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# Design of Set-Invariant Estimators for Linear Discrete-Time Systems

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***Abstract*— The concept of set-invariance is applied to the design of full-order state observers with limitation of the estimation error, for discrete-time linear systems subject to unknown-but-bounded persistent disturbances and measurement noise. It is shown that if the initial error belongs to a polyhedral $\mathcal{D}$-$(C, A)$-invariant set, then it can be kept in this set by means of a piece-wise affine output injection law, for all admissible disturbances and noise. Numerical algorithms are presented for the computation of a $\mathcal{D}$-$(C, A)$-invariant polyhedron containing the one the initial error belongs to. Easy-to-check necessary $\mathcal{D}$-$(C, A)$-invariance conditions are derived for symmetrical polyhedra. Then, it is shown that such conditions happen to be sufficient as well in special cases, for which optimal error limitation can be achieved. The results are illustrated by means of numerical examples.**

## I. Introduction

Several problems of control systems subject to constraints on their state, control or output variables have been solved in the last years through the so-called set-invariance approach, mainly when such constraints are linear, corresponding, hence, to polyhedral sets defined in the state space (see e.g. [1] for a survey). An important limitation of such techniques, however, is the fact that most of the proposed solutions assume the use of state feedback control laws, requiring the full measurement of the state, which is not always possible due to physical or economical reasons.

Very often, this difficulty can be circumvented by building an observer which estimates the unaccessible states. This is the case of the approaches based on set-membership techniques [2], [3], [4]. Considering discrete-time linear systems, for each time instant, the set of states which could generate the measured output (or an outer approximation of it) is computed and a point-wise optimal state is selected. Since such a set has to be computed on-line, the computational burden can be excessive, and its practical implementation in fast systems can become infeasible.

Another class of observers, based on the concept of $(C, A)$-invariant sets, has been recently proposed in [5], for discrete-time, single output deterministic systems. It was shown that full-order asymptotic state observers can be constructed in order to confine the trajectory of the estimation error into a $(C, A)$-invariant polyhedron. In [6], the results of [5] were extended to systems subject to disturbances and measurement noise. $\mathcal{D}$-$(C, A)$-invariance was defined as the possibility of keeping the estimation error in a given set in spite of the action of disturbances and noise belonging to polyhedral sets. Set invariancec was then achieved by means of an output injection computed "on-line".

In this paper, it is shown that the trajectory of the estimation error can be enforced to a $\mathcal{D}$-$(C, A)$-invariant set through a piece-wise affine output injection law. Moreover, for a particular class of systems, it is shown that optimal error limitation can be achieved through a linear output injection law, as long as the set of possible initial states is a symetrical polyhedron. The proposed results are illustrated by means of numerical examples.

**Notation:** In mathematical expressions, the symbol ":" stands for "such that". $\mathbf{1}$ represents a vector of appropriate dimensions whose components are all equal to 1. $M_i$ represents the i-th row of matrix $M$. $\mathrm{Conv}(\Omega)$ represents the convex hull of the set $\Omega$, i.e., the smallest convex set which contains $\Omega$.

## II. Set-Invariant State Estimation

Consider the linear, time-invariant, discrete-time, single-output system, described by:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 d(k), \\ y(k) &= Cx(k) + \eta(k), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^r$ is the disturbance, $y \in \mathbb{R}$ is the output, $\eta \in \mathbb{R}$ is the measurement noise and $k$ is the sampling time, with $k \in \mathbb{N}$. Matrix $C$ is supposed to be full rank and the pair $(C, A)$ is supposed to be detectable.

An estimation of the state can be obtained by means of the following full-order observer:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) - v(z(k)), \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{2}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state, $\hat{y} \in \mathbb{R}$ is the estimated output and $v(.)$ is the output injection.

The estimation error and the difference between the measured output and the estimated output are respectively defined as:

$$\begin{aligned} e(k) &= x(k) - \hat{x}(k), \\ z(k) &= y(k) - \hat{y}(k). \end{aligned}$$

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Then, the error dynamics is given by:

$$\begin{array}{c} e(k+1) = Ae(k) + B_1 d(k) + v(z(k)), \\ z(k) = Ce(k) + \eta(k), \end{array} \qquad (3)$$

The disturbance $d$ is assumed to be unknown but bounded to a compact (closed and limited) set $\mathcal{D} \subset \mathbb{R}^r$. The measurement noise is assumed to belong to the set $\mathcal{N} = \{\eta \ : \ |\eta| \le \bar{\eta}\}$.

