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On Invariance of Cyclic Group Symmetries in Multiagent Formations

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Abstract— **This paper explores how the interconnection topology among individuals of a multiagent system influences symmetry in its trajectories. It is shown how circulant connectivity preserves cyclic group symmetries in a formation of simple planar integrators. Moreover, it is revealed to what extent circulant connectivity is necessary in order that symmetric formations remain invariant under the system's dynamics.**

I. Introduction

"It's a basic principle: Structure always affects function."
— Steven Strogatz in *Sync* [1, p. 237]

This paper explores how the interconnection structure of a multiagent system influences, in particular, symmetry in its trajectories. A current research emphasis in the multiagent systems and cooperative control literature is to generalize: What are the connectivity conditions for achieving consensus [2], [3]? What happens if the interconnection topology among agents is dynamic [4]? These are matters of fundamental theoretical significance. On the other hand, practical issues arise when designing multiagent systems required to perform specific tasks. For instance, consider the problem of dynamic target tracking using a team of $n > 1$ autonomous robots. This task requires the team to act as a mobile and reconfigurable sensor array. Suppose each agent is equipped with a target-tracking sensor (e.g., ultrasonic sensors, a laser range finder, or a CCD camera) that, when combined with the sensor readings of other agents, can be utilized to estimate the location of a target. If the sensors measure distances to the target, then it can be shown that a configuration that optimizes the estimate is one in which the sensors are uniformly placed in a circular fashion around the target [5]. Notice how this optimal sensor placement is "symmetrical," in the sense that the configuration remains optimal under rotations by $2\pi/n$ about the target.

The problem of achieving and maintaining symmetry in multiagent formations is not a new endeavor. For example, [6] investigates distributed heuristic algorithms for the formation of geometric patterns in the plane (e.g., circles and polygons). In [7], artificial potentials are used to generate stable symmetric formations by inserting virtual leaders among the agents. How information flow influences the stability of formations is studied in [8]. In [9], the authors demonstrate how local pursuit strategies can generate regular polygon formations in systems of kinematic unicycles. Symmetry in the interconnection structure is exploited in [10], where the problem studied is distributed controller synthesis for large arrays of spatially interconnected systems. Of particular relevance to the current work is [11], wherein the symmetry in a network of coupled identical dynamical systems is exploited to classify invariant manifolds of the overall system dynamics with respect to their stability. Hence, "stability in the network descends from its topology" [11, p. 67].

The present research is especially influenced by the work of [12] and [13], wherein a circulant interconnection structure among multiple agents is utilized to deduce the overall steady-state behavior of the agents. In particular, [12] studies the asymptotic behavior of a collection of points in discrete-time circulant pursuit. Similarly, [13] studies the stability of certain geometric patterns for a collection of continuous-time fixed-speed agents in cyclic pursuit.

This paper studies connectivity as it relates to the problem of choosing distributed controllers that *inherently* preserve symmetric formations. Designing stable formations, e.g., as in [7], [8], [13], is not studied here. The paper begins by providing some terminology and background. Next, in Sec. III, it is shown how circulant connectivity preserves cyclic group symmetries in a formation of $n > 1$ simple planar integrators, each endowed with only relative sensing capabilities. In Sec. IV, it is revealed to what extent circulant connectivity is necessary in order that symmetric formations remain invariant under the system's dynamics.

II. Symmetry Groups, Graphs, and Pursuit

This section introduces some terminology and background material relating to symmetry groups, algebraic graph theory, and the class of multiagent systems studied in this paper.

A. Cyclic Group Symmetry

It is assumed that the reader is familiar with some basic group theory; e.g., as in [14]. Recall that the set of isometries in $\mathbb{R}^2$ form a group, denoted $I(\mathbb{R}^2)$. A subgroup G of $I(\mathbb{R}^2)$ is called a *symmetry group* of a subset $\mathcal{U} \subset \mathbb{R}^2$ if $\mathcal{U}$ remains invariant under every element of G. A group is called *cyclic* when all its elements are powers g^k of some one element g. For any element g in a group G, the set $\{g^k : k \in \mathbb{Z}\}$ is the cyclic subgroup of G generated by g. If $g^m = 1$ for some positive integer m, then the group generated by g consists of a finite number of elements. If m is the least positive integer for which this is true, then m is called the group's order.

Definition 1 (Rotation Group): The *rotation group* of order m, denoted C_m, is the cyclic group generated by a rotation through $2\pi/m$ about the origin.

Therefore, it is said that $\mathcal{U} \subset \mathbb{R}^2$ has symmetry C_m if the rotation group C_m is a symmetry group of $\mathcal{U}$.

