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Encoder-Decoder Design for Perfect Reconstruction: A Robust Control Perspective

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***Abstract*— In this paper we consider the transmission of discrete data via a communication channel that is subject to (additive) noise with a known upper bound on its magnitude but otherwise completely unknown. We are interested in designing transmitter-receiver pairs that perfectly reconstruct the discrete data with a given delay under all possible realizations of channel noise. A Decision Feedback Equalizer (DFE) structure is assumed for the receiver while a linear structure is imposed on the transmitter along with the requirement that the power of the transmission is limited. Under these circumstances, we build on our previous work to provide necessary and sufficient conditions for perfect reconstruction in terms of the ℓ^1 norms of appropriate maps. An ℓ^1 iteration procedure that results in parametric linear programs is developed to optimize the design parameters for the transmitter-receiver pair. This is done for both the (standard) case when no feedback from the receiver to the transmitter is available and for the case when feedback is available. When only a delayed binary decision is fed back to the transmitter, which is a special instance of the second case, we also provide an implementation for finite time error recovery in terms of an additional ℓ^1 optimization.**

Keywords: Equalization, ℓ^1 optimality, worst case, discrete data reconstruction.

I. INTRODUCTION

The study of data transmission and reconstruction has been based almost entirely on stochastic formulations of the various problems involved (e.g., [1], [2]). In these formulations, the measure of performance for a communication system is characterized primarily in terms of the probability of error under stochastic assumptions on the noise and channel behavior. Designing a system that minimizes this probability is a hard problem and the proposed algorithms are characterized by high complexity (e.g., Viterbi's algorithm [1]). In our earlier works in [3], [4] we presented a deterministic worst-case framework for perfect reconstruction of discrete (source) data transmissions. Our framework is applicable to a number of applications where unknown but bounded noise models are more realistic than additive white Gaussian noise (AWGN) channels. For example, recent studies on modeling of high speed links in chip-to-chip or board-to-board communication that consider CMOS components to generate, receive and recover timing of high-speed data [5] shows that the dominant noise sources are colored and bounded. Furthermore, there are applications where quantization is a dominant noise source and, as such, it is of a bounded non-AWGN type. Such applications can be found in signal and image processing [6] as well as in the estimation literature [7]. Another motivation for a worst-case approach comes from applications where security to attacks by malicious agents (e.g., jammers [8]) is of paramount importance and therefore "hard" (non-probabilistic) guarantees are required.

In this paper we consider the transmission of discrete data via a communication channel that is subject to additive noise. For the noise we assume that we have no data other than a knowledge of an upper bound on its magnitude. We are interested in designing transmitter-receiver systems that perfectly reconstruct the discrete data with a given delay under all possible realizations of channel noise. A DFE structure is assumed for the receiver while a linear transmitter structure is imposed together with the requirement that the power of the transmission is limited. Under these circumstances, we built on our previous framework and results [3], [4] to provide necessary and sufficient conditions for perfect reconstruction in terms of the ℓ^1 norms of appropriate maps. An ℓ^1 iteration procedure that results in parametric linear programs is developed to optimize the design parameters for the transmitter-receiver pair. This is done for the case when no feedback from the receiver to the transmitter is available and for the case when feedback is available. In a special instance of the second case, when only a delayed binary decision is fed back to the transmitter, we also provide an implementation for finite time error recovery in terms of an additional ℓ^1 optimization. We would like to mention that consideration of feedback communication schemes from a combined control- and information-theoretic viewpoint have recently received considerable attention (e.g., [9], [10]).

The notation in the paper is as follows: $\|x\| := \sup_k |x(k)|$ is the ℓ^∞ norm of a sequence $x = \{x(k)\}_{k=0}^\infty$; $\|T\|_1 := \sum_{k=0}^\infty |t(k)|$ is the ℓ^1 norm of the linear time-invariant (LTI) system T having unit pulse response $\{t(k)\}_{k=0}^\infty$; $\hat{T}(\lambda) := \sum_{k=0}^\infty t(k)\lambda^k$ is the λ-transform of T. For a vector-valued signal $x = (x_1, x_2, \ldots, x_n)'$, $\|x\| := \max_i \|x_i\|$ and for MIMO systems $T = \{T_{ij}\}$ where T_{ij} are SISO, $\|T\|_1 := \max_i \sum_j \|T_{ij}\|_1$; T will be called stable if $\|T\|_1 < \infty$.

