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Smooth and Analytic Normal and Canonical Forms for Strict Feedforward Systems

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Abstract— **Recently we proved that any smooth (resp. analytic) strict feedforward system can be brought into its normal form via a smooth (resp. analytic) feedback transformation. This will allow us to identify a subclass of strict feedforward systems, called systems in *special strict feedforward form*, shortly (SSFF), possessing a canonical form which is an analytic counterpart of the formal canonical form. For (SSFF)-systems, the step-by-step normalization procedure of Kang and Krener leads to smooth (resp. convergent analytic) normalizing feedback transformations. We illustrate the class of (SSFF)-systems by a model of an inverted pendulum on a cart.**

I. Introduction

In this paper we study the problem of analytic normal forms for analytic strict feedforward systems. A single-input nonlinear control system of the form

$$\Pi: \ \dot{x} = f(x,u),$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$, is in *strict feedforward form* if we have

$$(SFF) \qquad \begin{array}{rcl} \dot{x}_1 & = & f_1(x_2,\ldots,x_n,u) \\ & \cdots & \\ \dot{x}_{n-1} & = & f_{n-1}(x_n,u) \\ \dot{x}_n & = & f_n(u). \end{array}$$

A basic structural property of systems in strict feedforward form is that their solutions can be found by quadratures. Indeed, knowing $u(t)$ we integrate $f_n(u(t))$ to get $x_n(t)$, then we integrate $f_{n-1}(x_n(t),u(t))$ to get $x_{n-1}(t)$, we keep doing that, and finally we integrate $f_1(x_2(t),\ldots,x_n(t),u(t))$ to get $x_1(t)$.

In view of the above, systems in strict feedforward form can be considered as duals of flat systems. In the single-input case, flat systems are feedback linearizable and are defined as systems for which we can find a function of the state that, together with its *derivatives*, gives all the states and the control of the system [5]. In a dual way, for systems in strict feedforward form (SFF), we can find all states via a successive *integration* starting from a function of the control.

Another property, crucial in applications, of systems in (strict) feedforward form is that we can construct for them a stabilizing feedback. This important result goes back to Teel [37] and has been followed by a growing literature on stabilization and tracking for systems in (strict) feedforward form (see e.g. [11], [19], [28], [38], [3], [20]).

The problem of transforming a system into a (strict) feedforward form has recently been studied using various techniques: in [18] only state transformations are applied, the notion of (controlled) invariant distributions is used in [2], a step-by-step constructive method to bring a system into a feedforward form [35] and strict feedforward form [33], [34] has been developed by the authors who also described in [25] relations between strict feedforward forms and symmetries.

The general problem of transforming the nonlinear control single-input system

$$\Pi: \ \dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

by an invertible feedback transformation of the form

$$\Gamma: \quad \begin{array}{rcl} z & = & \phi(x) \\ u & = & \gamma(x,v) \end{array}$$

to a simpler form has been extensively studied during the last twenty years. The transformation Γ brings Π into the system

$$\tilde{\Pi}: \ \dot{z} = \tilde{f}(z,v),$$

whose dynamics are given by

$$\tilde{f}(z,v) = \mathrm{d}\phi(\phi^{-1}(z)) \cdot f(\phi^{-1}(z), \gamma(\phi^{-1}(z),v)).$$

If the control u is not present, that is, the system Π is actually a dynamical system of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

then the transformation Γ consists solely of a change of coordinates $z = \phi(x)$. A classical problem addressed by Poincaré is whether it is possible to find local coordinates $z = \phi(x)$ around an equilibrium point in which the dynamical system becomes linear. Poincaré has solved it by applying, step by step, homogeneous changes of coordinates in order to normalize the corresponding homogeneous parts of the same degree of the system. If all homogenous parts can be annihilated (no resonances), we formally linearize the system. If not, the result of this normalization procedure gives a *formal normal form*, which contains nonlinearizable terms only (called *resonant terms*, (see e.g. [1]).

