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Homogeneous Sliding Modes in The Presence of Fast Actuators

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Abstract— **It is known that the presence of actuators leads to the deterioration of sliding modes with chattering appearance. It is shown in the paper that the higher the order r of the homogeneous sliding mode the less sensitive it is to fast stable actuators. In particular, with the actuator time constant $\mu << 1$ the sliding variable magnitude is proportional to μ^r.**

I. Introduction

CONTROL under heavy uncertainty conditions remains one of the main subjects of the modern control theory. One of the most popular approaches to the problem is based on the sliding-mode control. The idea is to react immediately to any deviation of the system from some properly chosen constraint steering it back by a sufficiently energetic effort. Sliding mode is accurate and insensitive to disturbances [6, 23, 24]. The main drawback of the standard sliding modes is mostly related to the so-called chattering effect and much research was devoted to avoiding it [1, 4, 8, 10, 11, 20, 21, 22, 23].

Chattering is caused by the high, theoretically infinite frequency of control switching. The control signal does not directly influence the system, either influences it by means of a special device called actuator, being itself a dynamic system. The purpose of the actuator is to properly transmit the input, and it performs well when the input changes smoothly and slowly. For this end the actuator is to be fast, exact and stable. High frequency discontinuous input causes uncontrolled vibrations of the actuator and of its output. In its turn this causes vibration of the system.

High order sliding modes [3, 5, 9, 15] were historically created to remove the chattering effect. Recent research [4, 8-11] shows that the presence of an actuator between the discontinuous entity and the dynamic system causes vibrations of the output. The same research also shows that the vibrations of sliding variable are rather small when the sliding order exceeds 1. It is shown in this paper that the higher the sliding-mode order the less sensitive it is to the fast actuator. The consideration requires the r-sliding controller to be r-sliding homogeneous [15, 19]. This requirement is satisfied for practically all known high-order sliding controllers. The aim of this paper is to evaluate these vibrations.

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II. The Problem Statement and the Main Lemma

Consider a smooth dynamic system with a smooth output function σ. Let the system be closed by some possibly-dynamical discontinuous feedback and be understood in the Filippov sense [7]. Then, provided that successive total time derivatives $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables; and the set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ is a non-empty integral set, the motion on the set is called r-sliding (rth order sliding) mode [14, 16]. The standard sliding mode used in the most variable structure systems, is of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous).

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)v, \quad \sigma = \sigma(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$, a, b and σ: $\mathbf{R}^{n+1} \to \mathbf{R}$ are unknown smooth functions, $v \in \mathbf{R}$, n is also uncertain. The relative degree r of the system is assumed to be constant and known. That means that the input variable v appears explicitly for the first time in the rth total time derivative of σ [12]. The task is to provide in finite time for keeping $\sigma \equiv 0$. It is easy to check [12] that

$$\sigma^{(r)} = h(t,x) + g(t,x)v, \tag{2}$$

where $h(t,x) = \sigma^{(r)}|_{v=0}$, $g(t,x) = \frac{\partial}{\partial v}\sigma^{(r)}$. Following are the assumptions.

1º. It is supposed that

$$0 < K_m \le \tfrac{\partial}{\partial v}\sigma^{(r)} \le K_M, \ | \sigma^{(r)}|_{v=0} | \le C \tag{3}$$

for some K_m, K_M, $C > 0$. Note that conditions (3) are formulated in terms of input-output relations. It is also assumed that trajectories of (2) are infinitely extendible in time for any Lebesgue-measurable bounded input. Although it is formally not needed, the weakly minimum-phase property is often required in practice.

The actuator is described by the equations

$$\mu \dot{z} = f(z, u), \qquad v = v(z) \tag{4}$$

where $z \in \mathbf{R}^m$, $u \in \mathbf{R}$ is the control and the input of the actuator, v is a continuous output function, the time constant $\mu > 0$ is a small parameter.

The control u is determined by a dynamic feedback

$$u = U(\sigma, \dot{\sigma}, ..., \sigma^{(r)}) \tag{5}$$

where U is a function continuous almost everywhere, and bounded by some constant u_M, $u_M > 0$, in its absolute value. Being applied directly to (1), i.e. with

$$v = u, \tag{6}$$

it provides for $\sigma \equiv 0$ in finite time. All differential equations are understood in the Filippov sense [7]. The aim is to estimate σ and its derivatives in the presence of the actuator (4) with $\mu << 1$.

2º. The actuator features Bounded-Input-Bounded-State (BIBS) property with $\mu = 1$. Initial values of z belong to some compact set. Since $|u| \leq u_M$ this provides for infinite extension in time of any system solution and z belonging to some compact region Ω independent of μ. Indeed, μ can be excluded by the time transformation $\tau = t/\mu$. This assumption causes also the actuator output v to be bounded in its absolute value by some constant $v_M > u_M > 0$.

