K_D B^{-1}A\eta &\le -\underline{\lambda}(K_D B^{-1}A)\|\eta\|^2\\
\epsilon\tilde{\nu}^\top\mathcal{M}\tilde{\nu} &\le \epsilon\overline{\lambda}(\mathcal{M})\|\tilde{\nu}\|^2\\
-\epsilon\tilde{\pi}^\top K_P\tilde{\pi} &\le -\epsilon\underline{\lambda}(K_P)\|\tilde{\pi}\|^2\\
-\epsilon\tilde{\pi}^\top K_D\eta &\le \epsilon\overline{\lambda}(K_D)\|\tilde{\pi}\|\|\eta\|\\
\epsilon\eta^\top A\mathcal{M}\tilde{\nu} &\le \epsilon\overline{\lambda}(A)\overline{\lambda}(\mathcal{M})\|\eta\|\|\tilde{\nu}\|\\
-\epsilon\tilde{\nu}^\top B\mathcal{M}\tilde{\nu} &\le -\epsilon\underline{\lambda}(B)\underline{\lambda}(\mathcal{M})\|\tilde{\nu}\|^2\\
\epsilon\eta^\top K_P\tilde{\pi} &\le \epsilon\overline{\lambda}(K_P)\|\eta\|\|\tilde{\pi}\|\\
\epsilon\eta^\top K_D\eta &\le \epsilon\overline{\lambda}(K_D)\|\eta\|^2\,.
\end{aligned}$$

Let us define some constants $\beta_i > 0$ such that $\beta_1 + \beta_2 = 1$ and $\gamma_i > 0$ such that $\gamma_1 + \gamma_2 = 1$. Then, using the previous bounds, (24) can be upper bounded as

$$\begin{aligned}
\dot{V} \le\;& -\frac{\epsilon}{2}\begin{bmatrix}\|\tilde{\pi}\|\\ \|\eta\|\end{bmatrix}^\top \overbrace{\begin{bmatrix}2\underline{\lambda}(K_P) & -\overline{\lambda}(K_P) - \overline{\lambda}(K_D)\\ -\overline{\lambda}(K_P) - \overline{\lambda}(K_D) & \frac{\gamma_1}{\epsilon}\underline{\lambda}(K_D B^{-1}A)\end{bmatrix}}^{Q_1}\begin{bmatrix}\|\tilde{\pi}\|\\ \|\eta\|\end{bmatrix}\\
& -\frac{\epsilon}{2}\begin{bmatrix}\|\eta\|\\ \|\tilde{\nu}\|\end{bmatrix}^\top \overbrace{\begin{bmatrix}\frac{\gamma_1}{\epsilon}\underline{\lambda}(K_D B^{-1}A) & -\overline{\lambda}(A)\overline{\lambda}(\mathcal{M})\\ -\overline{\lambda}(A)\overline{\lambda}(\mathcal{M}) & 2\beta_1\underline{\lambda}(B)\underline{\lambda}(\mathcal{M})\end{bmatrix}}^{Q_2}\begin{bmatrix}\|\eta\|\\ \|\tilde{\nu}\|\end{bmatrix}\\
& -\epsilon\overbrace{\left[\beta_2\underline{\lambda}(B)\underline{\lambda}(\mathcal{M}) - \overline{\lambda}(\mathcal{M})\right]}^{\lambda_1}\|\tilde{\nu}\|^2\\
& -\overbrace{\left[\gamma_2\underline{\lambda}(K_D B^{-1}A) - \epsilon\overline{\lambda}(K_D)\right]}^{\lambda_2}\|\eta\|^2\,.
\end{aligned}$$

Now, sufficient conditions for $\dot{V}$ to be globally negative definite are derived. First, considering the conditions of Proposition 1, the matrix Q_1 is positive definite if

$$\frac{2\gamma_1\underline{\lambda}(K_P)\underline{\lambda}(K_D B^{-1}A)}{[\overline{\lambda}(K_P) + \overline{\lambda}(K_D)]^2} > \epsilon\,. \tag{25}$$

In a similar way, Q_2 is positive definite if

$$\frac{2\gamma_1\beta_1\underline{\lambda}(K_D B^{-1}A)\underline{\lambda}(\mathcal{M})\underline{\lambda}(B)}{[\overline{\lambda}(A)\overline{\lambda}(\mathcal{M})]^2} > \epsilon\,. \tag{26}$$

On the other hand, the positivity of the constants λ_1 and λ_2 respectively holds by condition (16) and

$$\frac{\gamma_2\underline{\lambda}(K_D B^{-1}A)}{\overline{\lambda}(K_D)} \ge \epsilon\,. \tag{27}$$

Notice that, (22), (23), (25), (26) and (27) are satisfied for ϵ sufficiently small. Therefore, (24) is globally negative definite. ■

B. Planar Manipulator System

In this subsection, our main stability result for the planar manipulator system is presented. We show that the same stabilizing feedback structure and the same Lyapunov function for establishing uniform global asymptotic stability of the unconstrained system (12), respectively, stabilizes and is applicable to the manipulator system (11). To that end, we require the following assumption regarding the existence of the Cartesian space coordinate system.