Similarly, for control systems, the natural question of feedback equivalence of Π to a linear system $\tilde{\Pi}$ has been studied and solved in [6] and [9]. If the geometric linearizability conditions are not satisfied, a natural problem is to find normal forms for non linearizable systems. Various approaches have been proposed, based on the singularity

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theory, Cartan equivalence method, hamiltonian formalism etc (see, e.g. [26] for references). In this paper, we will use a very fruitful approach of Kang and Krener [14], [12], [13] who have proposed to analyze (following Poincaré), step by step, the action of the Taylor series expansion of the feedback transformation Γ on the Taylor series expansion of the system Π and have obtained for single-input control systems with controllable linearization normal forms for the quadratic terms [14] and then for higher order terms [12]. The results of Kang and Krener [14], [12] have been completed by the authors who obtained canonical forms and dual canonical forms for single-input nonlinear control systems with controllable [31] and then with uncontrollable linearization [32] (see also [17]). Recently those results have been generalized by Tall [29], [30] to multi-input nonlinear control systems.

Although these normal and canonical forms are formal, they are very useful in studying bifurcations of nonlinear systems [15], [17], in obtaining a complete description of symmetries around equilibria [23], [24], and in characterizing systems equivalent to feedforward forms [33], [35], [34].

Challenging questions are thus whether these normal forms have their counterparts in the C^∞-smooth and real analytic (C^ω) categories and what are conditions for the normalizing procedure to be convergent. In other words, what are obstructions for obtaining smooth and analytic normal forms for control systems?

It is well known that the problems of convergence of the normalizing transformations is difficult already for dynamical systems. It was solved (in terms of locations of the eigenvalues of the linearization) by Sternberg and Chen in the C^∞-category and by Poincaré, Dulac, Siegel, and others in the C^ω- category (see [1] for details and references).

For control systems, the eigenvalues of the linearization are not invariant under feedback and the convergence problem seems to be even more involved. The only known results relating formal and C^ω-normal forms are in Kang [12]: for feedback linearizable systems (based on [16]) and for a class of non linearizable 3-dimensional systems. Other normal forms in the C^∞- and C^ω-categories have been obtained in [4], [8], [10], [27], [40] via singularity theory methods.

Very recently we showed in [36] that any smooth (resp. analytic) strict feedforward system can be brought to its normal form via a smooth (resp. analytic) feedback transformation. This allowed us to identify in [36] a subclass of strict feedforward systems, called *special strict feedforward systems*, possessing a smooth normal form. In this paper we will show that an analytic special strict feedforward system can be brought to an analytic canonical form. These normal and canonical forms are, respectively, smooth and analytic counterparts of the corresponding formal forms obtained, respectively, by Kang [12] (normal form) and the authors [31] (canonical form).

The paper is organized as follows. In Section II we will recall the Kang normal form and the canonical form of the authors for single-input systems. Our main results: smooth and analytic normal and canonical forms for strict feedforward and special strict feedforward systems are given in Section III and the proof of the canonical form in Section V. Finally, in Section IV we illustrate our strict feedforward normal forms by a model of inverted pendulum on a cart.

II. Notation and definitions

All objects, i.e., functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be either smooth (which will always mean C^∞-smooth) or real analytic (denoted by C^ω). Let h be a smooth function. By

$$h(x) = h^{[0]}(x) + h^{[1]}(x) + h^{[2]}(x) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(x)$$

we denote its Taylor expansion around zero, where $h^{[m]}(x)$ stands for a homogeneous polynomial of degree m.

Similarly, for a map ϕ of an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ (resp. for a vector field f on an open subset of $\mathbb{R}^n$) we will denote by $\phi^{[m]}$ (resp. by $f^{[m]}$) the term of degree m of its Taylor expansion at zero, i.e., each component $\phi_j^{[m]}$ of $\phi^{[m]}$ (resp. $f_j^{[m]}$ of $f^{[m]}$) is a homogeneous polynomial of degree m in x.

Together with the system Π, we will also consider its infinite Taylor series expansion, given by

$$\Pi^\infty : \dot{x} \quad = \quad f(x,u) = Fx + Gu + \sum_{m=2}^{\infty} f^{[m]}(x,u), \tag{II.1}$$

where $F = \frac{\partial f}{\partial x}(0)$ and $G = \frac{\partial f}{\partial u}(0)$. We will assume throughout the paper that $f(0,0) = 0$.