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II. Setup and Problem Formulation

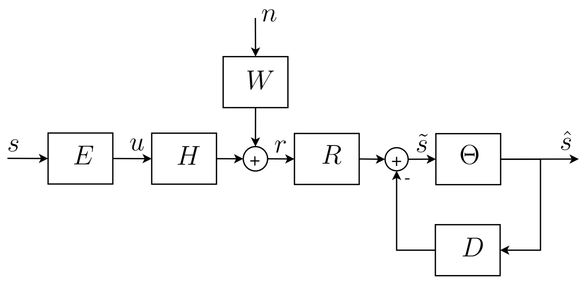

Fig. 1. Basic encoder-decoder structure, $R = \Lambda^{-K}Q$.

The basic problem we are concerned with is depicted in Figure 1. Signal s is a binary signal to be transmitted with $s(k) \in \{-1, 1\}$ for all $k = 0, 1, \ldots$. The noise sequence n satisfies $|n(k)| \leq b$ where b is a known constant and enters the channel through a stable known LTI system W. System $H = \{h_0, h_1, \ldots\}$ is a stable (real coefficient) LTI system and represents the channel dynamics which are also assumed known *a priori*. LTI system E is a preprocessor that can be thought of as a transmitter or an encoder (to be designed) that generates the signal u entering the channel. At the receiving end, we have a Decision Feedback Equalizer (DFE) structure consisting of a feedforward filter R operating on the received signal r, a feedback filter D, and a threshold element Θ. System D operates on past decisions $\hat{s}(k-1), \hat{s}(k-2), \ldots$ and is of the form $D = \Lambda F$ where Λ is the one step delay (one step right shift operator) and F a stable LTI system. The threshold element Θ produces the binary decision $\hat{s}(k)$ which is -1 or 1 depending on which one has the closest distance to the "soft decision" $\tilde{s}(k)$; in this particular case, $(\Theta\tilde{s})(k) = \text{ sgn } [\tilde{s}(k)]$.

Our goal is to derive necessary and sufficient conditions for the existence and design of E, R and D so that signal s is perfectly reconstructed for all possible realizations of the noise sequence n with a given delay K, i.e., $\hat{s}(k) = s(k-K)$ for all times $k \geq K$. The decision delay K is reflected in our setup in terms of allowing the feedforward filter R in the DFE to be noncausal of the form $R = \Lambda^{-K}Q$ where Q is causal LTI and Λ^{-K} is K step left shift operator. To make the problem physically meaningful, we assume that the power of transmission is limited (otherwise, perfect reconstruction is always possible by making the transmission power sufficiently high). We will consider both the case when no feedback from receiver to transmitter is available (as shown in Figure 1) as well as the case when feedback is available.

III. DFE Receiver with No Feedback to Transmitter

We capture the requirement of limiting the transmitted power by requiring that $|u(k)|$ remains within predefined limits at all times k. In particular, we require $|u(k)| \leq 1$. This translates to an ℓ^1 constraint that requires $\|E\|_1 \leq 1$. The following for perfect reconstruction (PR) can be established.

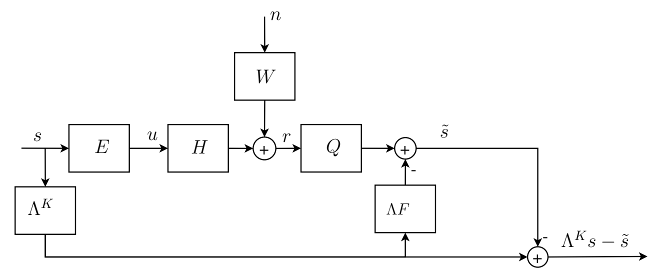

Fig. 2. Equivalent system for perfect reconstruction.

Theorem 3.1: PR is possible with delay K for some E, Q and $D = \Lambda F$ if and only if in Figure 2

$$\mu = \inf_{Q,D,E \text{ with } \|E\|_1 \leq 1} \|(\Lambda^K - QHE + \Lambda^K D \quad QWb)\|_1 < 1. \quad (1)$$

Moreover, if E, Q and D satisfy Condition 1 then they form a PR system in Figure 1.