Consider now a compact set $\Omega$ whose interior contains the origin, defined on the estimation error space. The following set of *admissible outputs* is associated to $\Omega$:

$$\mathcal{Z}(\Omega) = \{z : \ z = Ce + \eta \text{ for some } e \in \Omega, \ \eta \in \mathcal{N}\}.$$

$\mathcal{Z}(\Omega)$ is the set, also compact, of all values of $z$ which can be generated by $e \in \Omega$ and $\eta \in \mathcal{N}$. Therefore, if $e(k) \in \Omega$, then $z(k) \in \mathcal{Z}(\Omega)$.

*Definition 2.1: The set* $\Omega \subset \mathbb{R}^n$ *is said to be* $\mathcal{D}$-$(C, A)$-invariant *with respect to system (3) if* $\forall z \in \mathcal{Z}(\Omega)$, $\exists v \ : \ Ae + B_1 d + v \in \Omega, \forall d \in \mathcal{D}, \forall e \in \Omega : z = Ce + \eta,$ *for some* $\eta \in \mathcal{N}$.

After the application of the output injection, $\Omega$ is simply said to be positively $\mathcal{D}$-invariant.

*Definition 2.2: Given* $0 < \lambda < 1$*, the set* $\Omega \subset \mathbb{R}^n$ *is said to be* $\mathcal{D}$-$(C, A)$-invariant $\lambda$-contractive (or simply $\mathcal{D}$-$(C, A)$-$\lambda$-contractive) *with respect to system (3) if* $\forall z \in \mathcal{Z}(\Omega)$, $\exists v : Ae + B_1 d + v \in \lambda\Omega, \forall d \in \mathcal{D}, \forall e \in \Omega : z = Ce + \eta,$ *for some* $\eta \in \mathcal{N}$.

In words, if the observation error at time $k$ belongs to $\Omega$, with $\Omega$ $\mathcal{D}$-$(C, A)$-$\lambda$-contractive, then, the knowledge of only $z(k)$ is sufficient to enforce $e(k+1) \in \lambda\Omega$ through the computation of $v(z(k))$, in spite of the disturbance and the noise. As a consequence, if the initial observation error $e(0)$ is known to belong to $\Omega$ then, by means of a suitable output injection $v(z(k))$, it is possible to keep it always limited to this set.

One should notice that a necessary condition for $\mathcal{D}$-$(C, A)$-$\lambda$ contractivity is that $\lambda\Omega$ contains the set $B_1\mathcal{D} = \{w \ : \ w = B_1 d, \text{ for some } d \in \mathcal{D}\}$.

## III. $\mathcal{D}$-$(C, A)$-Invariance of Convex Polyhedra

Assume now that $\Omega$ and $\mathcal{D}$ are compact, convex polyhedra containing the origin, defined by:

$$\Omega = \{e : Ge \le \mathbf{1}\}, \ \mathcal{D} = \{d : Sd \le \mathbf{1}\},$$

with $G \in \mathbb{R}^{g \times n}$, $S \in \mathbb{R}^{s \times n}$.

The set of related outputs is also a compact and convex polyhedron defined by:

$$\begin{array}{rcl} \mathcal{Z}(\Omega) & = & \{z : z = Ce + \eta \text{ for some} \\ & & e : \ Ge \le \mathbf{1} \text{ and } \eta : |\eta| \le \bar{\eta}\}. \end{array}$$

In the case of single-output systems, $\mathcal{Z}(\Omega)$ is a line segment in $\mathbb{R}$.

Considering Definition 2.2, it is clear that $\Omega$ is $\mathcal{D}$-$(C, A)$-$\lambda$-contractive if and only if, $\forall z \in \mathcal{Z}(\Omega)$:

$$\begin{array}{c} \exists v(z) : G(Ae + B_1 d + v(z)) \le \lambda \mathbf{1}, \\ \forall e, \eta : z = Ce + \eta, \ Ge \le \mathbf{1}, \ |\eta| \le \bar{\eta}, \\ \forall d : Sd \le \mathbf{1} \end{array} \qquad (4)$$

Since the same $v(z)$ must work for all $d \in \mathcal{D}$, then the effect of disturbances can be taken into account by considering their worst case row by row. Let the elements of vector $\delta \in \mathbb{R}^g$ be defined by the following linear programming problems (LP):

$$\begin{array}{rcl} \delta_i & = & \max_d G_i B_1 d \\ & & \text{under: } Sd \le \mathbf{1} \end{array} .$$

Then, condition (4) becomes:

$$\begin{array}{c} \exists v(z) : G(Ae + v(z)) \le \lambda \mathbf{1} - \delta, \\ \forall e, \eta : z = Ce + \eta, \ Ge \le \mathbf{1}, \ |\eta| \le \bar{\eta}. \end{array}$$