Consider also the Taylor series expansion Γ^∞ of the feedback transformation Γ given by

$$\Gamma^\infty : \begin{array}{rcl} z &=& \phi(x) = Tx + \sum_{m=2}^{\infty} \phi^{[m]}(x) \\ u &=& \gamma(x,v) = Kx + Lv + \sum_{m=2}^{\infty} \gamma^{[m]}(x,v), \end{array} \tag{II.2}$$

where the matrix T is invertible and $L \neq 0$. The action of Γ^∞ on the system Π^∞ step by step leads to formal normal forms. The following normal form was obtained by Kang [12] (see also [14], [31]) and then completed by the authors who obtained the canonical forms (see [31] for details):

Theorem II.1 *Consider the system Π^∞, defined by (II.1).*

(i) Π^∞ *is feedback equivalent, by a formal transformation Γ^∞ of the form (II.2), to the formal normal form*

$$\Pi_{NF}^\infty \ : \dot{z} = Az + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(z,v),$$

where for any $m \geq m_0 \geq 2$, we have

$$\bar{f}_j^{[m]}(z,v) = \left\{ \begin{array}{ll} \sum_{i=j+2}^{n+1} z_i^2 P_{j,i}^{[m-2]}(\bar{z}_i), & 1 \leq j \leq n-1, \\ 0, & j = n, \end{array} \right. \tag{II.3}$$

with $P_{j,i}^{[m-2]}(\bar{z}_i)$ being homogeneous polynomials of degree $m-2$ of $\bar{z}_i = (z_1, \ldots, z_i)$, and $z_{n+1} = v$.

(ii) *The system Π^∞ given by* (II.1) *is equivalent by a formal feedback Γ^∞ to its canonical form*

$$\Pi_{CF}^\infty \ : \ \dot{z} = Az + Bv + \sum_{m=m_0}^{\infty} \bar{f}^{[m]}(z),$$

where, for any $m \geq m_0$, the components $\bar{f}_j^{[m]}(z)$ of $\bar{f}^{[m]}(z)$ are given by (II.3); additionally, we have

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m_0]}}{\partial z_1^{i_1} \cdots \partial z_{n-s}^{i_{n-s}}} = \pm 1 \qquad \text{(II.4)}$$

and, moreover, for any $m \geq m_0 + 1$,

$$\frac{\partial^{m_0} \bar{f}_{j^*}^{[m]}}{\partial z_1^{i_1} \cdots \partial z_{n-s}^{i_{n-s}}}(z_1, 0, \ldots, 0) = 0. \qquad \text{(II.5)}$$

(iii) *Two systems Π_1^∞ and Π_2^∞ are formally feedback equivalent if and only if their canonical forms $\Pi_{1,CF}^\infty$ and $\Pi_{2,CF}^\infty$ coincide.*

In (II.4) and (II.5), m_0 is the degree of the first nonlinearizable term and the integers j^* and tuple $(i_1, \ldots, i_{n-s})$ are defined in [31]. The form Π_{CF}^∞ satisfying (II.3), (II.4) and (II.5) is called the *canonical form* of Π^∞ because of its uniqueness property (iii).

The problem whether an analogous result holds in the smooth (resp. analytic) category is actually a challenging question, which can be formulated as whether for a smooth (resp. analytic) system Π the normalizing feedback transformation Γ^∞ gives rise to a smooth (resp. convergent) Γ and thus leads to a smooth (resp. analytic) normal form Π_{NF} and/or canonical form Π_{CF}. One of the difficulties resides in the fact that it is not clear at all how to express, in terms of the original system, homogeneous invariants transformed via an infinite composition of homogeneous feedback transformations. We will study in this paper a special class of analytic control systems, namely strict feedforward systems, that can be brought to their canonical (thus normal) forms by analytic transformations.