Proof: The "if" part follows from the fact that the signals s and n are ℓ^∞ bounded (by 1 and b respectively) sequences and therefore Condition 1 guarantees that the soft error $(\Lambda^K s - \tilde{s})(k)$ is always bounded away from 1 for all k. Hence $\hat{s}(k) = \Theta\tilde{s}(k) = s(k-K)$ for all time instants $k \geq K$. The "only if part" can be essentially shown as in Proposition 3.1 of [3]; the details are omitted here for brevity. ■

The implication of the above theorem is that the PR problem associated with the nonlinear system in Figure 1 is equivalent to checking Condition 1 in the linear system of Figure 2. If we define $G := QHE = \{g_0, g_1, \ldots\}$, the optimal D for any Q and E is $D = \Lambda F$ with $F = \{g_{K+1}, g_{K+2}, \ldots\}$. Thus, if $G_K := \{g_0, g_1, \ldots, g_K, 0, 0, \ldots\}$ the condition becomes

$$\mu = \inf_{Q,E \text{ with } \|E\|_1 \leq 1} \|(\Lambda^K - G_K \quad QWb)\|_1 < 1. \quad (2)$$

The underlying problem in checking Condition 2 is convex in Q and E but it is not jointly convex. Hence, a Q-E type of iteration can be considered where we fix one of Q and E and optimize for the other until no further improvement is made. Each step involves solving a ℓ^1 problem and the method is guaranteed to converge to a local minimum. Reliable software for solving these ℓ^1 optimizations exists [11]. It should also be clear from the form of the optimization problem that the best E will always be FIR of order K, i.e., $E = \{e_0, e_1, \ldots, e_K, 0, 0, \ldots\}$. Hence, one can alternatively think of the overall problem as a parametric optimization in the parameters $\{e_0, e_1, \ldots, e_K\}$. For a small delay K (e.g., 2 or 3) these types of problems are quite tractable.

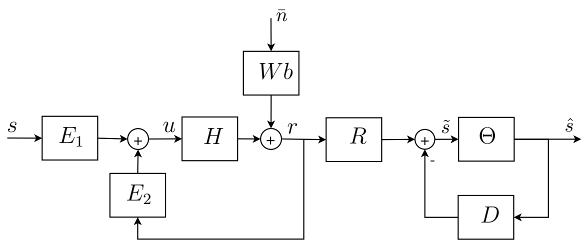

Fig. 3. Basic structure of receiver with feedback.

IV. DFE Receiver with Feedback to Transmitter

We now assume that the transmitter receives feedback from the receiver as depicted in Figure 3. In particular, we assume that the received signal r becomes known to the transmitter with one step delay. We require the signal u generated by the transmitter and sent to channel H to be composed as

$$u = E_1 s + E_2 r$$

where E_1 and E_2 are LTI with $E_2 = E_3 \Lambda$ with E_3 LTI (and causal). The constraint on transmission power is captured by requiring that in the closed loop $|u(k)| \leq 1$ at all times k. This of course means that $\|(s, \bar{n}) \to u\|_1 \leq 1$, where $\bar{n} = b^{-1} n$ is normalized noise with $\|\bar{n}\| \leq 1$ introduced for convenience.

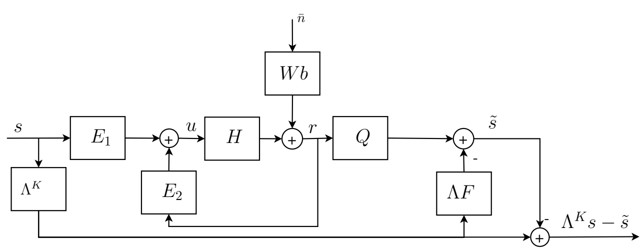

Fig. 4. Equivalent system for perfect reconstruction with feedback.