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B. Agents in Pursuit

In this paper, it will be useful to view the agents as points in the complex plane, $\mathbb{C}$. Consider a collection of $n > 1$ agents, $z_1(t), z_2(t), \ldots, z_n(t) \in \mathbb{C}$, evolving in time t. Suppose that each agent is a simple integrator; i.e., $\dot{z}_i(t) = u_i(t) \in \mathbb{C}$, $i = 1, 2, \ldots, n$, where $u_i(t)$ is the control input. Here, we shall assume that the agents have only *relative* sensing capabilities (i.e., there is no global reference frame) and, therefore, that the inputs $u_i(t)$ are of the type

$$u_i(t) = \sum_{k \neq i} a_{ik} \left(z_k(t) - z_i(t) \right), i = 1, 2, \ldots, n. \tag{1}$$

That is, the aggregate multiagent system is of the form

$$\dot{z}(t) = Az(t), \tag{2}$$

where $z(t) = (z_1(t), z_2(t), \ldots, z_n(t)) \in \mathbb{C}^n$. A direct consequence of the relative sensing limitation is:

Property 1: The matrix A has zero row-sums.

In other words, $A[1, 1, \ldots, 1]^\top = 0$. This implies that if the agents are all collocated, then there is no motion. Given A, one can define a *digraph*, denoted $\Gamma(A)$. That is, if $a_{ik} \neq 0$, then there exists a directed edge in $\Gamma(A)$ from vertex i to k, implying that agent i receives information about agent k. In the present context, for convenience, we abuse notation and simply neglect the elements a_{ii} when constructing $\Gamma(A)$. This paper concerns itself with the trajectories of (2), and we address the following question: What fixed interconnection topologies $\Gamma(A)$ and associated weights $A = [a_{ik}]$ preserve symmetries in multiagent formations $z(t) \in \mathbb{C}^n$ for all $t \geq 0$?

C. Circulant Interconnections

It will be shown in Secs. III and IV that of fundamental significance to the topic of symmetry is a particular structure in the sensing topology: namely, *circulant connectivity*. If a system has circulant connectivity we mean that the system matrix A is a *circulant matrix* [15]; i.e., of the form

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix} =: \mathrm{circ}(a_0, a_1, \ldots, a_{n-1}).$$

Each row is merely the row above, shifted one element to the right (modulo n). The matrix is entirely determined by its first row. Also, intrinsic to the theory of circulants is the *fundamental permutation matrix* of order n, $\Pi_n = \mathrm{circ}(0, 1, 0, \ldots, 0)$. If A is a circulant matrix, then it can be written as a sum of fundamentals, $A = \sum_{i=0}^{n-1} a_i \Pi_n^i$ [15, p. 68].

A matrix A_1 is said to have the same *structure* as another matrix A_2, of the same dimensions, if for every zero entry of A_1 the corresponding entry in A_2 is also zero, and vice versa. Accordingly, if a square matrix A is such that there exists a circulant matrix A_c of the same order and structure as A, then we call A *structurally circulant*.

D. Formation Graphs

At each instant t, one can define a set of locations $\mathcal{V}_t = \{z_1(t), z_2(t), \ldots, z_n(t)\}$ and a set $\mathcal{E}_t$ of edge vectors $e_{ik}(t) : \mathcal{V}_t \times \mathcal{V}_t \to \mathbb{C}$ such that an edge $e_{ik}(t) := z_k(t) - z_i(t)$ exists in $\mathcal{E}_t$ only if there exists a corresponding edge in $\mathcal{E}$. Abusing terminology, it is convenient to refer to the pair $(\mathcal{V}_t, \mathcal{E}_t) =: \Gamma(A, z(t))$ as the *formation graph* (or often just *graph* for short). Fig. 1 provides two example formation graphs. In particular, the graph in Fig. 1a and its corresponding adjacency matrix both exhibit a circulant structure.

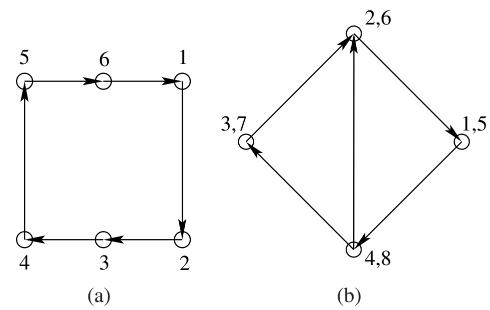


Fig. 1. Example formation graphs $\Gamma(A, z(t))$.