3º. The dynamic output-feedback (5) is supposed r-sliding homogeneous [18], which means that the identity

$$U(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \equiv U(\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)}) \qquad (7)$$

is kept for any $\kappa > 0$. Most known high-order sliding mode controllers [3, 13, 15-17] satisfy this assumption. It is also assumed that the control function U is locally Lipschitzian everywhere except a finite number of manifolds comprising a closed set Γ in the space with coordinates $\Sigma = (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$. Note that due to the homogeneity property (7) the set Γ includes the origin $\Sigma = 0$, where the function U is inevitably discontinuous [18].

As follows from (2), (3)

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]\, v\,. \qquad (8)$$

This inclusion does not "remember" anything on system (1) except the constants r, C, K_m, K_M.

4º. It is assumed that with control (5) applied directly to inclusion (8) a finite-time stable inclusion (5), (6), (8) is created. Note that this is the standard way to implement high-order sliding controllers [13, 16, 18].

Differential inclusions are understood here in the Filippov sense [7]. This means that the right-hand vector set is enlarged in a special way [7, 18] at the discontinuity points of U in order to satisfy certain convexity and semicontinuity conditions. The following Lemma actually formulates the main result.

Lemma 1. *Under assumptions* 1º - 4º *suppose that for some* $\mu = \mu_0$ *there is a bounded invariant set attracting all trajectories of the inclusion* (4), (5), (8) *in finite time. Then controller* (5) *provides for the establishment in finite time and keeping the inequalities* $|\sigma| < a_0\mu^r$, $|\dot{\sigma}| < a_1\mu^{r-1}$, ..., $|\sigma^{(r-1)}| < a_{r-1}\mu$ *with some positive constants* a_0, a_1, ..., a_{r-1} *independent of* μ.

Here and further all the proofs are presented in Section IV. Some additional assumptions are needed to provide for the Lemma conditions.

5º. The actuator is assumed exact in the following sense. With $\mu = 1$ and any constant value of u the output v uniformly tends to u. That means that for any u, u = const, $|u| \leq u_M$, $z(0) \in \Omega$ and any $\delta > 0$ there exists $t_0 > 0$ such that $|v - u| \leq \delta$ is kept starting from the moment $t = t_0$. It is required also that the function $f(z, u)$ in (4) be uniformly continuous in u, which means that $\|f(z, u) - f(z, u + \Delta u)\|$ uniformly tends to 0 with $\Delta u \to 0$.

Note that any linear actuator with the transfer function $P(\mu s)/Q(\mu s)$, where deg Q – deg $P > 0$, Q is a Hurwitz polynomial, $P(0)/Q(0) = 1$, satisfies Assumptions 2º, 5º.

6º. It is supposed that the change of (5), (6) to

$$v \in \begin{cases} U(\Sigma), & \Sigma \notin \Gamma \\ [-v_M, v_M], & \Sigma \in \Gamma \end{cases} \qquad (9)$$

does not interfere with the finite-time convergence, i.e. (8), (9) is finite-time stable. Recall that $\Sigma = (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$.

Note that while Filippov solutions of discontinuous differential equations do not depend on the values of the right-hand side on any set of the measure 0, it is not true with respect to differential inclusions. Since $v_M > u_M$, solutions of (8), (9) contain all solutions of (5), (6), (8). Note also that the inclusion (8), (9) is also r-sliding homogeneous, and, therefore, its asymptotic stability is equivalent to the finite-time stability [18, 20].

III. Main Results

Theorem 1. *Under assumptions* 1º - 6º *controller* (5) *provides for the establishment in finite time and keeping of the inequalities* $|\sigma| < a_0\mu^r$, $|\dot{\sigma}| < a_1\mu^{r-1}$, ..., $|\sigma^{(r-1)}| < a_{r-1}\mu$ *in the system* (4), (5), (8) *with some positive constants* a_0, a_1, ..., a_{r-1} *independent of* μ.

While assumptions 1º - 5º can be considered natural, assumption 6º is to be checked for each controller. Fortunately it holds for almost all known high-order sliding controllers. Consider some of them.

Three known families of high-order sliding controllers are defined by recursive procedures. In the following α, β_1,..., $\beta_{r-1} > 0$ and $i = 1,..., r-1$.

1. The following procedure defines the "nested" r-sliding controller [14, 16], based on a pseudo-nested structure of 1-sliding modes. Let q be the least common multiple of 1, 2, ..., r. Define

$$N_{i,r} = (|\sigma|^{q/r} + |\dot{\sigma}|^{q/(r-1)} + ... + |\sigma^{(i-1)}|^{q/(r-i+1)})^{(r-i)/q};$$
$$\Psi_{0,r} = \text{sign}\,\sigma, \quad \Psi_{i,r} = \text{sign}(\sigma^{(i)} + \beta_i N_{i,r} \Psi_{i-1,r}).$$
$$u = -\alpha\,\Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}).$$

Controller (5) is called quasi-continuous [17] if it can be redefined according to continuity everywhere except the r-sliding set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$.