Theorem 4.1: In Figure 3 PR is possible with delay K for some $E := (E_1 \;\; E_2)$, Q and $D = \Lambda F$ if and only if there exist Q, F, E_1 and E_2 in Figure 4 such that

$$\left\|(s, \bar{n}) \to \Lambda^K s - \tilde{s}\right\|_1 < 1 \text{ and } \|(s, \bar{n}) \to u\|_1 \leq 1. \quad (3)$$

Moreover, if E, Q and D satisfy Condition 3 then they form a PR system in Figure 3.

Proof: The proof is along similar lines as in Theorem 3.1 and is omitted for brevity. ∎

Thus the resulting optimization for selecting E, Q and D transforms to an ℓ^1 performance problem in a closed loop system. We can view this in the standard context of controller design as shown in Figure 5. The generalized plant P is depicted in this figure along with the structured controller C that defines Q, F, E_1 and E_2. For PR, the closed loop maps $\Phi_1 := (s, \bar{n}) \to \Lambda^K s - \tilde{s}$ and $\Phi_2 := (s, \bar{n}) \to u$ should satisfy the ℓ^1 constraints $\|\Phi_1\|_1 < 1$ and $\|\Phi_2\|_1 \leq 1$. In general, the underlying ℓ^1 optimization is not convex. Yet it can be viewed as a parameterized family of convex problems where the parameters are the first $K+1$ coefficients of Q. Indeed, Q can be parameterized as $Q = Q_K + \Lambda^{K+1}\tilde{Q}$ where $Q_K = \{q_0, \ldots, q_K, 0, 0, \ldots\}$ and $\tilde{Q}$ is arbitrary (stable, LTI). Then we can construct an equivalent loop by absorbing Q_K in the generalized plant as shown in Figure 6. The new generalized plant P_{Q_K} is stabilized by

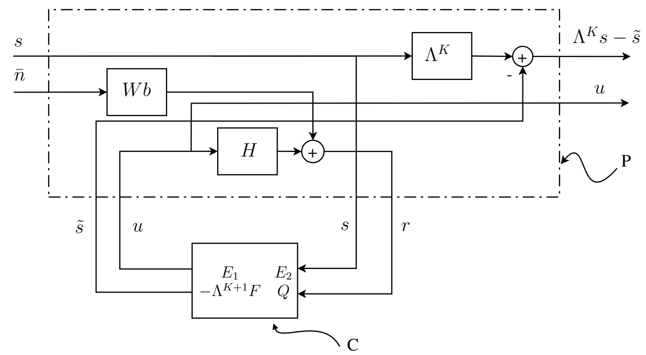

Fig. 5. Standard controller design setup.

$$C_{Q_K} = \begin{pmatrix} E_1 & E_2 \\ -F & \tilde{Q} \end{pmatrix}$$

and the only structural condition on C_{Q_K} comes from E_2. Specifically, E_2 has to be of the form $E_2 = E_3\Lambda$. Note that this condition leads to a convex problem for any fixed Q_K. This can be seen as follows. Define P_{22} to be the map $(u, \sigma) \to (r, s)$ in the open loop plant P_{Q_K}. Then

$$P_{22} = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}.$$

All stabilizing controllers for P_{22} and hence for P_{Q_K} are given (e.g., [12]) as $C_{Q_K} = Z(I + P_{22}Z)^{-1}$ where

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

is any stable (and causal) system. From this expression, it readily follows that $E_2 = E_3\Lambda$ if and only if Z_{12} is of the form $Z_{12} = Z_2\Lambda$ where Z_2 is arbitrary and stable. In terms of this parameterization

$$\Phi_1 = T_{11} - T_{21}ZT_{31} \quad \text{and} \quad \Phi_2 = T_{12} - T_{22}ZT_{32}$$

where the T_{ij}'s are stable and depend on H, Wb and Q_K. Thus, the relevant problem for PR is

$$\mu = \inf_{Z:\; Z_{12} = Z_2\Lambda,\; \|T_{12} - T_{22}ZT_{32}\|_1 \leq 1} \|T_{11} - T_{21}ZT_{31}\|_1 < 1, \quad (4)$$

which is an ℓ^1 optimization problem that corresponds to an infinite LP. These types of (structured) ℓ^1 problems can be readily solved with available software [11] within any predefined accuracy.