Let now $\phi(z)$ be the vector whose components are given by the solution of the following LP:

$$\begin{array}{c} \phi_i(z) = \max\limits_{e, \eta} \ G_i A e \\ \text{under: } Ge \le \mathbf{1}, \ |\eta| \le \bar{\eta}, \ Ce + \eta = z. \end{array} \qquad (5)$$

which can be rewritten as:

$$\begin{array}{c} \phi_i(z) = \max\limits_{e} \ G_i A e \\ \text{under: } Ge \le \mathbf{1}, \ |Ce - z| \le \bar{\eta}. \end{array} \qquad (6)$$

Since the same $v(z)$ must work for all $e \in \Omega$ which could have generated the output $z$, then the worst case $e$ can be computed row by row. Hence, condition (4) is equivalent to:

$$\exists v(z) : \phi(z) + Gv(z) \le \lambda \mathbf{1} - \delta \qquad (7)$$

From the numerical point of view, the treatment of this condition is difficult, because the functions $\phi_i(z)$ are concave, piece-wise linear and continuous with respect to $z$ [7]. Hence the computation of their break points (for which the linear function defining $\phi_i(z)$ changes) would be necessary.

Consider now the external representation of the compact polyhedron $\Omega$ in terms of its vertices $e^j$, $j = 1, ..., n_v$. For each $e^j$, two outputs are associated: $z_-^j = Ce^j - \bar{\eta}$ and $z_+^j = Ce^j + \bar{\eta}$. Let the discrete set $\mathcal{Z}_d(\Omega) \subset \mathcal{Z}(\Omega)$ be composed by all such outputs as follows:

$$\mathcal{Z}_d(\Omega) = \{z \ : \ z = z_-^j, \ z = z_+^j, \ j = 1 \cdots, n_v\},$$

and let $n_z$ be the cardinality of $\mathcal{Z}_d(\Omega)$.

It is assumed that the elements $z^l$ of $\mathcal{Z}_d(\Omega)$ are organized in increasing order, i.e., $z^1 \le z^2 \le ... \le z^{n_z}$.

The following necessary and sufficient conditions can be established [6]:

*Theorem 3.1: The polyhedron* $\Omega = \{Ge \le \mathbf{1}\}$ *is* $\mathcal{D}$-$(C, A)$-$\lambda$*-contractive if and only if:*

$$\forall l = 1, ..., n_z, \ \exists v(z^l) : \phi(z^l) + Gv(z^l) \le \lambda \mathbf{1} - \delta. \qquad (8)$$

*Proof:* The proof is based on the fact that, for $z \in \mathcal{Z}(\Omega)$ between two consecutive $z \in \mathcal{Z}_d(\Omega)$, the functions $\phi_i(z)$ are linear. As a consequence, the following function

$$\begin{array}{c} \varepsilon(z) = \min\limits_{\varepsilon, v} \ \varepsilon \\ \text{under: } \phi(z) + Gv \le \varepsilon \mathbf{1} - \delta, \end{array} \qquad (9)$$

is continuous, piece-wise linear and convex. Thus, for $z^l \le z \le z^{l+1}$, the maximum value of $\varepsilon(z)$ is obtained for either $z^l$ or $z^{l+1}$. Considering all intervals of $\mathcal{Z}(\Omega)$, one can conclude that the maximum value of $\varepsilon(z)$ corresponds to one of the $z^l \in \mathcal{Z}_d(\Omega)$. Therefore, if $\max_l \varepsilon(z^l) \le \lambda$, then $\Omega$ is $\mathcal{D}$-$(C,A)$-$\lambda$-contractive.

The reader is refered to [6] for further details. □

From this Theorem, in order to check for $\mathcal{D}$-$(C,A)$-$\lambda$-contractivity of $\Omega$ it is enough to solve the LP (9) for all $z^l \in \mathcal{Z}_d(\Omega)$, which are associated to the vertices of $\Omega$. Then, $\Omega$ is $\mathcal{D}$-$(C,A)$-invariant if, and only if the optimal solution for all $z^l \in \mathcal{Z}_d(\Omega)$ is such that $\varepsilon(z^l) \le 1$.

## IV. Output Injection Law

Given a $\mathcal{D}$-$(C,A)$-invariant polyhedron $\Omega = \{Ge \le \mathbf{1}\}$, it is necessary to derive an observation law $v(z(k))$ so as to enforce the observation error trajectory to belong to the polyhedron. For sufficiently slow systems, the output injection $v(y)$ can be computed on-line, as proposed in [6], from the following LP:

$$\begin{gathered}\min_{\varepsilon, v(k)} \ \varepsilon \\ \text{under: } \phi(z(k)) + Gv(k) \le \varepsilon \mathbf{1} - \delta.\end{gathered}$$

For fast systems though, such a computation can become infeasible. Consider, then, the system (3), a $\mathcal{D}$-$(C,A)$-$\lambda$-contractive compact polyhedron, $\Omega = \{e : Ge \le \mathbf{1}\}$, with $\lambda < \mathbf{1}$, the set of admissible outputs associated to $\Omega$, $\mathcal{Z}(\Omega)$, and the outputs $z^j \in \mathcal{Z}_d(\Omega)$, organized in increasing order. Then, the following output injection law can be proposed:

*Proposition 4.1: Let $\Omega = \{e : Ge \le \mathbf{1}\}$ be a $\mathcal{D}$-$(C,A)$-$\lambda$-contractive polyhedron, with $\lambda < 1$, for system (3). Then, there is a piece-wise affine time-variant output injection law, given by:*

$$v(z(k), k) = L^j z(k) + \lambda^k w^j \tag{10}$$

*where $L^j \in \mathbb{R}^{n\times 1}$ and $w^j \in \mathbb{R}^n$ are constant for $z^j \le z(k) \le z^{j+1}$, such that $e(k) \in \Omega$, $\forall k$. Moreover, for $d(k) = 0$ and $\eta(k) = 0$ $\forall k$, $e(k) \to 0$ when $k \to \infty$.*

*Proof:* An output $z$ such that $z^j \le z \le z^{j+1}$ can be written in the following form:

$$z = z^j + \alpha(z^{j+1} - z^j), \ 0 \le \alpha \le 1 \ . \tag{11}$$

Since $\Omega$ is $\mathcal{D}$-$(C,A)$-$\lambda$-contractive, then $\forall z^j$, $\exists v^j$ such that $\phi(z^j) + Gv^j + \delta \le \lambda \mathbf{1}$.

Consider now the law:

$$\bar{v}(z) = v^j + \alpha(v^{j+1} - v^j) \tag{12}$$

From the proof of Theorem 3.1, for $z^j \le z \le z^{j+1}$, $\phi(z)$ is linear in $z$, given by $\phi(z) = \Lambda(z)\mathbf{1} + \mu(z)z$, where $\Lambda(z)$ and $\mu(z)$ are constant in this interval. Thus: $\Lambda(z)\mathbf{1} + \mu(z)[(1-\alpha)z^j + \alpha z^{j+1}] + G[(1-\alpha)v^j + \alpha v^{j+1}] + \delta \le \lambda \mathbf{1}$. Then: $\Lambda(z)\mathbf{1} + \mu(z)z + G\bar{v}(z) + \delta \le \lambda\mathbf{1}$. Hence, $\bar{v}(z)$ (12) guarantees that if $e(k) \in \Omega$, then $e(k+1) \in \lambda\Omega$.

Consider now $d(k) = 0$ and $\eta(k) = 0$ $\forall k$. From (11), $\alpha = \frac{z - z^j}{z^{j+1} - z^j}$. Replacing in (12), it results in: $\bar{v}(z) = L^j z + w^j$ with $L^j = \frac{v^{j+1} - v^j}{z^{j+1} - z^j}$ and $w^j = v^j - \frac{z^j}{z^{j+1} - z^j}(v^{j+1} - v^j)$. Replacing $\bar{v}(z(k)) = L^j z + w^j$ in (3), results in $e(k+1) = (A + L^j C)e(k) + w^j$. As a consequence, if $w^j \ne 0$, clearly, $e(k)$ may not converge towards the origin. Such a convergence can be achieved by multiplying $w^i$ by $\lambda^k$, i.e., by means of the observation law $v(z(k), k) = L^j z(k) + \lambda^k w^j$. In this case, $e(k+1) = (A + L^j C)e(k) + \lambda^j w^j$.

By induction, suppose that, at time $k$, $Ge(k) \le \lambda^k \mathbf{1}$. Thus, $G\frac{1}{\lambda^k}e(k) \le \mathbf{1}$, i.e., $\frac{1}{\lambda^k}e(k) \in \Omega$. Then:

$$\begin{gathered}Ge(k+1) = G[(A + L^i C)e(k) + \lambda^k w^i] = \\ \lambda^k G[(A + L^i C)\tfrac{1}{\lambda^k}e(k) + w^i] \le \lambda^k.\lambda\mathbf{1} = \lambda^{k+1}\mathbf{1}.\end{gathered}$$

Therefore, if $e(k) \in \lambda^k\Omega$, then $e(k+1) \in \lambda^{k+1}\Omega$. It can be easily noticed that the hypothesis $e(k) \in \lambda^k\Omega$ is satisfied for $k = 1$, which proves, by induction, that for undisturbed systems without measurement noise, $e(k) \in \Omega$, $\forall k$ and $e(k) \to 0$ when $k \to \infty$, because $e(k) \in \lambda^k\Omega$, $\forall k$ and $\lambda < 1$. □

The output injection law is explicitely constructed in the proof. It must be pointed out that even though it may seem complex, such a law can be computed off-line. Its practical implementation is not hard, once it is quite easy to detect the line segment the measured output belongs to.