III. Main Results

Consider the class of *smooth* or *analytic* single-input control systems

$$\Pi \ : \ \dot{x} = f(x, u),$$

in *strict feedforward form* (SFF), that is, such that

$$f_j(x, u) = f_j(x_{j+1}, \ldots, x_n, u), \quad 1 \leq j \leq n.$$

Notice that for any $1 \leq i \leq n$, the subsystem Π_i, defined as the projection of Π onto $\mathbb{R}^{n-i}$ via $\pi(x_1, \ldots, x_n) = (x_{i+1}, \ldots, x_n)$, is a well defined system whose dynamics are given by

$$\dot{x}_j = f_j(x_{j+1}, \ldots, x_n, u), \qquad \text{for } i \leq j \leq n.$$

Define the linearizability index of the (SFF)-system to be the largest integer p such that the subsystem Π_r, where $p+r = n$, is feedback linearizable. Clearly, the linearizability index is feedback invariant and hence the linearizability indices of two feedback equivalent systems coincide. In this paper we will assume that the linear approximation around the origin is controllable. In this case $p \geq 2$. The general case of uncontrollable linearization will be considered elsewhere.

Each component of a strict feedforward system (SFF) decomposes uniquely, locally or globally, as:

$$f_j(x, u) = h_j(x_{j+1}) + F_j(x_{j+1}, \ldots, x_n, u), \qquad \text{(III.1)}$$

for $1 \leq j \leq n$ (we put $F_n = 0$), where

$$F_j(x_{j+1}, 0, \ldots, 0) = 0. \qquad \text{(III.2)}$$

A strict feedforward form for which

$$h_j(x_{j+1}) = k_j x_{j+1}, \quad 1 \leq j \leq r-1, \qquad \text{(III.3)}$$

for some non zero real numbers $k_1, \ldots, k_{r-1}$, will be called a *special strict feedforward form* (SSFF).

The main result of this paper is as follows.

Theorem III.1 *Consider an analytic special strict feedforward form (SSFF) given by (III.1)-(III.2)-(III.3), locally around $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}$ (resp. globally on $\mathbb{R}^n \times \mathbb{R}$). There exists a local around (x_0, u_0) (resp. global on $\mathbb{R}^n \times \mathbb{R}$) analytic feedback transformation that maps (x_0, u_0) into $(0, 0)$ and brings the system (III.1)-(III.2)-(III.3) into the canonical form*

$$\Pi_{SSFCF} : \begin{cases} \dot{z}_1 = z_2 + \sum_{i=3}^{n+1} z_i^2 P_{1,i}(z_2, \ldots, z_i) \\ \quad \cdots \\ \dot{z}_j = z_{j+1} + \sum_{i=j+2}^{n+1} z_i^2 P_{j,i}(z_{j+1}, \ldots, z_i) \\ \quad \cdots \\ \dot{z}_r = z_{r+1} + \sum_{i=r+2}^{n+1} z_i^2 P_{r,i}(z_{r+1}, \ldots, z_i) \\ \dot{z}_{r+1} = z_{r+2} \\ \quad \cdots \\ \dot{z}_{n-1} = z_n \\ \dot{z}_n = v\,, \end{cases} \qquad \text{(III.4)}$$

where $P_{j,i}(z_{j+1}, \ldots, z_i)$ are analytic functions of the indicated variables, $z_{n+1} = v$ and

$$\frac{\partial^{m_0} z_s^2 P_{r,s}}{\partial z_{r+1}^{i_{r+1}} \cdots \partial z_s^{i_s}}(z_1, 0, \ldots, 0) = \pm 1. \qquad \text{(III.5)}$$

The meaning of the integers $m_0 \geq 2$, $k \geq 0$, s and of the tuple $(i_{r+1}, \ldots, i_s)$ will be made precise in the proof.

The main observation is that the canonical form Π_{SSFCF} given by (III.4)-(III.5) is itself a (SSFF)-system. Recall that, by Theorem II.1, any system (not necessarily (SSFF)) can be brought to its formal canonical form Π_{CF}^∞ via a formal feedback transformation Γ^∞. If the system is in special strict feedforward form (SSFF), then its formal canonical form Π_{CF}^{∞} is actually the formal power series of the analytic canonical form Π_{SSFCF} (whose existence is assured by Theorem III.1), which is, moreover, strict feedforward. In other words, for a (SSFF)-system put into its canonical form, the formal series expansions $\sum_{m=m_0}^{\infty} \bar{f}^{[m]}(z,v)$ (with the components $\bar{f}_j^{[m]}$ of $\bar{f}^{[m]}$ given by (II.3)) can be replaced by analytic functions of (III.4), exhibiting additionally a strict feedforward form. A counterpart of Theorem III.1 for C^{∞}-smooth systems was proved in [36] for normal forms Π_{SSFCF} satisfying (III.4) only.