2. The following procedure defines a family of quasi-continuous controllers [17]:

$$\varphi_{0,r} = \sigma,\ N_{0,r} = |\sigma|, \quad \Psi_{0,r} = \varphi_{0,r}/N_{0,r} = \text{sign}\,\sigma,$$
$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-1)/(r-i+1)} \Psi_{i-1,r},$$
$$N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-1)/(r-i+1)},$$
$$\Psi_{i,r} = \varphi_{i,r}/N_{i,r},$$
$$u = -\alpha\,\Psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}).$$

3. Another family of quasi-continuous controllers [18] is obtained from the first family by the following homogeneous regularization:

$$N_r = (|\sigma|^{q/r} + |\dot\sigma|^{q/(r-1)} + \ldots + |\sigma^{(r-1)}|^{q})^{1/q},$$
$$\mathrm{sat}(z, \varepsilon) = \min[1, \max(-1, z/\varepsilon)],$$
$$\psi_{0,r} = \mathrm{sign}\,\sigma,\ \ \psi_{i,r} = \mathrm{sat}([\sigma^{(i)} + \beta_i N_{i,r}\psi_{i-1,r}]/ N_r^{r-i}, \varepsilon_i),$$
$$u = -\alpha\, \Psi_{r-1,r}(\sigma, \dot\sigma, \ldots, \sigma^{(r-1)}).$$

Following are the nested sliding-mode controllers (of the first family) for $r \le 4$ with tested β_i :

1. $u = -\alpha\,\mathrm{sign}\,\sigma$,
2. $u = -\alpha\,\mathrm{sign}(\dot\sigma + |\sigma|^{1/2}\mathrm{sign}\,\sigma)$,
3. $u = -\alpha\,\mathrm{sign}(\ddot\sigma + 2\,(|\dot\sigma|^{3} + |\sigma|^{2})^{1/6}\,\mathrm{sign}(\dot\sigma + |\sigma|^{2/3}\mathrm{sign}\,\sigma))$,
4. $u = -\alpha\,\mathrm{sign}\{\dddot\sigma + 3\,(\ddot\sigma^{6} + \dot\sigma^{4} + |\sigma|^{3})^{1/12}\,\mathrm{sign}[\ddot\sigma + (\dot\sigma^{4} + |\sigma|^{3})^{1/6}\,\mathrm{sign}(\dot\sigma + 0.5\,|\sigma|^{3/4}\mathrm{sign}\,\sigma)]\}$.

It can be shown that the above sets of parameters β_i with $r \le 4$ are valid for all 3 families of controllers, $\varepsilon_i = 0.2$ is chosen in that case for the 3rd family. Note that while enlarging α increases the class (3) of systems to which the controller is applicable, parameters β_i, ε_i are tuned to provide for the needed convergence rate [18].

Theorem 2. *The listed* 3 *families of arbitrary-order sliding-mode controllers satisfy Assumption* 6º. *Thus, in each case under assumptions* 1º - 5º *(i.e. with sufficiently large* α *and properly chosen* $\beta_1,\ldots,\ \beta_{r-1}$*) the controller*

$$u = -\alpha\, \Psi_{r-1,r}(\sigma, \dot\sigma, \ldots, \sigma^{(r-1)}), \qquad (10)$$

provides for the establishment in finite time and keeping of the inequalities $|\sigma| < a_0\mu^{r}$, $|\dot\sigma| < a_1\mu^{r-1}$, ..., $|\sigma^{(r-1)}| < a_{r-1}\mu$ *in the system* (4), (5), (10) *with some positive constants* a_0, a_1, ..., a_{r-1} *independent of* μ.

All 2-sliding controllers from [13,15] which do not require switching on the axis $\dot\sigma = 0$ satisfy Assumption 6º. Thus, Theorem 1 is applicable for them with $r = 2$. The popular sub-optimal 2-sliding controller [2, 3] does satisfy Assumption 6º, though it has memory and, therefore, does not have the form (5). As follows from the proof in the next Section, the statement of Theorem 1 is true also for it with $r = 2$.

Unfortunately, the twisting controller

$$u = -\alpha_1\,\mathrm{sign}\,\sigma - \alpha_2\,\mathrm{sign}\,\dot\sigma,\ \alpha_1 > \alpha_2 > 0 \qquad (11)$$

does not satisfy Assumption 6º. Indeed, the set Γ consists of the axes $\sigma = 0$, $\dot\sigma = 0$, and (10) having been applied, the possibility of the sliding mode $\dot\sigma = 0$ appears, preventing the convergence of σ to 0. Nevertheless, the switching logic can be changed preserving the same trajectories, if the twisting controller (11) is considered as a particular case of the generalized sub-optimal controller with the traditional parameter $\beta = 0$ [3]. Another way is to require the following assumption.