## V. Computation of a $\mathcal{D}$-$(C,A)$-invariant Polyhedron

In a typical state observer design problem, the initial state of the system is not known, but it is possible to define a region to which it belongs. Assume that this region is defined by linear inequalities which generate a compact symmetrical polyhedron $\Omega_x = \{x : \ |Qx| \le \mathbf{1}\}$. Then, initializing the observer with $\hat{x}(0) = 0$, the initial error $e(0)$ is such that $|Qe(0)| \le \mathbf{1}$, thus $e(0) \in \Omega = \{e : |Qe| \le \mathbf{1}\}$.

If $\Omega$ is $\mathcal{D}$-$(C,A)$-invariant, then it is possible to achieve optimal error limitation. Indeed, in this case, there would be an output injection $v(z(k))$ such that $e(k) \in \Omega$ $\forall k$ and $\forall d \in \mathcal{D}$, $\eta \in \mathcal{N}$. Therefore the estimation error would not exceed the known limits of the initial error, the set $\Omega$.

However, it is quite rare that the polyhedron defined by the uncertainty on the initial state is $\mathcal{D}$-$(C,A)$-$\lambda$-contractive. Thus, it is necessary to construct a polyhedron which satisfies this propriety, the smallest possible one containing $\Omega$.

As discussed in [5], [6], for a general polyhedra, such a smallest set may not exist. In this sense, $(C,A)$-invariant polyhedra as defined here are not dual to $(A,B)$-controled-invariant ones [1], [9]. Therefore, standard controled-invariant set computation cannot be used in the general $(C,A)$-invariance setting. Nevertheless, a polyhedron which is $\mathcal{D}$-$(C,A)$-$\lambda$-contractive and results in a suitable limitation of the observation error can be

computed by the following algorithm, which has been proposed in [6]:
Given: $\Omega = \{e : \ |Qe| \leq \mathbf{1}\}$, the initial set of estimation errors; $\lambda$, the desired contraction rate.

1) Define the tolerance $\Delta\delta$. Initialize $i = 0$, $Q^0 = Q$, $Q^0 \in \mathbb{R}^{q^0 \times n}$, $\mathcal{C}^0 = \{e : \ Q^0 e \leq \mathbf{1}\}$
2) Compute the vertices of $\mathcal{C}^i$, $e^{i^j}$;
3) Compute the set $\mathcal{Z}_d(\mathcal{C}^i)$; Set $n_{z_i}$ equal to the cardinality of $\mathcal{Z}_d(\mathcal{C}^i)$.
4) For $l = 1, ..., n_{z_i}$:
   a) Compute $\phi^{i^l}(Ce^l)$ from (6);
   b) Compute $\varepsilon(Ce^l)$ from (9);
5) Set $\varepsilon^i = \max\limits_l \varepsilon^l$; Set $v^i$ and $e^i$ as the optimal values of $v$ and $e$ in (9), associated to $\varepsilon^i$;
6) If $\varepsilon^i \leq \lambda(1 + \Delta\delta)$, STOP! $\mathcal{C}^i = \{e : \ |Q^i e| \leq \mathbf{1}\}$ is $\mathcal{D}$-$(C, A)$-$\lambda$-contractive;
7) Compute the set:
$$\begin{aligned} \mathcal{Q}^i \ &= \{x : x = Ae + B_1 d + v^i, \text{ for some } e, d \\ &\quad \text{such that } |Q^i e| \leq \mathbf{1}, |Ce - Ce^i| \leq \bar{\eta}, \\ &\quad |Ed| \leq \mathbf{1}\}. \end{aligned}$$
8) Compute $\mathcal{C}^{i+1} = \mathrm{Conv}(\mathcal{C}^i \cup \frac{1}{\lambda}\mathcal{Q}^i)$
9) Do $i = i + 1$ and return to step 2.

The key point of this algorithm is step 8. It picks up the worst case $\varepsilon$, computes the corresponding optimal $v$ which tries to place the set $\mathcal{Q}^i$ (the one-step propagation of all possible points $e$ associated to the output $z^i$) inside $\mathcal{C}^i$. If it succeeds, then $\mathcal{C}^i$ is $\mathcal{D}$-$(C, A)$-$\lambda$-contractive. Otherwise, another candidate set is computed through the convex hull of the union of $\mathcal{C}^i$ and $\frac{1}{\lambda}\mathcal{Q}^i$. Even though the convergence of this algorithm has not been proved, no example for which it does not converge has been found. It has been tested on more than 20 ramdomly generated 3rd and 4th order systems, with initial polyhedron given by $\{e : |e| \leq \mathbf{1}\}$.

Stronger results can be established if the polyhedron is symmetrtical with respect to the origin.