To justify the name *canonical form*, consider another analytic system

$$\tilde{\Pi} \;:\; \dot{\tilde{x}} = \tilde{f}(\tilde{x},\tilde{u}),$$

in *strict feedforward form*, that is, such that

$$\tilde{f}_j(\tilde{x},\tilde{u}) = \begin{cases} \tilde{h}_j(\tilde{x}_{j+1}) + \tilde{F}_j(\tilde{x}_{j+1},\ldots,\tilde{x}_n,\tilde{u}), & 1 \le j \le \tilde{r} \\ 0, & \tilde{r}+1 \le j \le n \end{cases} \tag{III.6}$$

where

$$\tilde{F}_j(\tilde{x}_{j+1},0,\ldots,0) = 0, \qquad \text{and} \qquad d\tilde{F}_j(0) = 0. \tag{III.7}$$

It is in the special strict feedforward form (SSFF) if

$$\tilde{h}_j(\tilde{x}_{j+1}) = \tilde{k}_j \tilde{x}_{j+1}, \qquad 1 \le j \le \tilde{r}-1, \tag{III.8}$$

for some non zero real numbers $\tilde{k}_1,\ldots,\tilde{k}_{\tilde{r}-1}$. We then have the following result justifying the name of canonical form:

Theorem III.2 *Two analytic special strict feedforward systems $(SSFF)$ given, respectively by, (III.1)-(III.2)-(III.3) and (III.6)-(III.7)-(III.8) are analytic feedback equivalent if and only if their canonical forms Π_{SSFCF} and $\tilde{\Pi}_{SSFCF}$ coincide.*

The proof of this theorem is given in Section V. A natural question to ask is whether it is always possible to transform a strict feedforward form, given by (III.1)-(III.2), into a special strict feedforward form (III.1)-(III.2)-(III.3).

Theorem III.3 *If two analytic (SFF)-systems given by, respectively, (III.1)-(III.2) and (III.6)-(III.7) are feedback equivalent, then $r = \tilde{r}$ and*

$$\tilde{h}_j(l_{j+1}\tilde{x}_{j+1}) = l_j h_j(\tilde{x}_{j+1}), \qquad 1 \le j \le r-1,$$

for some non zero real numbers $l_1,\ldots,l_{r-1}$.

Corollary III.4 *An analytic strict feedforward system (SFF), given by (III.1)-(III.2), is feedback equivalent to the special strict feedforward form (SSFF), given by (III.6)-(III.7)-(III.8), if and only if*

$$h_j(x_{j+1}) = k_j x_{j+1},$$

for $1 \le j \le r-1$, that is, the nonlinearizable part of the system is already in (SSFF) in its original coordinates.

Basically, Theorem III.3 or Corollary III.4 imply that if the nonlinearizable part of a (SFF)-system is not in a (SSFF), then it cannot be brought to that form by any smooth (in particular, analytic) feedback transformation. This means that special strict feedforward forms (SSFF) define the only subclass of strict feedforward systems that can be brought to the Kang normal form Π_{NF} (actually, canonical form Π_{CF}) still being in the strict feedforward form. Whether it is possible to bring a smooth (resp. analytic) (SFF)-system into its normal form Π_{NF} or canonical form Π_{CF} by a smooth (resp. analytic) transformation is unclear but if true, then the normal form Π_{NF} (or canonical form Π_{CF}) will loose the structure of (SFF) (unless the system is (SSFF)). On the other hand, any smooth (resp. analytic) strict feedforward form (SFF) can be brought to a smooth (resp. analytic) form Π_{SFNF}, called *strict feedforward normal form* (introduced by the authors in [33] in the formal category), which is close as much as possible to the normal form Π_{NF}. Indeed, we have the following result proved in [36] (which, together with Theorem III.3, implies Theorem III.1):