Assumption 1 *Assume that the Cartesian coordinates π of the unconstrained system (12) always belong to the set Ω_π during closed loop operation.* □

Also, it is assumed that the desired trajectory $\pi_d(t)$ of the unconstrained system belongs to Ω_π for all $t \ge 0$. This means that $\pi_d(t) = \pi(q_d(t))$ and $\nu_d(t) = \mathcal{J}(q_d(t))\,\dot{q}_d(t)$ for given values of $q_d(t), \dot{q}_d(t)$.

On the basis of these assumptions, the control law (13)-(15) can be projected on the submanifold of movement of the manipulator system (3) to obtain

$$\begin{aligned} \tau &= \mathcal{MJ}(q_d)\,\ddot{q}_d + \mathcal{M}\dot{\mathcal{J}}(q_d,\dot{q}_d)\,\dot{q}_d + v \\ &\quad -K_P(\pi(q)-\pi(q_d)) - K_D\eta \qquad (28) \\ \dot{\pi}_c &= -A[\pi_c + B(\pi(q)-\pi(q_d))] \qquad (29) \\ \eta &= \pi_c + B(\pi(q)-\pi(q_d)) \qquad (30) \end{aligned}$$

where (4), $\tilde{\pi} = \pi - \pi_d, \pi_d = \pi(q_d)$ and $\nu_d = \mathcal{J}(q_d)\,\dot{q}_d$ have been utilized. In the same manner, the Lyapunov function (19) can also be written as a function of the tracking errors $(\tilde{q}, \dot{\tilde{q}}, \eta)$.

Our main result for the planar manipulator system (11) is stated by the following proposition.

Proposition 2 *Consider the manipulator system (11) in closed loop with the dynamic control law (28)-(30). Then, under the conditions of Proposition 1, the closed loop system with state $(\tilde{q}, \dot{\tilde{q}}, \eta)$ is uniformly globally asymptotically stable for any tracking initial conditions.*

Proof:
It will be shown that the function (19) is also a Lyapunov function for establishing the uniform global asymptotic stability of the manipulator system (11) in closed loop with the dynamic control law (28)-(30). To that end, we proceed as follows.

From Lemma 1 and the definitions of the position and velocity tracking errors $\tilde{\pi}$ and $\tilde{\nu}$ in terms of $(\tilde{q}, \dot{\tilde{q}})$, it is straightforward to see that

1) $\tilde{q} = 0 \Leftrightarrow \tilde{\pi} = 0$
2) $\dot{\tilde{q}} \neq 0 \Rightarrow \tilde{\nu} \neq 0$
3) $(\tilde{q}, \dot{\tilde{q}}) = (0, 0) \Rightarrow \tilde{\nu} = 0$
4) $\|\tilde{q}\| \to \infty \Rightarrow \|\tilde{\pi}\| \to \infty$
5) $\|\dot{\tilde{q}}\| \to \infty \Rightarrow \|\tilde{\nu}\| \to \infty$.

Under these observations and the conditions (22), (23), (25), (26) and (27), it is easy to verify that (19) is positive definite and radially unbounded with respect to the state $(\tilde{q}, \dot{\tilde{q}}, \eta)$, and its time derivative along the trajectories of the closed loop system (11) and (28)-(30) is globally negative definite. Uniform global asymptotic stability follows by invoking the Lyapunov second method.

An alternative but more intuitive proof of the proposition above is to show that the term of constraint τ_c in the redundant dynamics (3), in closed loop operation, converges to zero. Indeed, when τ_c converges to zero, the systems (3) and (12) become equivalent. As a result, with the same control law, the trajectories of these systems will behave simultaneously at the time of convergence of τ_c. Finally, under conditions of Proposition 1, π and ν of the redundant dynamics (3) are bounded and converge asymptotically, as π and ν of the unconstrained dynamics (12), to π_d and ν_d, respectively. Asymptotic convergence of $(\tilde{q}, \dot{\tilde{q}}, \eta)$ follows by using Lemma 1 and the definitions $\pi_d(t) = \pi(q_d(t))$ and $\nu_d(t) = \mathcal{J}(q_d(t))\,\dot{q}_d(t)$. A new method of proof supported by modern stability analysis tools rather the usual Lyapunov analysis can be used to establish this result. This is the I&I method introduced in [1] that combines the classical tools of system immersion and manifold invariance.