## VI. $\mathcal{D}$-$(C, A)$-INVARIANCE OF SYMMETRICAL POLYHEDRA

Assume now that $\Omega$ and $\mathcal{D}$ are symmetrical with respect to the origin. Hence, they can be represented as:
$$\Omega = \{e : |Qe| \leq \mathbf{1}\}, \ \mathcal{D} = \{d : |Ed| \leq \mathbf{1}\}.$$
$\Omega$ (and $\mathcal{D}$ accordingly) can be written in the standard form $Ge \leq \mathbf{1}$, with $G = \begin{bmatrix} Q \\ -Q \end{bmatrix}$.

Let also the elements of the vector of worst case disturbances be now defined as:
$$\xi_i \ = \ \begin{array}{c} \max_d Q_i B_1 d \\ \text{under: } |Ed| \leq \mathbf{1} \end{array}. \tag{13}$$

In this case, considering $Q \in \mathbb{R}^{q \times n}$, for $i \leq q$, $\phi_i(z) = \max\limits_e Q_i Ae$ under $|Qe| \leq \mathbf{1}$, $|Ce - z| \leq \bar{\eta}$. Therefore, $\phi_{i+q}(z) = \max\limits_e -Q_i Ae$ under the same constraints.

One can then conclude that, for $z = 0$, $\phi_{i+q}(0) = \phi_i(0)$. Hence, $\phi(0) = \begin{bmatrix} \phi_q(0) \\ \phi_q(0) \end{bmatrix}$, where $\phi_q(z)$ corresponds to the $q$ firsts rows of $\phi(z)$. Thus condition (7) becomes, for $z = 0$:
$$\exists v(0) : \begin{bmatrix} \phi_q(0) \\ \phi_q(0) \end{bmatrix} + \begin{bmatrix} Q \\ -Q \end{bmatrix} v(0) \leq \lambda \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} - \begin{bmatrix} \xi \\ \xi \end{bmatrix}$$

As an immediate consequence, the following necessary condition can be stated:

*Lemma 6.1:* $\Omega = \{e : |Qe| \leq \mathbf{1}\}$ *is* $\mathcal{D}$-$(C, A)$-$\lambda$-*contractive only if:*
$$\phi_q(0) \leq \lambda\mathbf{1} - \xi \tag{14}$$
One should notice that this necessary condition is much easier to be verified than the necessary and sufficient ones of Theorem 3.1, insofar as the computation of vertices is no more required. Moreover, in many systems tested along the development of this research it turned out to be sufficient as well. Condition (14) is also very useful to compute a $\mathcal{D}$-$(C, A)$-$\lambda$-contractive polyhedron.

Consider now the following class of polyhedra which satisfy the necessary condition (14):
$$\begin{aligned} \mathcal{K}(\Omega, \mathcal{D}, \lambda) = \{&\text{set of symmetrical polyhedra} \\ &\text{containing } \Omega \text{ such that } \phi_q(0) \leq \lambda\mathbf{1} - \xi\}. \end{aligned}$$

*Lemma 6.2: The intersection of two polyhedra belonging to* $\mathcal{K}(\Omega, \mathcal{D}, \lambda)$ *also belongs to* $\mathcal{K}(\Omega, \mathcal{D}, \lambda)$.
*Proof:* Immediate (see [6]). □

This Lemma assures the existence of the set:
$$\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, \lambda) = \text{ infimal set in } \mathcal{K}(\Omega, \mathcal{D}, \lambda),$$
which is the smallest set containing $\Omega$ satisfying the necessary condition $\phi_q(0) \leq \lambda\mathbf{1} - \xi$. Therefore, if $\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, \lambda)$ satisfies the sufficient condition too, it can be assured that it is the smallest $\mathcal{D}$-$(C, A)$-$\lambda$-contractive set containing $\Omega$.

$\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, \lambda)$ can be computed by means of a simplified algorithm, with the following main modifications with respect to the general algorithm:

- replace steps 2 to 5 by *compute* $\phi_q(0)$ *and* $\varepsilon = \max_i \phi_{q_i}$;
- $$\begin{aligned} \mathcal{Q}^i \ &= \{x : x = Ae + B_1 d, \text{ for some } e, d \\ &\quad \text{such that } |Q^i e| \leq \mathbf{1}, |Ce| \leq \bar{\eta}, |Ed| \leq \mathbf{1}\}. \end{aligned}$$

For $d(k) = 0$ and $\eta(k) = 0$, it can be shown that this algorithm is dual to the algorithm for computation of the largest $(A, B)$-$\lambda$-contractive set contained in a given polyhedron [10], [9]. This assures its convergence towards $\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \lambda)$.

The implementation of the proposed algorithms only requires the solution of linear programming problems and the manipulation of polyhedra (computation vertices and convex hulls) for which several methods are available (see, e.g. [11], [12]).