Theorem III.5 *Any smooth (resp. analytic) strict feedforward form (SFF), given by (III.1)-(III.2), is smooth (resp. analytic) feedback equivalent to the* strict feedforward normal form *(SFNF):*

$$\Pi_{SFNF} : \begin{cases} \dot{z}_1 = \bar{h}_1(z_2) + \sum_{i=3}^{n+1} z_i^2 P_{1,i} \\ \quad\cdots \\ \dot{z}_j = \bar{h}_j(z_{j+1}) + \sum_{i=j+2}^{n+1} z_i^2 P_{j,i} \\ \quad\cdots \\ \dot{z}_{n-1} = \bar{h}_{n-1}(z_n) + z_{n+1}^2 P_{n-1,n+1} \\ \dot{z}_n = \bar{h}_n(v)\,, \end{cases} \tag{III.9}$$

where $z_{n+1} = v$, $\bar{h}_j(z_{j+1}) = h_j(z_{j+1})$ and $P_{j,i} = P_{j,i}(z_{j+1},\ldots,z_i)$ are smooth (resp. analytic) functions of the indicated variables.

Provided that the linear approximation is controllable, the linearizability index of a general (SFF)-system on $\mathbb{R}^2$ is at least one while the linearizability index of a general control-affine system on $\mathbb{R}^3$ is at least two. It follows that in those two cases the functions h_j are not invariant (compare Theorem III.3), which implies the following:

Corollary III.6 (i) *Any smooth (resp. analytic) strict feedforward form (SFF) on $\mathbb{R}^2$, given by (III.1)-(III.2), is special and is feedback equivalent to the* normal form

$$\begin{aligned} \dot{z}_1 &= z_2 + v^2 P_{1,3}(z_2,v) \\ \dot{z}_2 &= v, \end{aligned}$$

where $P_{1,3}$ is a smooth (resp. analytic) function of the indicated variables.

(ii) *Any smooth (resp. analytic) control-affine strict feedforward system (SFF) on $\mathbb{R}^3$ is special and is feedback* *equivalent to the* normal form

$$\begin{array}{rcl}\dot{z}_1 & = & z_2 + z_3^2 P_{1,3}(z_2, z_3)\\ \dot{z}_2 & = & z_3\\ \dot{z}_3 & = & v,\end{array}$$

where $P_{1,3}$ is a smooth (resp. analytic) function of the indicated variables.

IV. EXAMPLE

Example IV.1 Cart-Pole System. In this example we consider a cart-pole system that is represented by a cart with an inverted pendulum on it [21], [39]. The Lagrangian equations of motion for the cart-pole system are

$$\begin{array}{rcl}(m_1+m_2)\ddot{q}_1 + m_2 l\cos(q_2)\ddot{q}_2 & = & m_2 l \sin(q_2)\dot{q}_2^2 + F\\ \cos(q_2)\ddot{q}_1 + l\ddot{q}_2 & = & g\sin(q_2),\end{array}$$

where m_1 and q_1 are the mass and position of the cart, m_2, l, $q_2 \in (-\pi/2, \pi/2)$ are the mass, length of the link, and angle of the pole, respectively.

Taking $\ddot{q}_2 = u$ and applying the feedback law (see [21])

$$\begin{aligned}F = &-ul(m_1 + m_2\sin^2(q_2))/\cos(q_2)\\ &+ (m_1+m_2)g\tan(q_2) - m_2 l\sin(q_2)\dot{q}_2^2\end{aligned}$$

the dynamics of the cart-pole system are transformed into

$$\begin{array}{ll}\dot{x}_1 = x_2, & \dot{x}_2 = g\tan(x_3) - lu/cos(x_3)\\ \dot{x}_3 = x_4, & \dot{x}_4 = u,\end{array}$$

where we take $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, and $x_4 = \dot{q}_2$.