E. Permutations

Of particular utility when studying formations and symmetry is the theory of permutations. Let $\mathcal{N} := \{1, 2, \ldots, n\}$ and consider a bijection $\sigma : \mathcal{N} \to \mathcal{N}$, which is called a *permutation* of the set $\mathcal{N}$. Associated with every permutation σ is a square matrix, denoted P_σ, of order n. Given an $n \times n$ matrix $A = [a_{ik}]$, P_σ is such that $P_\sigma A = [a_{\sigma(i),k}]$ and, therefore, that $P_\sigma A P_\sigma^\top = [a_{\sigma(i),\sigma(k)}]$ (e.g., Π_n is the matrix corresponding to $\sigma(i) = i+1$). Let $\sigma^l(i) := \sigma \circ \sigma \circ \cdots \circ \sigma(i)$, the permutation σ applied l times to element $i \in \mathcal{N}$. Every $i \in \mathcal{N}$ generates a subset of $\mathcal{N}$ called a *cycle* of length l, where l is the least positive integer such that $\sigma^l(i) = i$. In general, a permutation σ can be factored (or partitioned) into a product of disjoint cycles, denoted $\sigma = (i_1, i_2, \ldots, i_m) \cdots (i_p, i_{p+1}, \ldots, i_n)$, where $i_1, i_2, \ldots, i_n \in \mathcal{N}$. This factorization is unique up to the ordering of factors, which are disjoint cycles. A permutation is called *primitive* if it has only one factor (of full length n). For more about permutations, see [15, Sec. 2.4].

III. Symmetric Formations and Invariance

We refer to the configuration of points $z(t) \in \mathbb{C}^n$ at time t as a multiagent *formation*, irrespective of the interagent connections. The principal result of this section is Theorem 2, which states that if the system matrix A is circulant, then symmetric formations remain symmetric. Let $j := \sqrt{-1}$.

Definition 2 (Formation Symmetry): The formation $z(t) \in \mathbb{C}^n$ at time t is said to have symmetry C_m if there exists a permutation $\sigma : \mathcal{N} \to \mathcal{N}$ such that

$$e^{j2\pi/m} z(t) = P_\sigma z(t). \tag{3}$$

That is, by rotating the agents $z(t) \in \mathbb{C}$ through angle $2\pi/m$ one obtains the same set of points in $\mathbb{C}$, but (generally) with a different labeling. Since agents at the origin play no role in symmetry, for simplicity's sake, it is assumed throughout this paper that there are no agents located at the origin. Henceforth, it will simply be said that a formation $z(t) \in \mathbb{C}^n$ has symmetry C_m "with P_σ" if the vector $z(t)$ satisfies Definition 2 with associated permutation matrix P_σ. Following Definition 2, several remarks are in order.

Remark 1: If at time t a formation $z(t)$ has symmetry C_m, then m divides n. Apply the constraint (3) m times, yielding

$$e^{j2\pi m/m} z(t) = z(t) = P_\sigma^n z(t)$$

(i.e., $\sigma^m(i) = i$ for every $i \in \mathcal{N}$). Thus, σ factors into n/m disjoint cycles of length m. Hence, C_m is a subgroup of C_n.

Remark 2: If a formation $z(t)$ has symmetry C_n, then the associated permutation σ is primitive. For if not (i.e., σ has a cycle of length $l < n$), then one obtains at the l-th iteration

$$e^{j2\pi l/n} z(t) = P_\sigma^l z(t) = z(t),$$

which can only be true for $l < n$ if $z(t) \equiv 0$.

Remark 3: Suppose a formation $z(t)$ has symmetry C_m, where $m < n$. If there are collocated agents, then it is possible that there exists more than one permutation σ such that (3) is satisfied. For instance, the $n = 8$ agents in Fig. 1b have symmetry C_4 with the primitive permutation $\sigma = (1, 2, \ldots, 8)$. However, the constraint (3) also holds with $\sigma = (1, 2, 3, 4)(5, 6, 7, 8)$. Following Remark 1, it is clear from the geometry of symmetry C_m that any factors of σ must have a length that is a multiple of m.

Consequent to Remark 3, it is assumed in this paper that if a formation $z(t)$ has symmetry C_m according to Definition 2, then its associated permutation σ is one that factors into exactly n/m cycles of length m. Let $\gcd(n, q)$ denote the greatest common divisor of the integers n and q. The following is a useful fact.

Remark 4: If m divides n, then there always exists an integer $q \in \{1, 2, \ldots, n-1\}$ such that $\gcd(n, q) = n/m$ since one can always choose $q = n/m$.

A. Canonical Ordering

The following theorem establishes a connection between formation symmetry C_m and a canonical ordering of the agents, often simply assumed; e.g., as in [14].

Theorem 1: Consider a formation $\tilde{z}(t)$ with symmetry C_m and let $q \in \{1, 2, \ldots, n-1\}$ satisfy $\gcd(n, q) = n/m$ (cf. Remark 4). Then, there exists a permutation τ of the agent locations $z(t) = P_\tau \tilde{z}(t)$ such that (3) holds with $P_\sigma = \Pi_n^q$.