Remarks: The case of arbitrary (but known) communication delay in the feedback loop can be handled similarly:

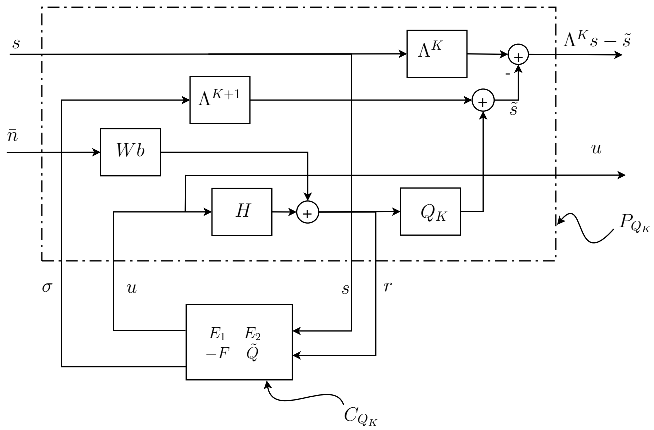

Fig. 6. Controller design setup parameterized by Q_K.

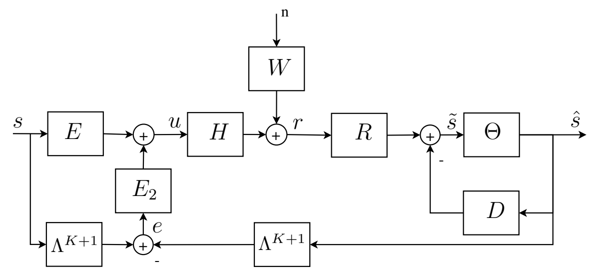

Fig. 7. Hard decision feedback with a DFE receiver and $R = \Lambda^{-K}Q$.

if there is an M-step delay in passing accurately to the transmitter the information about r through the feedback path, then this is equivalent to requiring that $E_2 = E_3\Lambda^M$. From the parameterization of all stabilizing controllers, given earlier by $C_{Q_K} = Z(I + P_{22}Z)^{-1}$, it readily follows that

$$E_2 = E_3\Lambda^M \Leftrightarrow Z_{12} = Z_2\Lambda^M,$$

where Z_2 is arbitrary and stable. Thus, the underlying problem is a structured ℓ^1 optimization. We should also point out that the no feedback case is the special case when $E_2 = 0$. It is easy to see that this corresponds to having $Z_{12} = 0$ which leads again to a structured ℓ^1 optimization that can be solved effectively [11] for any given parameter set $\{q_0, \ldots, q_K\}$. This is an alternative to the methods proposed in the previous section.

We should also note at this point that the situation where the soft decision $\tilde{s}$ is fed back instead of r leads to similar formulations of the underlying problems. For the sake of brevity, these are not presented here.

Finally, the possibility of some (small) noise in the feedback path should be investigated in terms of its effects on the achievable PR channel noise bounds. This is a theme that is currently under investigation by the authors.

V. Hard Decision Feedback

In this section we consider the case where the feedback to the transmitter is the decision $\hat{s}(k)$ delayed by one step as shown in Figure 7. This is a case where the feedback information is limited only to a single bit. Clearly, the previous feedback scheme implicitly contained this information as knowledge of $\{r(t)\}_{t=0}^{k-1}$ and $R = \Lambda^{-K}Q$ and F allow the transmitter to determine the decision $\hat{s}(k-1)$.

We would like to consider how to use this one-bit information in order to contain the effect of erroneous past decisions. In particular, we consider the scheme of Figure 7 where the encoder/transmitter operates as

$$u = Es + E_2\Lambda^{K+1}(s - \hat{s}) \qquad (5)$$

where E_2 is stable. The reason why there is a $K+1$-step delay in the above expression is because the decision $\hat{s}(k)$ for $s(k)$ is in reality taken K time steps later. Thus, defining the error $e := s - \hat{s}$, the transmitter cannot have $\hat{s}(k)$ (and thus $e(k)$) earlier than K time steps plus one more (due to the delay propagation in the feedback path). The equation for the soft error $s - \tilde{s}$ becomes