7º. There is a constant $k > 0$ such that for any $\varepsilon > 0$ with sufficiently small $\delta > 0$ the reaction of the actuator output $v(t)$ to a step-wise change Δu of any constant input u at the moment t_0 with $|v(t_0) - u| < \delta$ satisfies the inequality $|v(t) - u| < \varepsilon + k\,|\Delta u|$ for any $t > t_0$.

In the case of a linear actuator this assumption is satisfied if the overshoot of the reaction to a step function does not exceed 100 k %.

Theorem 3. *Under assumptions* 1º - 5º, 7º *with*

$$K_m(\alpha_1 - (1+k)\alpha_2) > C,\ K_m(\alpha_1 + \alpha_2) - C > K_M(\alpha_1 - \alpha_2) + C$$

controller (11) *provides for the establishment in finite time and keeping of the inequalities* $|\sigma| < a_0\mu^2$, $|\dot\sigma| < a_1\mu$ *in the system* (4), (5), (11) *with some positive constants* a_0, a_1 *independent of* μ.

As follows from the Assumptions 1º, 2º, the rth derivative of the output σ is uniformly bounded by $C + K_M v_M$. Thus r-th order exact robust differentiator [16] with finite-time convergence can be applied here, producing exact estimations of $\dot\sigma$, ..., $\sigma^{(r-1)}$ and preserving the asymptotics from Lemma 1 and Theorems 1 – 3.

Remark 1. In practice, one cannot expect the output of the actuator to perform in the only possible way to prevent the convergence of the twisting controller by establishing sliding motion on the axis $\dot\sigma = 0$. Therefore, Assumption 7º is probably redundant. Unfortunately, the authors do not know how to formalize this reasoning.

Remark 2. A slightly generalized Assumption 5º can be considered, when the actuator instead of tracking the input u tracks $\gamma\,u$, where $\gamma > 0$ is some uncertain constant. All known high-order sliding controllers are capable to compensate for such a systematic actuator error if their output is proportionally increased.

Remark 3. In many practical cases Assumption 1º is only locally satisfied. That means that all the coordinates are restricted to some operational region where the Assumption holds. The Theorems can also be naturally reformulated for such local case as in [16].

IV. Proofs

Proof of Lemma 1. It is easy to check that due to the homogeneity property (7) with $\kappa > 0$ the combined time-coordinate-parameter transformation

$$G_\kappa: (t, \Sigma, z, \mu) \mapsto (\kappa\, t, d_\kappa\,\Sigma, z, \kappa\mu), \qquad (12)$$

$$d_\kappa: (\sigma, \dot\sigma, \ldots, \sigma^{(r-1)}) \mapsto (\kappa^{r}\sigma, \kappa^{(r-1)}\dot\sigma, \ldots, \kappa\,\sigma^{(r-1)}), \qquad (13)$$

preserves the trajectories of the inclusion (4), (5), (8) transferring them into the trajectories of the same inclusion with the actuator parameter changed from μ to $\kappa\mu$.

Let the invariant set Θ of (4), (5), (8) with $\mu = \mu_0$ satisfy the inequality $|\sigma^{(i)}| \le \gamma$, $i = 1,\ldots, r$-1, and $\|z\| \le \gamma$, then taking an arbitrary parameter μ and $\kappa = \mu/\mu_0$ obtain that the transformation transfers Θ into $d_\kappa\Theta$ being the invariant set of (4), (5), (8). The set $d_\kappa\Theta$ satisfies the inequalities $|\sigma| < a_0\mu^{r}$, $|\dot\sigma| < a_1\mu^{r-1}$, ..., $|\sigma^{(r-1)}| < a_{r-1}\mu$ with $a_i = \gamma\mu_0^{\,i-r}$. ■

Lemma 2. *Under the Assumptions* 2º, 5º *let the input* $u(t)$ *of the actuator* (4) *be a Lipschitz function of time with some fixed Lipschitz constant, then for any* $\delta, \varepsilon > 0$ *with sufficiently small* μ *the inequality* $|v - u| \leq \varepsilon$ *is established in the time* δ *and is kept afterwards.*

Proof. Let the Lipschitz constant of $u(t)$ be $L > 0$. Consider the time transformation $t = \mu\, \tau$. Then (4) takes the form

$$\dot{z} = f(z, u_1(\tau)), \qquad v = v(z), \quad u_1(\tau) = u(\mu\, \tau).$$

The function $u_1(\tau)$ is also Lipschitzian, but with the Lipschitz constant μL. Fix some initial value t_0 of the time t corresponding to $\tau = t_0/\mu$. Let $T > 0$ be the time τ needed to establish the inequality $|v - u_1| \leq \varepsilon/2$ with $u_1 = u(t_0)$. Since the function f is uniformly continuous in u_1, and with sufficiently small μ changes of u_1 are arbitrarily small during the time 2T, the inequality $|v - u_1| \leq \varepsilon$ is established in time T and is kept during the next interval of the same length. Applying the same reasoning from the moment $\tau = t_0/\mu + \mathrm{T}$ obtain that $|v - u_1| \leq \varepsilon$ holds also during the third interval. Continuing this reasoning obtain that $|v - u_1| \leq \varepsilon$ is kept forever. Returning to the original time $t = \mu\, \tau$ obtain the statement of the Lemma. ■