## VII. Optimal Error Limitation in a Class of Systems

In this section, multiple-output systems are also considered. Consider the system (3) without measurement noise ($\eta(k)=0\ \forall k$), $z \in \mathbb{R}^p$ and the following hypothesis:

$$p = n-1,$$

i.e. the number of measured outputs is equal to the number of states minus one.

*Proposition 7.1:* If $p=n-1$, a symmetrical polyhedron $\Omega=\{e : |Qe| \le \mathbf{1}\}$ is $\mathcal{D}$-$(C,A)$-$\lambda$-contractive if, and only if:

$$\phi_q(0) \le \lambda \mathbf{1} - \xi \ . \tag{15}$$

Moreover, there exists a linear output injection $v(z(k)) = Lz(k)$, such that $\Omega$ is $\lambda$-contractive.

*Proof:* The dimension of the polyhedral set:

$$\Omega \cap \{Ce = z\} = \{e \ : \ |Qe| \le \mathbf{1},\ Ce = z\}$$

is $n-p$ [11]. Since $p=n-1$, then $\Omega \cap \{Ce=z\}$ is an one-dimensional polyhedron, i.e. a line segment in $\mathbb{R}^n$.

Therefore, since $\Omega$ is symmetrical, the set

$$\Omega \cap \ker C = \{|Qe| \le \mathbf{1}\ ,\ Ce = 0\}$$

can be written as:

$$\Omega \cap \ker C = \{|Q_I e| \le \mathbf{1}\ ,\ Ce = 0\}$$

where $Q_I$ is the only non-redundant row of $Q$ in $\Omega \cap \ker C$. Hence, the matrix $\begin{bmatrix} Q_I \\ C \end{bmatrix}$ is invertible and:

$$\exists q \in \Re^n : Q_I q = 0,\ Cq = -z.$$

Let now $v(z) = Aq$. Then:

$$QAe + Qv(z) = QA(e+q) = QA\tilde{e},$$

with $\tilde{e} = e+q$.

Clearly, $\tilde{e}$ is such that:

$$\begin{array}{l} |Q_I\tilde{e}| = |Q_I(e+q)| = |Q_I e| \le \mathbf{1} \\ C\tilde{e} = C(e+q) = 0 \end{array} . \tag{16}$$

Therefore, if $e(k) \in \Omega$, then

$$\begin{array}{rcl} e(k+1) & = & Ae(k) + v(z(k)) + B_1 d(k) \\ & = & A(e(k)+q) + B_1 d(k) \\ & = & A(\tilde{e}) + B_1 d(k) \end{array}$$

Then,

$$\begin{array}{rcl} |Qe(k+1)| & = & |QA(\tilde{e}) + QB_1 d(k)| \\ & \le & |QA(\tilde{e})| + |QB_1 d(k)| \end{array} .$$

Hence, from (5), (16) and (13):

$$\begin{array}{rcl} |Qe(k+1)| & \le & \phi(0) + \xi \\ & \le & \lambda \mathbf{1} \end{array} ,$$

and the $\lambda$-contractivity of $\Omega$ is proved. The output injection is given by:

$$v(z(k)) = Lz(k) = Aq,$$

where:

$$q = \begin{bmatrix} Q_I \\ C \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -z \end{bmatrix} = \begin{bmatrix} Q_I \\ C \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I \end{bmatrix} z.$$

Then, $L = A \begin{bmatrix} Q_I \\ C \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I \end{bmatrix}$. □

As a consequence of this result and the discussion of section VI, for this class of systems it is possible to achieve optimal error limitation through the computation of the set $\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, \lambda)$.

*Example 7.1:* Consider the system (3), with $\eta(k)=0$, for which:

$$A = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The symmetrical polyhedron $\Omega_x = \{x \ : \ |Qx| \le \mathbf{1}\}$ which represents the uncertainty on the initial state $x(0)$ is represented by: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The disturbance set is given by: $\mathcal{D} = \{d : |d| \le 1\}$.

Initializing the observer with $\hat{x}(0) = 0$, the initial estimation error $e(0)$ belongs to the polyhedron $\Omega = \{e : \ |Qe| \le \mathbf{1}\}$.

A set-valued observer was synthesized for this system in [3] and a set-invariant one in [6], considering the existence of measurement noise.