The proof has been omitted for brevity's sake.

Let τ be the permutation described in Theorem 1. Clearly, if a formation has symmetry C_m then any permutation of the agent locations does not change this; it only changes the permutation σ with which (3) holds. By simultaneously permuting the rows and columns of A (i.e., compute $P_\tau A P_\tau^\top$) one can view this as just a coordinate transformation given by P_τ or, equivalently, simply a *relabeling* of the agents.

B. Symmetry Invariance

The focus of this paper is on identifying certain interconnection structures that inherently result in invariant manifolds corresponding to formation symmetry. This naturally leads to the following definition.

Definition 3 (Formation Symmetry Invariance): Let m be a divisor of n. Formation symmetry C_m is said to be *invariant* under the system dynamics (2) if for every $q \in \{1, 2, \ldots, n-1\}$ such that $\gcd(n, q) = n/m$ and for every initial formation $z(0) \in \mathbb{C}^n$ with $P_\sigma = \Pi_n^q$, the formation $z(t)$ has symmetry C_m with $P_\sigma = \Pi_n^q$ for all $t \geq 0$.

What follows is the first principal result of this paper. It shows that with the proper ordering, rotation group symmetry of a formation is invariant under circulant dynamics.

Theorem 2: If A is a circulant matrix, then formation symmetry C_m is invariant under the dynamics (2) for every m that divides n.

Proof: For every m that divides n, associated with the constraint (3) at time $t = 0$ is a complex linear subspace $\mathcal{M} = \{z \in \mathbb{C}^n : Mz = 0\} \subset \mathbb{C}^n$, where $M = \Pi_n^q - e^{j2\pi/m} I_n$. It is well known that the subspace $\mathcal{M}$ is A-invariant if $MA = AM$. Since A is a circulant matrix, it can be written in the form $A = \sum_{i=0}^{n-1} a_i \Pi_n^i$ [15, p. 68], implying that

$$\begin{aligned} MA &= \left(\Pi_n^q - e^{j2\pi/m} I_n\right) \sum_{i=0}^{n-1} a_i \Pi_n^i \\ &= \sum_{i=0}^{n-1} a_i \Pi_n^{i+q} - e^{j2\pi/m} \sum_{i=0}^{n-1} a_i \Pi_n^i \\ &= \sum_{i=0}^{n-1} a_i \Pi_n^i \left(\Pi^q - e^{j2\pi/m} I_n\right) \\ &= AM. \end{aligned}$$

Therefore, the subspace $\mathcal{M}$ is invariant under the system's dynamics (2), which means that the formation constraint (3) holds with $P_\sigma = \Pi_n^q$ for all $t \geq 0$. ■

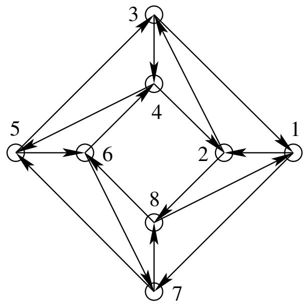
Example 1: Consider the $n = 8$ agents depicted in Fig. 2a. This formation $z(0)$ has symmetry C_4 with associated permutation $\sigma = (1, 3, 5, 7)(2, 4, 6, 8)$. Let

$$A = \mathrm{circ}(-1, -1, 0, 0, 0, 0, 2, 0) \tag{4}$$

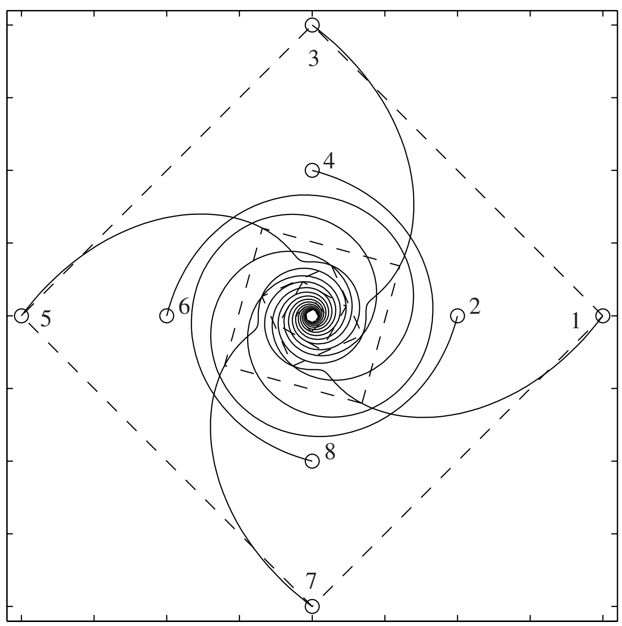
be the corresponding multiagent system matrix. Thus, every agent $i \in \mathcal{V}$ is repelled from agent $i+1$, but doubly attracted to agent $i+6$. The simulation in Fig. 1b shows the evolution of the formation starting at $z(0)$ under the dynamics (2) with (4). The fact that the agents converge to the origin is not of interest here. Rather, dashed lines connecting agents of the cycle $\{1, 3, 5, 7\}$ form a square at regular intervals during the simulation, highlighting that C_4 symmetry is preserved. ⋄

The following corollary to Theorem 2 addresses the more general case when the formation is not initially ordered.

