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Robust Output Command Tracking for Linear Systems with Nonlinear Uncertain Structure with Application to Flight Control

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Abstract—**The problem of asymptotic robust output command tracking is studied for the case of linear systems with nonlinear uncertain structure. The problem is solved using a compensator with static state feedback, unity output feedback and integral action. The controller is restricted to be independent from the uncertainties. The output of the closed loop system tracks polynomial type reference signals. Sufficient conditions for the problem to have a solution are established. The result of asymptotic robust output command tracking is applied to control the short period longitudinal motion of an aircraft. In the mathematical description of the aircraft an uncertain stability derivative is included. The steps to determine the controller are analytically presented. The effectiveness of the controller over a wide range of the system uncertainty, wind disturbances and actuator failure is illustrated through simulation to the nonlinear model.**

I. Introduction

THE problem of output tracking has attracted consider attention during the last decades (see [1]-[6] and the reference therein). It is a common practice to translate both the non uncertain and the robust tracking problem to a stabilizability problem (see [1]-[3]). Particularly for the case of robust tracking a variety of approaches, particularly optimal and adaptive, has been proposed (e.g. [4]-[6]).

The problem of tracking appears to be of exceptional interest to flight control systems and industrial applications [2]. In the present paper we focus on the general system category of linear systems with nonlinear uncertain structure. Furthermore, the derived results are successfully applied to control short period longitudinal motion of an aircraft. The contribution of the present paper consists in establishing sufficient condition for the aforementioned system category.

The material of the paper is organized in to two parts. In *Part A* the problem of robust output command tracking for polynomial reference inputs is solved for linear SISO systems with nonlinear uncertain structure. The problem is solved using a compensator with static state feedback, unity output feedback and integral action. The original uncertain system is augmented with error dynamics and the overall system is robustly stabilized using the approach in [7] and [8]. This way, sufficient conditions for the problem to have a solution are established. In *Part B* the results of asymptotic output robust tracking are applied to control the short period longitudinal motion of an aircraft having uncertain stability derivative. An analytic algorithm for the derivation of the controller is presented. Simulation results executed for the nonlinear aircraft mathematical description, illustrate the effectiveness of the controller over a wide range of the system uncertainty, wind disturbances and actuator failure.

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Part A: Control Design Procedure

II. Problem Formulation

Consider the linear time-invariant SISO system with non linear uncertain structure described by

$$\dot{x}(t) = A(q)x(t) + b(q)u(t) \ , \ y(t) = c(q)x(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the input and $y(t) \in \mathbb{R}$ is the output of the system. The matrices $A(q) \in [\wp(q)]^{n\times n}$, $b(q) \in [\wp(q)]^{n\times 1}$ and $c(q) \in [\wp(q)]^{1\times n}$ are function matrices depending upon the uncertainty vector $q = [q_1 \ \cdots \ q_l] \in \mathbb{Q}$ ($\mathbb{Q}$ denotes the uncertain domain). The set $\wp(q)$ is the set of nonlinear functions of q. The uncertainties $q_1,\dots,q_l$ do not depend upon the time. With regard to the nonlinear structure of the system matrices $A(q)$, $b(q)$ and $c(q)$ no limitation or specification is assumed (i.e. boundness, continuity etc.).

2.1 Polynomial Reference Signals

Consider the case where the reference output $y_r(t)$ is the output of a reference model described by

$$\dot{x}_r(t) = A_r x_r(t), \quad y_r(t) = c_r x_r(t), \quad x_r(0_-) = x_{r,0} \tag{2}$$

where $y_r(t) \in \mathbb{R}$, $x_r(t) \in \mathbb{R}^{r\times 1}$ and $x_{r,0}$ is an arbitrary vector of initial conditions and where

$$A_r = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad c_r = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

Clearly, it holds that $y_r^{(r)}(t) = 0$. Define the tracking error

$$e(t) = y(t) - y_r(t) \tag{3}$$

Differentiating the error r-times, we get

$$e^{(r)}(t) = y^{(r)}(t) = c(q)x^{(r)}(t) \quad (4)$$

Define the variables

$$z(t) = x^{(r)}(t) \quad , \quad \tilde{u}(t) = u^{(r)}(t) \quad (5)$$

According to (4) and (5), the system (1) can be augmented with the tracking error dynamics as follows:

$$\frac{d}{dt}\tilde{x}(t) = \tilde{A}(q)\tilde{x} + \tilde{b}(q)\tilde{u}(t) \quad (6)$$

where $\tilde{x}(t) = \begin{bmatrix} e(t) & e^{(1)}(t) & \cdots & e^{(r-1)}(t) & z(t) \end{bmatrix}^T$ and $\tilde{A}(q) = \begin{bmatrix} 0_{(r-1)\times 1} & I_{r-1} & 0 \\ 0 & 0_{1\times(r-1)} & c(q) \\ 0 & 0_{n\times(r-1)} & A(q) \end{bmatrix}$, $\tilde{b}(q) = \begin{bmatrix} 0_{r\times 1} \\ b(q) \end{bmatrix}$.