Proof of Theorem 1. Consider some closed vicinity of the origin $\Sigma = 0$. Let t^* be the maximal time of convergence to 0 for the trajectories of (8), (9) starting in this vicinity. The set Θ of points of the trajectory segments of the length t^* starting in the chosen vicinity of $\Sigma = 0$ is a compact region [6], which is obviously invariant with respect to (8), (9). Moreover, due to the finite-time stability of the inclusion it attracts any trajectory of (8), (9) in finite time. Note that, due to the homogeneity, any set $d_\kappa\Theta$, $\kappa > 0$, features the same properties and $d_\eta\Theta \subset d_\kappa\Theta$ with $\kappa > \eta > 0$.

Let $O_\delta(\Gamma)$ be the δ-vicinity of the control singularity set Γ. According to the definition of the set Γ, the function U has a Lipschitz constant valid in the whole compact set $d_3\Theta \setminus d_{1/2}\Theta \setminus O_{\delta/2}(\Gamma)$. Therefore, the function $U(\Sigma(t))$ calculated along any trajectory of the inclusion is Lipschitzian, for $d_3\Theta$ is compact and $\sigma^{(r)}$ is uniformly bounded. According to Lemma 2, with μ taken sufficiently small, the inequality $|v - u| \leq \delta$ is valid along any trajectory starting in $d_3\Theta \setminus d_{1/2}\Theta$ while the point is in $d_3\Theta \setminus d_{1/2}\Theta \setminus O_\delta(\Gamma)$ and before it first time leaves $d_3\Theta \setminus d_{1/2}\Theta$.

Thus, with sufficiently small μ any trajectory of (4), (5), (8) starting in $d_2\Theta \setminus \Theta$ satisfies the inclusion defined by (5), (8) and

$$\mu \dot{z} = f(z, u), \qquad v \in \begin{cases} V(z) + [-\delta, \delta], & \Sigma \notin O_\delta(\Gamma) \\ [-v_M, v_M], & \Sigma \in O_\delta(\Gamma) \end{cases}, \qquad (14)$$

until it leaves $d_3\Theta \setminus d_{1/2}\Theta$. It is easy to see that the *graph* of the inclusion (5), (8), (14) is close to the *graph* of (4), (5), (9) over the region $d_3\Theta \setminus d_{1/2}\Theta \setminus O_{\delta/2}(\Gamma)$. Let the trajectories of (8), (9) starting in $d_2\Theta$ terminate at 0 in some time T without leaving $d_2\Theta$ on the way. Then, due to the continuous dependence of solutions on the right-hand side of differential inclusion [7], any trajectory of (5), (8), (14) starting in $d_2\Theta \setminus \Theta$ enters $d_{2/3}\Theta$ at some moment during time T without leaving $d_3\Theta$ on the way. On the other hand $d_{2/3}\Theta$ is invariant for (5), (8), and, therefore, no trajectory of (5), (8), (14) starting in $d_{2/3}\Theta$ can ever leave Θ if δ is sufficiently small. Hence, the trajectories starting in $d_2\Theta \setminus \Theta$ enter Θ in the time T without leaving $d_3\Theta$ on the way and stay in Θ forever.

Applying the homogeneity transformation $(t, \Sigma) \mapsto (2^m t, d_{2^m}\Sigma)$ obtain that the trajectories starting in $d_{2^{m+1}}\Theta$ gather in the set $d_{2^m}\Theta$ in the time $2^m T$. The difference is that the trajectories satisfy (14) with increased perturbation vicinity of Γ:

$$\mu \dot{z} = f(z, u), \quad v \in \begin{cases} V(z) + [-\delta, \delta], & \Sigma \notin d_{2^m} O_\delta(\Gamma) \\ [-v_M, v_M], & \Sigma \in d_{2^m} O_\delta(\Gamma) \end{cases}.$$

Note that μ does not change, for the Lipschitz constant of the control function out of the above increased vicinity of Γ is 2^m times smaller due to the homogeneity property (5). Indeed, the same increment of $u(t)$, which corresponded previously to Δt corresponds now to the time interval $2^m \Delta t$. Let $M = d_2\Theta \setminus \Theta$ then, obviously,

$$\mathbf{R}^r = \Theta \cup M \cup d_2 M \cup d_4 M \cup \dots,$$

and the global convergence to the invariant set Θ is ensured. The Theorem follows now from Lemma 1. ■

Lemma 3. *All three families of arbitrary-order sliding-mode controllers defined in Section* III *satisfy Assumption* 6º.