Corollary 1: Given a permutation σ, let τ be such that $P_\tau P_\sigma P_\tau^\top = \Pi_n^q$ (cf. Theorem 1). Let m be any divisor of n and suppose $z(0) \in \mathbb{C}^n$ has symmetry C_m with permutation matrix P_σ. If $P_\tau A P_\tau^\top$ is a circulant matrix, then the formation $z(t)$ has symmetry C_m with P_σ for all $t \geq 0$.



(a) Initial graph $\Gamma(A, z(0))$.



(b) Simulation demonstrating symmetry invariance.

Fig. 2. Initial formation graph and simulation results for Example 1.

IV. Circulant Necessity

Thus far, it has been shown that circulant multiagent systems preserve rotation group symmetries. But, when is a circulant system matrix also necessary? In this section, we reveal that circulant connectivity is necessary if formation symmetry C_m is to be invariant under the system's dynamics for every m that divides n.

A. Counterexample

Firstly, for any single m dividing n, the condition of Theorem 2 that A be circulant is not, in general, necessary for symmetry invariance, as illustrated by the following example.

Example 2: Consider a system (2) of $n = 4$ agents, where the inputs (1) are given by

$$u_i(t) = z_{i+1}(t) - z_i(t),\ i = 1,3,4$$
$$u_2(t) = z_4(t) - z_2(t) - (z_1(t) - z_2(t))$$

The corresponding system matrix A is not circulant. Consider the initial formation $z(0)$ given by the graph $\Gamma(A, z(0))$ in Fig. 3. Although the graph $\Gamma(A, z(0))$ does not have symmetry C_4 (see Sec. V), the formation $z(0)$ does (and with $P_\sigma = \Pi_4$). Simulations confirm that $z(t)$ has symmetry C_4 for all $t \geq 0$.

However, it can also be verified by simulation that there exists an initial formation having symmetry C_2 (a subgroup of C_4) with $P_\sigma = \Pi_4^2$ such that symmetry C_2 is not preserved for all $t \geq 0$ (e.g., let $z_1(0) = z_3(0)$ and $z_2(0) = z_4(0)$). ◇

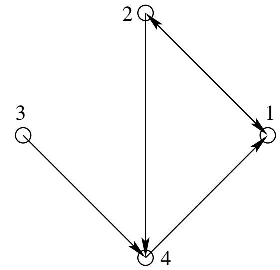


Fig. 3. Non-circulant graph $\Gamma(A, z(0))$ for Example 2.

B. A Special Class of Formations

When studying the necessity of circulant connectivity, it is helpful to employ a special class of formations; namely, those given by the constraint

$$\omega^q z(t) = \Pi_n z(t), \tag{5}$$

for some $q \in \{1, 2, \ldots, n-1\}$ and where $\omega := e^{j2\pi/n}$. Notice that the locations $z_i(t)$, $i = 1, 2, \ldots, n$, generated by the constraint (5) all have the same magnitude, and hence lie on a common circle. The following lemma associates a formation satisfying (5) with its symmetry.

Lemma 1: Suppose $\omega^q z(t) = \Pi_n z(t)$ holds for some $q \in \{1, 2, \ldots, n-1\}$ and $z(t) \in \mathbb{C}^n$. Then, the formation $z(t)$ has symmetry C_m, where $m = n/\gcd(n, q)$.

Proof: Let $p := \gcd(n, q)$ and define $m := n/p$ and $k_q := q/p$. To show the formation has symmetry C_m one must show there exists a permutation matrix P_σ such that (3) holds. From $\omega^q z(t) = \Pi_n z(t)$ one has

$$\left(e^{j2\pi/n}\right)^q z(t) = \left(e^{j2\pi/m}\right)^{k_q} z(t) = \Pi_n z(t). \tag{6}$$

By Bézout's identity[1], there exist integers l_q and l_m such that $1 = \gcd(k_q, m) = l_q k_q + l_m m$. This fact with (6) yields