Since $n-p = 2-1 = 1$, then the set $\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, \lambda)$ is the infimal $\mathcal{D}$-$(C,A)$-invariant set containing $\Omega$. With $\lambda = 0.9$, it is obtained after 2 iterations and is given by $\{e \ : \ |Q^2 e| \le \mathbf{1}\}$ with:

$$Q^2 = \begin{bmatrix} -0.4412 & 0.5588 \\ -0.8333 & 0.1667 \\ 0.9000 & 0 \end{bmatrix}$$

In figure 1, the polyhedron $\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, 0.9)$ and the initial polyhedron $\Omega$ are shown, together with a trajectory starting from the origin, with the following disturbance sequence:

$$\begin{array}{rl} \{d\} = & \{1,-1,-1,1,1,-1,1,-1,-1, \\ & -1,1,-1,1,-1,-1,-1,1\}, \end{array}$$

The output injection law is linear with $L = \begin{bmatrix} -0.7000 \\ -0.2212 \end{bmatrix}$.

*Example 7.2:* Consider now the system (3), with $\eta(k) = 0$, and:

$$A = \begin{bmatrix} -1.3335 & -0.0113 & 0.3966 \\ 1.0727 & -0.0008 & -0.2640 \\ -0.7121 & -0.2494 & -1.6640 \end{bmatrix}, B_1 = 0,$$

$$C = \begin{bmatrix} -1.0290 & 0.2431 & -1.2566 \end{bmatrix}.$$

In this example, $n-p = 3-1 = 2$, hence the condition $n-p=1$ is not satisfied. The symmetrical polyhedron $\Omega_x = \{x \ : \ |Qx| \le \mathbf{1}\}$ which models the

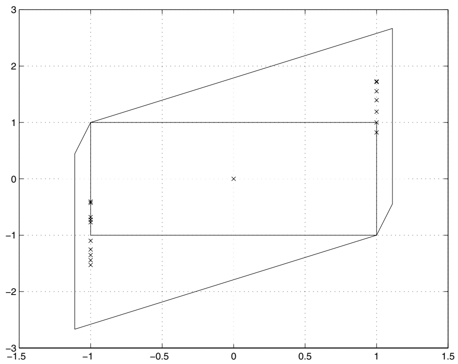


Fig. 1. $\Omega$ (.) and $\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, 0.9)$ and a trajectory of the error for Example 7.1.

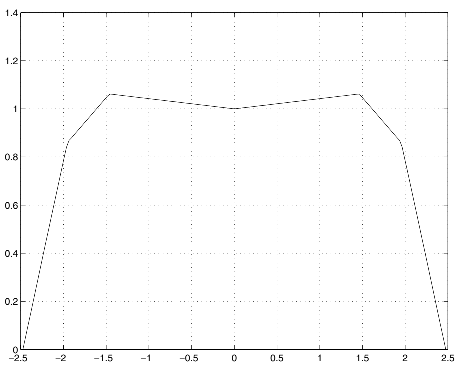


Fig. 2. $\varepsilon \times z$ for Example 7.2.

uncertainty on the initial state $x(0)$ is represented by:

$$Q = \begin{bmatrix} 0.2681 & 0.7595 & -0.0851 \\ 0.2650 & 0.3250 & -0.3861 \\ 0.5816 & 0.7042 & 0.4561 \\ 0.7181 & 0.2776 & 0.4182 \\ 0.2706 & 0.7595 & -0.0806 \\ 0.5344 & 0.2771 & 0.0786 \\ -0.2693 & 0.3721 & -0.5514 \\ -0.0299 & 0.1443 & -0.6200 \\ -0.4393 & 0.1386 & -0.5010 \\ -0.2217 & 0.4188 & -0.5016 \end{bmatrix}.$$

The set $\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, \lambda)$, with $\lambda = 1$ was computed and the plot of $\varepsilon(z) \times z$ (9) is depicted in Figure (2). Even though the necessary condition of Lemma 14 is verified, this polyhedron is not $\mathcal{D}$-$(C, A)$-invariant because $\varepsilon(z) > 1$ for some $z$. A $\mathcal{D}$-$(C, A)$-invariant polyhedron can however be obtained by the application of the algorithm for the general case.

## VIII. Conclusions

In this work a new approach for the design of full-order state estimators for discrete-time linear systems subject to persistent disturbances and measurement noise was presented, extending the results of [6] in two ways:

- it has been shown that the estimation error can be forced to remain inside a $\mathcal{D}$-$(C, A)$-invariant polyhedral set by means of a piece-wise affine output injection law;

- in a particular case, it has been shown that through a linear output injection law the optimal error limitation can be achieved, i.e., the error can be confined to the smallest $\mathcal{D}$-$(C, A)$-invariant polyhedron which contains the polyhedron the initial error is known to belong to.

Compared to set-membership observers, the main advantages of the set-invariant estimators are: the ability to impose a limitation to the estimation error, which is guaranteed to be optimal in some cases; off-line computations implying smaller numerical on-line effort.

Our research effort is now directed towards the study of the convergence of the general algorithm to compute a $\mathcal{D}$-$(C, A)$-invariant set, as well as its minimality properties; the extension of the results to multiple-output systems and the possibility of using the estimator for controling constrained systems.

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