$$s - \tilde{s} = (I - RHE + D)s - (RHE_2\Lambda^{K+1} + D)e - QWn \qquad (6)$$

Note that the error $e(k)$ takes values in $\{-2, 0, 2\}$. The parameters E, R and $D = \Lambda F$ can be designed to maximize the noise level b for which PR is possible using the methods of the previous sections. In this case it can be seen from Equation 6, that E_2 does not play any role as far as perfect reconstruction ($e = 0$) goes. The scenario we are interested in is when the noise suddenly exceeds the presupposed level and generates an initial wrong decision. In the absence of any further noise, we would like to investigate under what conditions the error propagation is limited to a finite time time interval i.e., the DFE recovers from the initial error in finite time. We should note here that analysis on error propagation for DFEs can be found in earlier works in the communications literature (e.g., [14], [16]) which however, do not treat the case of feedback.

From Equation 6, assuming that the noise becomes zero after the generation of an initial decision error, the effect to the future soft error $s - \tilde{s}$ of past noise through the term QW is going to be decaying to zero with time. Thus, even if more errors are building up after the initial one (due to past noise), after some long enough time we can guarantee no more errors if

$$\left\|(I - RHE + D \quad 2(RHE_2\Lambda^{K+1} + D))\right\|_1 < 1$$

or, equivalently,

$$\|I - RHE + D\|_1 + 2\left\|RHE_2\Lambda^{K+1} + D\right\|_1 < 1 \qquad (7)$$

Hence, one possible choice for E_2 is to pick the one that solves

$$\nu := \inf_{E_2} \left\|RHE_2\Lambda^{K+1} + D\right\|_1 = \inf_{E_2} \|QHE_2 + F\|_1 . \qquad (8)$$

If

$$2\nu + \|I - RHE + D\|_1 < 1 \qquad (9)$$

then the system has guaranteed finite error recovery. Obviously, this is in general a sufficient condition which can be conservative. Nonetheless, it can be useful for determining cases with guaranteed error recovery properties. For example, when QH is minimum phase, picking E_2 as $E_2 = -F(QH)^{-1}$ is stable and it will guarantee that $\nu = 0$ and thus recovery is always possible in finite time. Note that $\|I - RHE + D\|_1 < 1$ by the choice of E, Q and D. In fact, any E_2 of the form $E_2 = -\gamma F(QH)^{-1}$ with $0 \leq \gamma < 1$ and $(1-\gamma)\|F\|_1 < 1/2$ will work. In what follows we elaborate more on a special case for which more analytical and less conservative conditions can be obtained.

The case $K = 0$ and $W = I$

Consider the case when there is no delay ($K = 0$) in the decision $\hat{s}$ and the noise n enters the channel directly ($W = I$). In this case the underlying optimization (to assess the maximum b for perfect reconstruction) is expressed in the condition

$$\mu = \inf_{Q,E} \|(I - QHE + D \quad - Qb)\|_1 < 1$$

subject to $\|E\|_1 \leq 1$. This turns out (see also [3]) to be equivalent to the following LP feasibility condition

$$\mu = \min_{q_0, |e_0| \leq 1} |1 - q_0 h_0 e_0| + b|q_0| < 1$$

which leads to the necessary and sufficient condition $|h_0| > b$; a feasible solution in this case is $q_0 = h_0^{-1}$ and $e_0 = 1$ with the corresponding systems being the constant $Q = q_0$, $E = e_0$ while $D = \Lambda F$ with $F = \{h_1, h_2, \ldots\}$. This solution leads to $\mu = |b/h_0|$ and $\|I - QHE + D\|_1 = 0$. So it is sufficient to have $\nu < 1/2$. The resulting optimization for ν becomes

$$\nu = \inf_{E_2} \left\|\bar{H}E_2 + \bar{F}\right\|_1$$

where $\bar{H} = h_0^{-1}H$ and $\bar{F} = h_0^{-1}F$. If H is minimum phase then the optimal E_2 is $E_2 = -H^{-1}F$ leading to $\nu = 0$ and thus guaranteeing error recovery in finite time. In particular, since $Q = q_0$ past values of the noise do not contribute to the soft error $s - \tilde{s}$. Thus, if the noise becomes zero after an excursion at some time t that generates an initial error $e(t)$, and since $I - QHE + D = 0$, the propagation of this error depends only on $QHE_2 + F$ as follows from Equation 6. That is, in this case

$$(s - \tilde{s})(k + 1) = -((QHE_2 + F)e)(k), \quad k \geq t \; .$$

Since $QHE_2 + F = 0$, the soft error $s - \tilde{s}$ is zero for all subsequent time instants and hence the same holds true for e. Therefore, the initial error $e(t)$ is suppressed in one step.