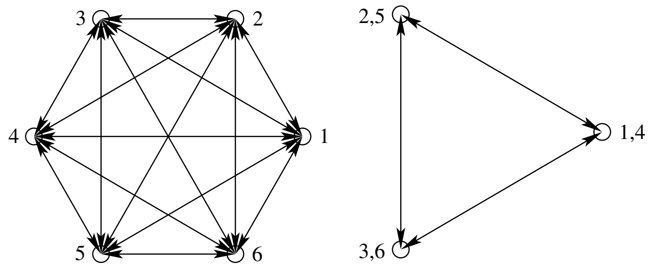
$$e^{j2\pi/m} z(t) = \left(e^{j2\pi/m}\right)^{l_q k_q} z(t) = \Pi_n^{l_q} z(t).$$

By letting $P_\sigma = \Pi_n^{l_q}$, one obtains the desired result. ■

Notice that the proof of Lemma 1 also reveals how formations satisfying the special constraint (5) have symmetry C_m with the canonical ordering introduced in Sec. III-A (i.e., (3) holds with $P_\sigma = \Pi_n^{l_q}$).

Example 3: Consider the example graphs $\Gamma(A, z(0))$ with $\omega^q z(0) = \Pi_n z(0)$ given in Fig. 4, where $n = 6$. In Fig. 4a, $q = 1$ and the formation has symmetry C_6 since $m = 6/\gcd(6, 1) = 6/1 = 6$. In Fig. 4b, $q = 2$ and the formation has symmetry C_3 since $m = 6/\gcd(6, 2) = 6/2 = 3$. ◇

[1]Given two nonzero integers a and b, Bézout's identity says there exist integers c and d such that $\gcd(a, b) = ac + bd$ [16, Sec. 1.2, Theorem 1.7].



(a) $q = 1$ (b) $q = 2$

Fig. 4. Example graphs $\Gamma(A, z(0))$ with $\omega^q z(0) = \Pi_6 z(0)$.

Let $v_q := (1, \omega^q, \omega^{2q}, \ldots, \omega^{(n-1)q})$, the $(q+1)$-th column of $\sqrt{n}F_n^*$, where F_n denotes the Fourier matrix [15, p. 32].

Lemma 2: For every $q \in \{1, 2, \ldots, n-1\}$, the vector $z \in \mathbb{C}^n$ satisfies $\omega^q z = \Pi_n z$ if and only if $z = v_q z_1$.

Proof: The statement $\omega^q z = \Pi_n z$ is equivalent to $z_2 = \omega^q z_1$, $z_3 = \omega^q z_2 = \omega^{2q} z_1, \ldots, z_n = \omega^{(n-1)q} z_1$, with $\omega^{nq} z_1 = z_1$. Equivalently, $z = v_q z_1$, which concludes the proof. ■

C. Necessary Conditions for Invariance

The following theorem is about the necessity of circulant connectivity and is the second principal result of this paper.

Theorem 3: If formation symmetry C_m is invariant under the dynamics (2) for every m that divides n, then the system matrix A is a circulant matrix.

Proof: Theorem 3.1.1 of [15] says that an $n \times n$ matrix A is circulant if and only if it commutes with the fundamental permutation matrix, Π_n. Therefore, it suffices to show that $\Pi_n A - A\Pi_n = 0$. Let $q \in \{1, 2, \ldots, n-1\}$ be arbitrary and pick an initial formation $z(0) = v_q z_1(0)$, with $z_1(0) \neq 0$. By Lemma 2, $z(0)$ satisfies $\omega^q z(0) = \Pi_n z(0)$. By Lemma 1, $z(0)$ has symmetry C_m with $m = n/\gcd(n, q)$. By assumption, $z(t)$ has symmetry C_m for all $t \geq 0$. By differentiating the constraint $\omega^q z(t) = \Pi_n z(t)$ with respect to time, one obtains

$$\begin{aligned}\omega^q A z(t) = \Pi_n A z(t) &\overset{(5)}{\Longleftrightarrow} (\Pi_n A - A\Pi_n)\, z(t) = 0 \\ &\Longleftrightarrow (\Pi_n A - A\Pi_n)\, v_q z_1(t) = 0,\end{aligned}$$

for all $t \geq 0$, using Lemma 2 again in the last step. In particular, since $z_1(0) \neq 0$, $(\Pi_n A - A\Pi_n) v_q = 0$. By Property 1, A has zero row-sums. Thus, $Av_0 = 0$. Also, because v_0 is an eigenvector of Π_n with corresponding eigenvector $\lambda = 1$, $\Pi_n v_0 = v_0$ [15, pp. 72–73]. Therefore,

$$(\Pi_n A - A\Pi_n)\, v_0 = \Pi_n A v_0 - A\Pi_n v_0 = -Av_0 = 0.$$

Recall that, $[v_0\; v_1\; \cdots\; v_{n-1}] = \sqrt{n}F_n^*$, where F_n is the Fourier matrix [15, p. 32]. Therefore, it has been shown that $(\Pi_n A - A\Pi_n) F_n^* = 0$. Since F_n^* is invertible, $\Pi_n A - A\Pi_n = 0$. Therefore, A is a circulant matrix. ■

The next example highlights the significance of the assumption that not only is symmetry C_n invariant, but also all of its subgroups are invariant under the system's dynamics.