Proof. The change of (5), (6), (8) to (8), (9) can generate new motions only on the singularity set Γ. It means that such motion can appear only in the points $\Sigma \in \Gamma$, where the vector set $(\dot{\sigma}, ..., \sigma^{(r-1)}, [-v_M, v_M])^t$ contains a vector tangential to Γ. The new motion is to take place on Γ itself.

In the case of the nested r-sliding controller (*the first family*) Γ consists of the discontinuity set of the control and of the points where $N_{i,r} = 0$, $i = 1,..., r - 1$, and, therefore, the control is not Lipschitzian. The set $N_{i,r} = 0$ is the set $\sigma = \dot{\sigma} = ... = \sigma^{(i-1)} = 0$, and the above tangentiality condition requires $\Sigma \equiv 0$. Such motion satisfies also (5), (6), (8) and is not a new one.

Consider now the discontinuity set. The main point of the convergence proof for the first family of controllers is that after some transient the trajectory never leaves some relatively small vicinity of the discontinuity set determined by the control gain α [14, 16]. Any new motions on the discontinuity set do not interfere with this proof, which establishes the Lemma for the first family.

In the case of *the second family* of controllers Γ contains only points where $N_{i,r} = 0$, $i = 1,..., r - 1$, with differently defined $N_{i,r}$ (see Section III). It can be shown by induction that $N_{i,r} = 0$ iff $\sigma = \dot{\sigma} = ... = \sigma^{(i)} = 0$. Thus, the tangentiality condition requires $\Sigma \equiv 0$. This is not the new motion. Thus in this case solutions of (5), (6), (8) and (8), (9) coincide.

The singularity set Γ of *the third family* consists only of the origin $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$, which makes the Lemma trivial in that case. ■

Proof of Theorem 2 follows immediately from Lemma 3.■

Proof of Theorem 3 is a simple modification of the convergence proof for the twisting controller [13]. ■

V. Computer simulation

Already traditional example of the kinematic car model

$$\dot{x} = v \cos \varphi, \quad \dot{y} = v \sin \varphi,$$
$$\dot{\varphi} = v/l \tan \theta,$$
$$\dot{\theta} = u_{\text{act}},$$

is chosen. Here x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle (Fig. 1a), u_{act} is the actuator output. The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while $g(x)$ and y are assumed to be measured in real time.

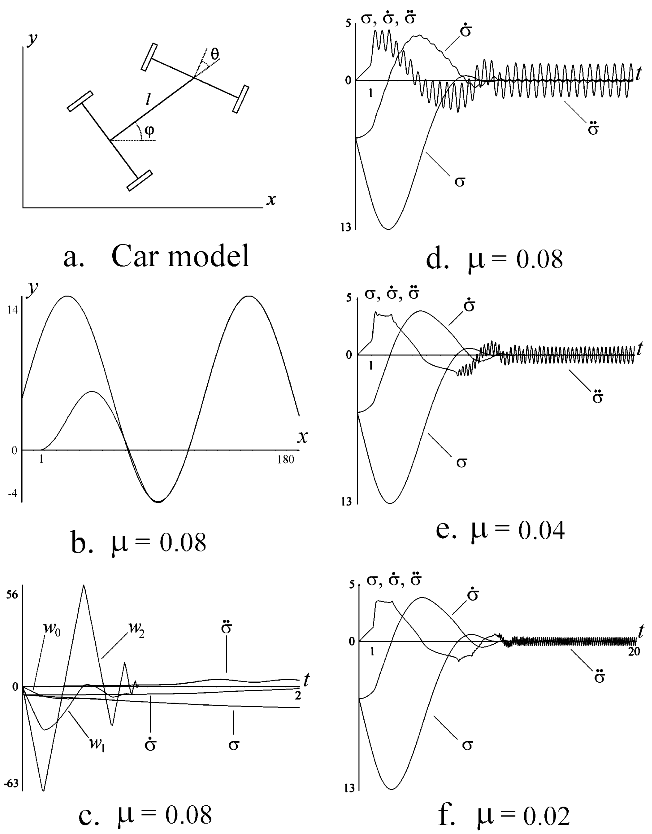

Fig. 1: Car model (a), trajectory tracking (b) and differentiator convergence (c) with $\mu = 0.08$; comparison of 3-sliding deviations with $\mu = 0.08$, 0.04, 0.02 (d, e, f)