In fact, based on the previous analysis for this particular case, having H minimum phase is unnecessarily demanding for achieving error recovery in one step. It is sufficient to ensure that the effect of an initial error $e(t)$ on the soft error $(s - \tilde{s})(t + k)$ is (strictly) bounded by 1 for all subsequent times $k = 1, 2, \ldots$. That is, it is sufficient to have that

$$\|(QHE_2 + F)e_t\| < 1/2$$

where $e_t = \{1, 0, 0, \ldots\}$ is a fixed input corresponding to an error at t. If the above condition is satisfied, then any initial error is recovered in one step (in the absence of further noise, i.e., $n(t+k) = 0$ for $k = 1, 2, \ldots$). Thus, the relevant optimization for E_2 is

$$\nu_\infty := \inf_{E_2} \left\|\bar{H}E_2 + \bar{F}\right\| < 1/2. \tag{10}$$

This is an ℓ^∞ optimization of the pulse response of $QHE_2 + F$ which obviously leads to $\nu_\infty = 0$ when H is minimum phase. If that is not the case, assuming for simplicity that the non-minimum phase zeros $\{z_i\}_{i=1}^N$ of H are real and strictly inside the unit disk (i.e., $|z_i| < 1$), we have using duality [12] that

$$\nu_\infty = \max_{\alpha_1, \ldots, \alpha_N} \sum_{i=1}^{N} \alpha_i \hat{\bar{F}}(z_i)$$

subject to an ℓ^1 constraint $\sum_{j=0}^{\infty} |\sum_{i=1}^{N} \alpha_i z_i^j| \leq 1$.

As $\hat{\bar{F}}(z_i) = -1/z_i$ the above optimization becomes

$$\nu_\infty = \max_{\alpha_1, \ldots, \alpha_N} \sum_{i=1}^{N} -\alpha_i / z_i \tag{11}$$

subject to $\sum_{j=0}^{\infty} |\sum_{i=1}^{N} \alpha_i z_i^j| \leq 1$.

This is an infinite LP which can be approximated by a finite LP to any a-priori defined accuracy (e.g., by truncating the tail of the infinite sum in the constraint). In the special case when there is only one $z_i = z_1$ the resulting optimization is $\nu_\infty = \max_{\alpha_1} -\alpha_1/z_1$ subject to $\frac{|\alpha_1|}{1-|z_1|} \leq 1$, which leads to $\nu_\infty = \frac{1-|z_1|}{|z_1|}$. Thus, $\nu_\infty < 1/2$ is equivalent to $2/3 < |z_1| (< 1)$.

Appropriate conditions that ensure error recovery in one step can also be found when the feedback information $\hat{s}$ to the transmitter is delayed in the feedback path more than one step. Consider the case where the decision $\hat{s}(k)$ is provided to the transmitting end with delay of $M + 1$ steps on top of the K-step delay in the decision at the receiving end. This means that E_2 in Equation 5 is constrained to be of the form $E_2 = E_3 \Lambda^M$. Following the previous developments, the optimization for ν_∞ now becomes

$$\nu_\infty = \inf_{E_2 = E_3 \Lambda^M} \left\|\bar{H}E_2 + \bar{F}\right\| = \inf_{E_3} \left\|\bar{H}E_3 \Lambda^M + \bar{F}\right\| .$$

Denoting by $\bar{h}_i = h_i/h_0$ we have that $\bar{H} = \{1, \bar{h}_1, \ldots\}$ and $\bar{F} = \{\bar{h}_1, \bar{h}_2, \ldots\}$. Splitting $\bar{F}$ as $\bar{F} = \bar{F}_{M-1} + \Lambda^M F_1$ where $\bar{F}_{M-1} = \{\bar{h}_1, \ldots, \bar{h}_M, 0, \ldots\}$ and $F_1 = \{\bar{h}_{M+1}, \bar{h}_{M+2}, \ldots\}$, we have that the resulting optimization in this case becomes

$$\nu_\infty = \left\|\bar{F}_{M-1}\right\| + \inf_{E_3} \left\|\bar{H}E_3 + F_1\right\|$$

or

$$\nu_\infty = \max_{i=1,\ldots,M} |\bar{h}_i| + \inf_{E_3} \left\|\bar{H}E_3 + F_1\right\| .$$