Example 4: Consider $n = 6$ agents initially configured such that $\omega z(0) = \Pi_6 z(0)$. Suppose the graph $\Gamma(A, z(0))$ is coupled in an all-to-all fashion, as in Fig. 4a. Let $\tilde{A} = \mathrm{circ}(-5, 1, 1, 1, 1, 1)$ and let A be the matrix $\tilde{A}$ but with its second row replaced by $(1/2, -4, 1/2, 1/2, 2, 1/2)$. For the initial formation $\omega z(0) = \Pi_6 z(0)$, Fig. 5a shows how the rotation group C_6 is invariant under the dynamics (2), despite the fact that A is not circulant. In Fig. 5a, the dashed lines connect agents $\{1, 2, 3, 4, 5, 6\}$, in sequence, at regular intervals during the simulation.

However, consider a different initial formation $\omega^2 z(0) = \Pi_6 z(0)$, which has symmetry C_3 (since $\gcd(6, 2) = 2$, implying that $m = 6/2 = 3$). C_3 is a subgroup of C_6. The associated formation graph is given in Fig. 4b. Formation symmetry C_3 is not invariant under the dynamics (2), as one can see from the simulation results of Fig. 5b, where the dashed lines connect agents $\{1, 2, 3\}$. As time evolves, the initial equilateral formation becomes only isosceles. ◇

V. Graph Symmetry and Invariance

Although Theorems 2 and 3 make no mention of *graph* symmetry, the condition that A is a circulant matrix implies the graph is also symmetric. Proposition 1 offers this last result, but a definition and example are helpful first.

Definition 4 (Graph Symmetry): The graph $\Gamma(A, z(t)) = (\mathcal{V}_t, \mathcal{E}_t)$ is said to have symmetry G at time t if it has the property that for every element $g \in G$, if $v(t) \in \mathcal{V}_t$, then $gv(t) \in \mathcal{V}_t$, and if $e(t) \in \mathcal{E}_t$, then $ge(t) \in \mathcal{E}_t$. Moreover, the maps $v(t) \mapsto gv(t)$ and $e(t) \mapsto ge(t)$ are permutations.

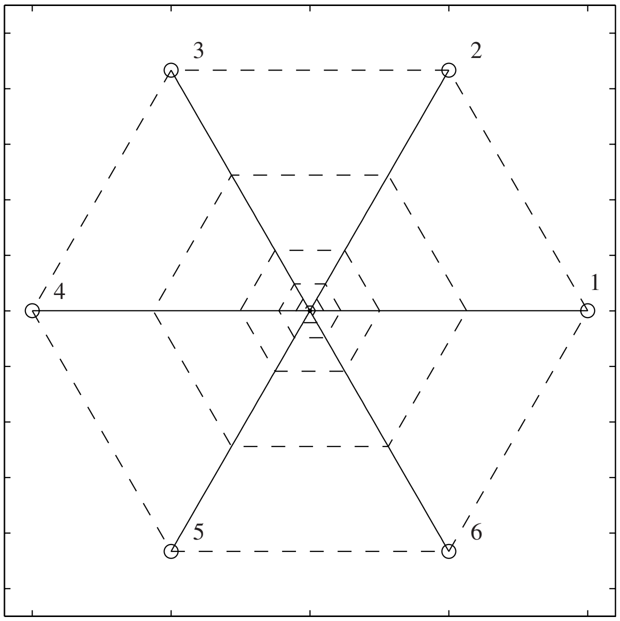
Example 5: Fig. 1a has symmetry C_2, but not C_4 because a rotation through $\pi/2$ does not map vertices to vertices. Fig. 1b has symmetry C_1, but not C_2 because a rotation through π changes the edge directions. ◇

Proposition 1: Suppose the formation $z(t)$ has symmetry C_m with $P_\sigma = \Pi_n^q$. If A is a structurally circulant matrix, then the graph $\Gamma(A, z(t))$ also has symmetry C_m.