Define

$$\sigma = y - g(x).$$

Let $v = const = 10$ m/s, $l = 5$ m, $g(x) = 10 \sin(0.05x) + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$. The relative degree of the system is 3 and 3-sliding controller can be applied here. A representative of the less known third family was chosen for demonstration. The resulting output-feedback controller (7), (8) is defined as

$$N_3 = (|w_0|^2 + |w_1|^3 + |w_2|^6)^{1/6}, \ \text{sat}(p, 0.2) = \min[1, \max(-1, 5p)],$$
$$u = -0.5 \ \text{sat}\{[w_2 + 2(|w_1|^3 + |w_0|^2)^{1/6} \text{sat}((w_1 + |w_0|^{2/3} \text{sign}\, \sigma)/N_3, 0.2)]/N_3, 0.2\},$$

where w_i are the real time estimations of the derivatives $\sigma^{(i)}$, $i = 0, 1, 2$, obtained by the differentiator

$$\dot{w}_0 = \xi_0, \ \xi_0 = -9\,|w_0 - \sigma|^{2/3} \text{sign}(w_0 - \sigma) + w_1,$$
$$\dot{w}_1 = \xi_1, \ \xi_1 = -15\,|w_1 - \xi_0|^{1/2} \text{sign}(w_1 - \xi_0) + w_2,$$
$$\dot{w}_2 = -110 \ \text{sign}(w_2 - \xi_1).$$

The initial conditions of the differentiator are $w_0(0) = \sigma(0)$, $w_1(0) = w_2(0) = 0$.

The control is applied only starting from $t = 1$ in order to provide some time for the differentiator convergence. The actuator is described by the transfer function

$$F(s) = \frac{\mu s + 1}{\mu^3 s^3 + 2\mu^2 s^2 + 2\mu s + 1}$$

realized in the form

$$\mu \dot{z}_1 = z_2,$$
$$\mu \dot{z}_2 = z_3,$$
$$\mu \dot{z}_3 = -z_1 - 2z_2 - 2z_3 + u,$$
$$u_{\text{act}} = z_1 + z_2,$$

with zero initial conditions.

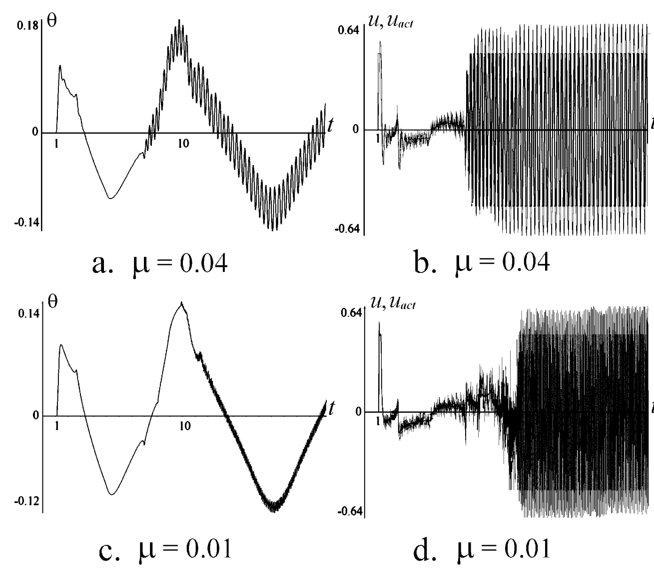

Fig. 2: Steering angle (a, c) and actuator performance (b, d) with $\mu = 0.04$, 0.02

The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics), the sampling step being equal to the integration step $\tau = 10^{-4}$. Tracking accuracies are listed in Table 1. It is seen that the accuracies of σ, $\dot{\sigma}$, $\ddot{\sigma}$ are proportional to μ^3, μ^2, and μ respectively (Fig. 1d, e, f). It is seen from Fig. 1c that the differentiator convergence takes about 0.9 s. The system performs remarkably well with a rather large actuator time constant $\mu = 0.08$. Indeed, the tracking deviation is only 4 cm. (Fig. 1b).

TABLE I

TRACKING ACCURACIES WITH DIFFERENT ACTUATOR TIME CONSTANTS

μ	Sup $\lvert\sigma\rvert$	Sup $\lvert\dot{\sigma}\rvert$	Sup $\lvert\ddot{\sigma}\rvert$
0.01	0.0000765	0.00294	0.189
0.02	0.000644	0.0102	0.374
0.04	0.00529	0.0408	0.746
0.08	0.0433	0.182	1.50

The actuator performance and the resulting steering angles are demonstrated in Fig. 2. Since sliding control signals are actually very fast, one can see from Fig. 2b, d that the actuator performs as a low-pass filter. On the first glance this contradicts the idea of the proofs that the actuator tracks the input with good precision, if the coordinates are distanced from the 3-sliding manifold. In fact, such tracking would be observed here only with very small μ, which requires in its turn also very small integration step.

Actuators with other transfer functions were also considered providing for similar simulation results.