This system is in special strict feedforward form (SSFF) with the linearizability index $p = 2$. The feedback transformation defined by

$$\begin{array}{ll}\tilde{x}_1 = x_1 + l\displaystyle\int_0^{x_3}\frac{ds}{\cos s}, & \tilde{x}_2 = x_2 + l\dfrac{x_4}{\cos x_3}\\ \tilde{x}_3 = g\tan x_3, & \tilde{x}_4 = g\dfrac{x_4}{\cos^2 x_3}\end{array}$$

and $\tilde{u} = gu/\cos^2(x_3) + 2gx_3x_4\sin(x_3)/\cos^3(x_3)$, takes the system into the normal form

$$\begin{array}{ll}\dot{\tilde{x}}_1 = \tilde{x}_2, & \dot{\tilde{x}}_2 = \tilde{x}_3 + \dfrac{l\tilde{x}_3}{(g^2+\tilde{x}_3^2)^{3/2}}\tilde{x}_4^2\\ \dot{\tilde{x}}_3 = \tilde{x}_4, & \dot{\tilde{x}}_4 = \tilde{u}.\end{array}$$

Taking a linear transformation $z = \lambda\tilde{x}_i$ followed by a linear feedback $v = \lambda\tilde{u}$, with $\lambda = \frac{1}{g}\sqrt{l/g}$, we obtain the canonical form Π_{SSFCF}:

$$\begin{array}{ll}\dot{z}_1 = z_2, & \dot{z}_2 = z_3 + \dfrac{z_3}{(1+(g/l)z_3^2)^{3/2}}z_4^2\\ \dot{z}_3 = z_4, & \dot{z}_4 = v.\end{array}$$

V. PROOFS

Theorems III.5 and III.3 are proved in details in [36], so we will show Theorems III.1 and III.2.

Proof of Theorem III.1 Consider the system (III.1)-(III.2)-(III.3). Since this system is in strict feedforward form, it follows (because of Theorem III.5 and Lemma 1(ii) of [36] and the fact that the linearizability index is invariant) that there exists an analytic feedback transformation (local or global) that takes the system into the strict feedforward normal form

$$\Pi_{SFNF}: \left\{\begin{array}{rcl}\dot{x}_1 & = & \bar{h}_1(x_2) + \sum\limits_{i=3}^{n+1} x_i^2 P_{1,i}\\ & \cdots & \\ \dot{x}_j & = & \bar{h}_j(x_{j+1}) + \sum\limits_{i=j+2}^{n+1} x_i^2 P_{j,i}\\ & \cdots & \\ \dot{x}_r & = & \bar{h}_r(x_{r+1}) + \sum\limits_{i=r+2}^{n+1} x_i^2 P_{r,i}\\ \dot{x}_{r+1} & = & x_{r+2}\\ & \cdots & \\ \dot{x}_{n-1} & = & x_n\\ \dot{x}_n & = & u\,,\end{array}\right.$$

where $x_{n+1} = u$, $P_{j,i} = P_{j,i}(x_{j+1}, \ldots, x_i)$ are smooth (resp. analytic) functions of the indicated variables, and $\bar{h}_j(x_{j+1}) = k_j x_{j+1}$ for some non zero real numbers $k_1, \ldots, k_{r-1}$.

Taking $z_1 = x_1$, $z_j = \lambda_j x_j$ for $2 \le j \le r+1$, with $\lambda_j = k_1 \cdots k_{j-1}$, completed by $z_{r+2} = \lambda_{r+1}x_{r+2}, \ldots, z_n = \lambda_{r+1}x_n$ and $v = \lambda_{r+1}u$, we obtain $\bar{h}_j(x_{j+1}) = x_{j+1}$.

Choose s to be the largest integer, $r+2 \le s \le n+1$, such that $P_{r,s}(x_{r+1}, \ldots, x_s) \ne 0$.