■

C. Discussion

1) The control strategy we deal with in this paper showed that we can drop the term of constraint τ_c from the manipulator redundant dynamics (3) and henceforth use the control design of the obtained dynamics (12) to deducting a stabilizing control law for the manipulator system (11). From a physical point of view, this can be argued by the fact that the mechanical constraints between the links can be eliminated in stability analysis and control design if we can ensure that the free links move in their places as if these constraints were present. Clearly, this is guaranteed by the stability result of Proposition 1, Lemma 1, Assumption 1 and the fact that the desired trajectory of the unconstrained system $\pi_d(t)$ belongs to Ω_π for all $t \geq 0$.

2) We conjecture that the manipulator system (11) and its corresponding unconstrained system (12) are equivalent in the sense of stability under the condition that the closed loop equilibrium point of the unconstrained system is generated by a certain equilibrium point of the manipulator system. In addition, *any* structure control and Lyapunov function of the unconstrained system are stabilizing control and appropriate Lyapunov function of the manipulator system. This generalizes the results of [7] in which it was shown that *there exists* a control law which stabilizes both holonomically constrained systems and their unconstrained systems.

D. Simulation Results

The performance of the stabilizing control law for the manipulator system has been tested by simulations. We have used the model of a two-link planar robot with y the vertical axis (hence $g_o = \text{col}[0\ -9.81\ 0]$).

For simplicity, the inertia contributions of the actuators of this robot have been neglected. Then, the vector of potential forces v and the Jacobian matrix $\mathcal{J}(q)$ are defined as $v = \text{col}[0\ 9.81 m_{\ell_1}\ 0\ 0\ 9.81 m_{\ell_2}\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ and $\mathcal{J}(q) = [E\ F]$ with $E = \text{col}[-l_{c_1}\sin(q_1)\ \ l_{c_1}\cos(q_1)\ \ 0\ \ -l_1\sin(q_1) - l_{c_2}\sin(q_1+q_2)\ \ l_1\cos(q_1) + l_{c_2}\cos(q_1+q_2)\ \ 0\ 0\ 0\ 1\ 0\ 0\ 1]$ and $F = \text{col}[0\ 0\ 0\ \ -l_{c_2}\sin(q_1+q_2)\ \ l_{c_2}\cos(q_1+q_2)\ \ 0\ 0\ 0\ 0\ 0\ 0\ 1]$, where the values of the parameters are as follows: $l_1 = 2$ m, $l_{c_1} = 1$ m, $l_{c_2} = 0.5$ m, $m_{\ell_1} = 5$ kg, $m_{\ell_2} = 3.5$ kg, $I^1_{\ell_1 zz} = 6.33$ kgm^2, $I^2_{\ell_2 zz} = 0.83$ kgm^2. It is worth noticing that the vector of Cartesian coordinates $\pi(q)$ is obtained by integration over time of (5) using the Jacobian matrix $\mathcal{J}(q)$ defined above.

The simulation was started from the initial conditions $q(0) = \text{col}[1, -1]$, $\dot{q}(0) = \text{col}[-2, 2]$ and $\pi_c(0) = 0_{12\times 1}$. The controller and filter gains are given by $K_D = 5\,I_{12}$, $K_P = 7\,I_{12}$, $A = 10\,I_{12}$ and $B = 15\,I_{12}$. We have used the desired trajectory $q_d(t) = \text{col}[\sin(\frac{1}{8}\pi t),\ \cos(\frac{1}{8}\pi t)]$. The resulting position-velocity tracking errors $(\tilde{q}, \dot{\tilde{q}}, \eta)$ are shown in Figures 1-2.

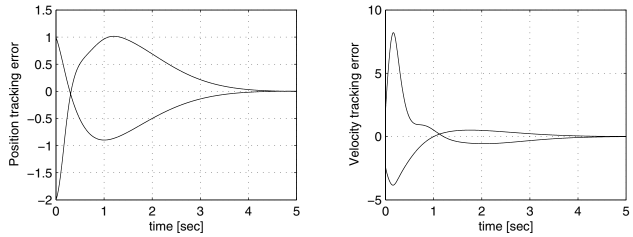

Fig. 1. Position and velocity tracking errors $\tilde{q}$, $\dot{\tilde{q}}$.

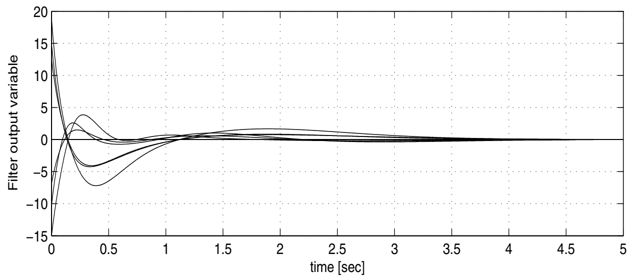

Fig. 2. Velocity tracking error η.

IV. CONCLUSION

A simple output feedback tracking controller for rigid-joint planar manipulators that exhibits uniform global asymptotic position and velocity tracking has been presented. Specifically, we utilize a linear filter to remove the need for velocity measurements. The novelty of our approach is that it is based on a new manipulator model which is simple for control design.

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