If H is minimum phase, then $\inf_{E_3} \left\|\bar{H}E_3 + F_1\right\| = 0$ and thus $\nu_\infty = |\bar{h}_m|$ where $|\bar{h}_m| = \max_{1,\ldots,M} |\bar{h}_i|$. Hence $\nu_\infty < 1/2$ is equivalent to have $|h_m| < |h_0|/2$. Otherwise, if H is non-minimum phase, (and assuming the same simplifications as before) the optimization to solve is an LP

$$\nu_\infty = |\bar{h}_m| + \max_{\alpha_1,\ldots,\alpha_N} \sum_{i=1}^{N} \alpha_i \hat{F}_1(z_i),$$

subject to an ℓ^1 constraint $\sum_{j=0}^{\infty} |\sum_{i=1}^{N} \alpha_i z_i^j| \leq 1$.

Since $\hat{F}_1(z_i) = -\frac{\hat{\bar{H}}_M(z_i)}{(z_i)^M}$ where $\bar{H}_M = \{1, \bar{h}_1, \ldots, \bar{h}_M, 0, 0, \ldots\}$ the above LP becomes

$$\nu_\infty = |\bar{h}_m| + \max_{\alpha_1,\ldots,\alpha_N} \sum_{i=1}^{N} -\alpha_i \frac{\hat{\bar{H}}_M(z_i)}{(z_i)^M} \tag{12}$$

subject to $\sum_{j=0}^{\infty} |\sum_{i=1}^{N} \alpha_i z_i^j| \leq 1$.

As before, this LP can be approximated within any a-priori defined accuracy with a finite dimensional LP. Note that when $|\bar{h}_i| < 1/2$ or equivalently $|h_i| < |h_0|/2$ for all $i = 1, 2, \ldots$, then error recovery in one step is always possible by selecting $E_3 = 0$, i.e., no feedback is necessary. **Remarks:** We would like to note that in the developments of this section, no power constraint on E_2 is imposed (of course, $\|E\|_1 \leq 1$ still holds). This means is that, while the transmitter is in the "normal" mode of operation (i.e., when the noise levels are smaller than the maximum possible for PR) no extra power is required; however, in the exceptional events of noise excursions, the transmitter has an "emergency" error-correction power right on demand. If that is not the case, imposing a power constraint of the form $\|E_2\|_1 \leq \gamma$ will lead to constrained optimizations for ν or ν_∞ which are still convex and can be effectively solved. However, analytical results of the form obtained in this section are harder to obtain. This is an ongoing research.

What is also on going research, is the situation where preliminary (one-bit) decisions can be fed back to the transmitter (with one step delay as before.) As it can be seen from Equation 5, initially it takes $K+1$ steps delay in using the information to affect the input u to the channel. In these initial K steps the system runs open loop; therefore, there is potentially some benefit in passing preliminary decisions during this initial phase.

VI. Conclusions

We considered transmitter-receiver systems that perfectly reconstruct binary data with a given delay under all possible realizations of channel noise that is limited in magnitude by a known bound. A DFE structure was assumed for the receiver while a linear transmitter structure was imposed with the requirement that the power of the transmission is limited. We provided necessary and sufficient conditions for perfect reconstruction in terms of the ℓ^1 norms of appropriate maps. An ℓ^1 iteration procedure that results in parametric linear programs was developed to optimize the design parameters for the transmitter-receiver pair in the cases where feedback from the receiver to the transmitter is not available and when it is available. In a special instance of the second case, when only a delayed binary decision is fed back to the transmitter, we also provided an implementation for finite time error recovery in terms of an additional ℓ^1 optimization. The analysis in the case of MIMO channels and more complicated alphabets, although not presented here, follows a similar path.

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