Let m_0 denote the degree of the first homogeneous nonzero terms in the Taylor series expansion of

$$F_{r,s}(x_{r+1}, \ldots, x_s) = x_s^2 P_{r,s}(x_{r+1}, \ldots, x_s).$$

Define $(i_{r+1}, \ldots, i_s)$ with $i_{r+1} + \cdots + i_s = m_0$ and $i_s \ge 2$ to be the largest $(s-r)$-tuple , in the lexicography ordering, such that

$$\frac{\partial^{m_0} F_r}{\partial x_{r+2}^{i_{r+2}} \cdots \partial x_s^{i_s}}(0) = c \ne 0.$$

By a linear transformation $z_i = \lambda x_i, v = \lambda u$ we transform the term $cx_{r+1}^{i_{r+1}} \cdots x_s^{i_s}$ of degree m_0 of $F_{r,s}(x_{r+1}, \ldots, x_s)$ into $\tilde{c}z_{r+1}^{i_{r+1}} \cdots z_s^{i_s}$ with $\tilde{c} = c\lambda^{1-m_0}$. We then choose λ so that $\tilde{c} = \pm 1$. It follows that

$$\frac{\partial^{m_0} F_r}{\partial x_{r+2}^{i_{r+2}} \cdots \partial x_s^{i_s}}(x_1, \ldots, 0) = \pm 1,$$

that is, the system is in canonical form. □

Proof of Theorem III.2 Consider two analytic special strict feedforward forms (SSFF) given, respectively by, (III.1)-(III.2)-(III.3) and (III.6)-(III.7)-(III.8).

Sufficiency. It is clear (using Theorem III.1) that the two systems are analytic feedback equivalent if their canonical forms Π_{SSFCF} and $\tilde{\Pi}_{SSFCF}$ coincide.

Necessity. Suppose that the two systems are analytic feedback equivalent. Theorem III.1 implies that their canonical forms Π_{SSFCF} and $\tilde{\Pi}_{SSFCF}$ are also analytic feedback equivalent, say, by $\tilde{z} = \phi(z), \tilde{v} = \gamma(z, v)$. Because of the (SFF)-structure $\tilde{v} = \lambda v$ and $\phi_j(z) = \phi_j(z_j, \ldots, z_n)$ for $1 \le j \le n$. We claim that $\phi = Id$. Indeed, let k be the smallest integer such that for $k+1 \le j \le n$, we have $\phi_j(z) = \lambda_j z_j$ for some nonzero real numbers. The integer k is well-defined because $\phi_n(z) = \lambda_n z_n$. If $k \ge 1$, the transformation ϕ will then modify the k-component of the canonical form Π_{SSFCF} according to

$$\dot{\tilde{z}}_k = \frac{\partial \phi_k}{\partial z_k}\dot{z}_k + \cdots + \frac{\partial \phi_k}{\partial z_l}\dot{z}_l + \hat{F}_r(z_{k+1}, \ldots, z_n, v),$$

where $\hat{F}_r(z_{k+1}, \ldots, z_n, v) = \sum_{i=r+2}^{n+1} z_i^2 \hat{P}_{r,i}(z_{r+1}, \ldots, z_i)$ for some analytic functions $\hat{P}_{r,i}$, and l is the largest integer such that $\phi_k(z) = \phi_k(z_k, \ldots, z_l)$. The transformed system is a (SFF)-system but NOT a (SSFNF) because of the terms

$$\frac{\partial \phi_k}{\partial z_l} z_{l+1} = \frac{\partial \phi_k(z_k, \ldots, z_l)}{\partial z_l} z_{l+1}$$

that invert as $\Theta(\tilde{z}_k, \ldots, \tilde{z}_l)\tilde{z}_{l+1}$. Thus $k = 0$ and hence $\phi_j(z) = \lambda_j z_j$, for $1 \le j \le n$. Since Π_{SSFCF} and $\tilde{\Pi}_{SSFCF}$ satisfy, respectively

$$\begin{aligned} \frac{\partial^{m_0} z_s^2 P_{r,s}}{\partial z_{r+1}^{i_{r+1}} \cdots \partial z_s^{i_s}}(z_1, 0, \ldots, 0) &= \pm 1 \\ \frac{\partial^{m_0} \tilde{z}_s^2 \tilde{P}_{r,s}}{\partial \tilde{z}_{r+1}^{i_{r+1}} \cdots \partial \tilde{z}_s^{i_s}}(\tilde{z}_1, 0, \ldots, 0) &= \pm 1, \end{aligned}$$

it follows that $\lambda_j = 1$ and $\lambda = 1$, and completes the proof. □

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