VI. CONCLUSIONS

The main conclusion is that stable fast actuators do not really destroy the performance of homogeneous high-order sliding-mode controllers. The resulting asymptotic sliding accuracy does not depend on the relative degree of the actuator and is determined by the sliding order. The only exclusion is a rare case, when an asymptotically stable sliding mode $\sigma \equiv 0$ arises with the sliding order being equal to the sum of the system and actuator relative degrees. In such a case the residual chattering gradually disappears, and, though the Theorems are surely still valid, the coefficients a_i can be taken arbitrarily small. Probably, it is possible only when both relative degrees equal one [9].

One can consider application of smoothing filters at the input of an actuator device, which does not accept discontinuous inputs. If the time constant of the additional artificial actuator is sufficiently small, the resulting actuators will still provide for good performance due to the high sliding order (Fig. 1b).

The most widely used application of high-order sliding modes is based on the artificial increase of the relative degree, when the control derivative is considered as a new control. This results in a smooth control entering an actuator. Preliminary results show that also in that case the sliding mode is not destroyed crudely.

REFERENCES

[1] G. Bartolini, "*Chattering phenomena in discontinuous control systems*," *Internat. J. Systems Sci.*,vol. 20, pp. 2471–2481, 1989.

[2] G. Bartolini, A. Ferrara and E. Usai "Output tracking control of uncertain non linear second order systems," *Automatica,* 33, 3, 2203-2210, 1997.

[3] G. Bartolini, A. Ferrara, A. Pisano and E. Usai, "On the convergence properties of a 2-sliding control algorithm for nonlinear uncertain systems", *Int. J. Control ,* vol. 74, pp. 718-731, 2001.

[4] I. Boiko, "Frequency domain analysis of fast and slow motions in sliding modes", *Asian Journal of Control,* vol. 5, pp. 445-453, 2003.

[5] M. Djemai, J.P. Barbot, "Smooth manifolds and high order sliding mode control," *Proceedings of the 41st IEEE Conference on Decision and Control,* pp. 335 - 339, 2002.

[6] Edwards, C., Spurgeon, S. K. "Sliding Mode Control: Theory and Applications", Taylor & Francis, 1998.

[7] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Dordrecht, The Netherlands, Kluwer Academic Publishers, 1988.

[8] L. Fridman, "An averaging approach to chattering, " *IEEE Trans. on Automatic Control,* vol. 46, pp.1260-1264, August 2001.

[9] L. Fridman , and A. Levant. "Higher order sliding modes, " in: *Sliding Mode Control in Engineering,* J.P Barbot , W.Perruquetti (eds.), Marcel Dekker, New York, 2002, ch. 3.

[10] L. Fridman, "Singularly Perturbed Analysis of Chattering in Relay Control Systems", *IEEE Trans. on Automatic Control*, vol. 47, pp. 2079-2084, December 2002.

[11] L. Fridman, "Chattering analysis in sliding mode systems with inertial sensors," *International J. Control,* vol . 76, no. 9/10, pp. 906-912, 2003.

[12] A. Isidori, *Nonlinear Control Systems*, second edition, Springer Verlag, New York, 1989.

[13] A. Levant, "Sliding order and sliding accuracy in sliding mode control", *Int. J. Control,* vol. 58, pp. 1247-1263, 1993.

[14] A. Levant, "Universal SISO sliding-mode controllers with finite-time convergence," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1447–1451, Sept. 2001.

[15] A. Levant, "Construction principles of output-feedback 2-sliding mode design", Proc. of the 42th *IEEE Conference on Decision and Control*, Las-Vegas, Nevada, December 10-13, vol.1 ,pp. 317 - 322, 2002.

[16] A. Levant, "Higher order sliding modes, differentiation and output-feedback control," *Int. J. Control* , 76, no. 9/10, pp. 924-941, 2003.

[17] A. Levant, **"**Quasi-continuous high-order sliding-mode controllers*"*, *IEEE Transactions on Automatic Control*, to appear, 2005.

[18] A. Levant, "Homogeneity approach to high-order sliding mode design." *Automatica*, vol. 41, no. 5, pp. 823-830, 2005.

[19] A. Levant, A. Michael, "Adjustment of high-order sliding-mode controllers", *Proc. of the 16th IFAC World Congress*, Prague, 2005

[20] Y. Orlov, "Finite time stability of switched systems," *SIAM J. Control and Optimization,* vol. 43, no. 4, pp. 1253-1271,2005.

[21] Y. B. Shtessel, I. A. Shkolnikov, M.D.J. Brown. *"An asymptotic second-order smooth sliding mode control", Asian J. of Control* ,vol. 5, no. 4, pp. 498-5043, 2003.

[22] J.-J. E. Slotine, W. Li, *Applied Nonlinear Control*, London: Prentice-Hall, Inc., 1991.

[23] V.I. Utkin *Sliding modes in control and optimization,* Berlin, Germany: Springer Verlag, 1992.

[24] V. Utkin, J. Guldner, and J Shi*, Sliding Modes in Electromechanical Systems,* Taylor and Francis, London, 1999.