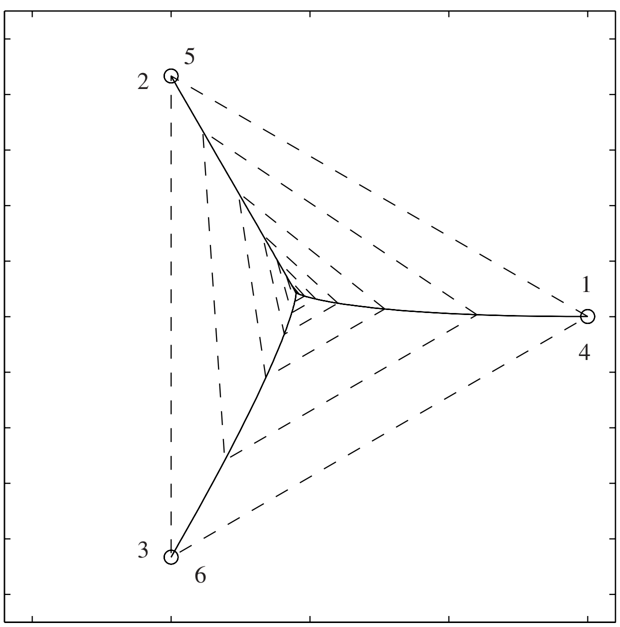
Proof: As per Definition 4, it is enough to show that the map induced by a generator of the cyclic group C_m maps vertices in $\mathcal{V}_t$ (resp., edges in $\mathcal{E}_t$) to vertices in $\mathcal{V}_t$ (resp., edges in $\mathcal{E}_t$) by a bijection. Rotation through $2\pi/m$ is a generator of the cyclic group C_m. Constraint (3) implies the map $z(t) \mapsto e^{j2\pi/m} z(t)$ is a bijection on $\mathcal{V}_t$, which means that vertices $z_i(t) \in \mathcal{V}_t$ are mapped to vertices in $\mathcal{V}_t$ by a bijection. Consider the rotation of an arbitrary edge $e_{ik}(t) \in \mathcal{E}_t$ through angle $2\pi/m$, yielding $e^{j2\pi/m} e_{ik}(t) = e^{j2\pi/m}(z_k(t) - z_i(t)) = z_{k+q}(t) - z_{i+q}(t) = e_{i+q,k+q}(t)$. Since $e_{ik}(t) \in \mathcal{E}_t$, $a_{ik} \neq 0$. But, since A is structurally circulant, $a_{i+q,k+q} \neq 0$, implying that $e_{i+q,k+q}(t) \in \mathcal{E}_t$. Hence, by the constraint (3), edges $e_{ik}(t) \in \mathcal{E}_t$ are mapped to edges in $\mathcal{E}_t$ by a bijection. ■

This final example illustrates the fact that graph symmetry is not sufficient to preserve cyclic group symmetries.

Example 6: Consider a system of $n = 6$ agents with $A = \mathrm{circ}(-3, 1, 2, -1, 3, -2)$ and corresponding graph at $t = 0$ given by $\Gamma(A, z(0))$ in Fig. 4a, which has symmetry C_6 with $P_\sigma = \Pi_n$. Following Theorem 2, formation symmetry C_6 is invariant. But, consider a new initial formation, given by a permutation of the original one, $\tilde{z}(0) = P_\tau z(0)$, where $\tau = (1)(2,3)(4)(5)(6)$. Since the coupling is all-to-all, the new graph $\Gamma(A, \tilde{z}(0))$ also has symmetry C_6. However, (3) does not hold with $P_\sigma = \Pi_n^q$ for any q, since $P_\tau \Pi_n P_\tau^\top$ is not of the form Π_n^q. It can be shown by simulation that symmetry



(a) C_6 symmetry preserved.



(b) C_3 symmetry not preserved.

Fig. 5. Simulations for Example 4.

C_6 of the formation $\tilde{z}(0)$ is not invariant under $\dot{\tilde{z}}(t) = A\tilde{z}(t)$, despite the fact that $\Gamma(A, \tilde{z}(0))$ has symmetry C_6. $\diamond$

VI. Conclusions

By combining the sufficiency result of Theorem 2 and the necessity result of Theorem 3, we have shown that for a multiagent system of the form (1)–(2), formation symmetry C_m is invariant under the system's dynamics for every m that divides n if and only if the system has circulant connectivity.

In light of our results, there exist a few open questions. Firstly, one might wonder about the necessity of the canonical labeling introduced in Sec. III-A and assumed in Definition 2. Is this ordering assumption without loss of generality? Do there exist other classes of ordering for which there is symmetry invariance if and only if the system matrix is circulant? Secondly, to what extent are the presented results specific to the simple integrator model (1)–(2)? And finally, multiagent systems design is often presented as the problem of synthesizing local control strategies that generate desired global behaviors. Instead, the contributions of this paper emphasize the importance of structure. It seems reasonable that structure could be exploited towards design. Given a set of fixed agent behaviors, can we control a multiagent system's function (e.g., its steady-state and transient behaviors) by intelligently switching the agent interconnection topology?

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