To the augmented system (6) apply the static state feedback control law

$$\tilde{u}(t) = f\,\tilde{x}(t) = f_1\tilde{e}(t) + f_2 z(t) \quad (7)$$

where $\tilde{e}(t) = \begin{bmatrix} e(t) & e^{(1)}(t) & \cdots & e^{(r-1)}(t) \end{bmatrix}^T$, $f_1 \in \mathbb{R}^{1\times r}$, and $f_2 \in \mathbb{R}^{1\times n}$. The robust output command tracking is formulated in the following Lemma:

Lemma 1 : The roblem of robust output command tracking is solvable, i.e. the output of the uncertain system (1) follows the output of the reference system (2) while the tracking error (3) decreases asymptotically to zero, if there exists a static state feedback control law of the form $\tilde{u}(t) = f\,\tilde{x}(t)$ such that the characteristic polynomial

$$\tilde{p}_{cl}(s,q,f) = \det\left[sI_{r+n} - \tilde{A}(q) - \tilde{b}(q)f\right] \quad (8)$$

is robustly stable.

Proof: If there exists a control law $\tilde{u}(t) = f\,\tilde{x}(t)$ making the augmented system robustly stable then the tracking error $e(t)$ is decreasing asymptotically to zero as $t \to \infty$, for arbitrary initial conditions, i.e $\lim_{t\to\infty}\left[y(t) - y_r(t)\right] = 0$. ■

The control law $\tilde{u}(t) = f\,\tilde{x}(t)$ can be expressed in terms of the original systems as follows

$$u(t) = f_{1,1}\overbrace{\int_{\tau=0}^{t}\cdots\int_{\tau=0}^{t}}^{r}\left[e(\tau)d\tau\right] + f_{1,2}\overbrace{\int_{\tau=0}^{t}\cdots\int_{\tau=0}^{t}}^{r-1}\left[e(\tau)d\tau\right] + \cdots + f_{1,r}\int_{\tau=0}^{t} e(\tau)d\tau + f_2 x(t) \quad (9)$$

where $f_{1,i}\ (i = 1,\dots,r)$ are the elements of f_1.

I. Solvability Conditions

The characteristic polynomial (8) can be rewritten as follows

$$\tilde{p}_{cl}(s,q,f) = \det\left[sI_{n+r} - \tilde{A}(q)\right] - f\,\mathrm{adj}\left[sI_{n+r} - \tilde{A}(q)\right]\tilde{b}(q) \quad (10)$$

Define:

$$\tilde{a}(q) = \begin{bmatrix} 1 & \tilde{a}_0(q) & \cdots & \tilde{a}_{n-1}(q) & 0_{1\times r} \end{bmatrix} \quad (11)$$

where $\tilde{a}_i(q)\ \ (i = 0,\dots,n-1)$ are the coefficients of the characteristic polynomial of the open loop system (1). Also define the polynomial matrix

$$\tilde{P}(s,q) = \Omega(q)[s^{\mu(q)} \cdots \ s^0]^T = \mathrm{adj}\left[sI_{n+r} - \tilde{A}(q)\right]\tilde{b}(q) \quad (12)$$

where $\mu(q) \le n + r$ is the maximum degree of the polynomial matrix $\mathrm{adj}\left[sI_{n+r} - A(q)\right]b(q)$ and where

$$\Omega(q) = \left[\omega_0(q)\cdots\omega_{\mu(q)}(q)\right];\ \omega_i(q) = \left[\tilde{\omega}_{i,1}(q)\cdots\tilde{\omega}_{i,n+r}(q)\right]^T \quad (13)$$

Since

$$\det\left[sI_{n+r} - \tilde{A}(q)\right] = s^r \det\left[sI_n - A(q)\right] = \tilde{a}(q)[s^{n+r}\cdots s^0]^T$$

and according to definitions (11), (12) and (13) the augmented closed loop characteristic polynomial can equivalently be expressed as follows:

$$\tilde{p}_{cl}(s,q,f) = \left[\tilde{a}(q) - f\tilde{\Omega}(q)\right][s^{n+r} \ \cdots \ s^0]^T \quad (14)$$

where $\tilde{\Omega}(q) = \left[0_{(n+r)\times(n+r-\mu(q))} \ \middle|\ \Omega(q)\right]$, or equivalently as

$$\tilde{p}_{cl}(s,q,f) = [s^{n+r} \ \cdots \ s^0]\left[-\left[\tilde{\Omega}(q)\right]^T f^T + \tilde{a}(q)^T\right] \quad (15)$$

Define

$$A^{**}(q) = \left[\begin{array}{c|c} 1 & 0_{1\times(n+r)} \\ \hline \hat{a}_1(q) & 0_{(n+r-\mu(q)-1)\times(n+r)} \\ \hline \hat{a}_2(q) & -\hat{\omega}_1(q) \\ \hline 0_{r\times 1} & -\hat{\omega}_2(q) \end{array}\right] \quad (16)$$

where $\hat{a}_1(q) = \begin{bmatrix} \tilde{a}_0(q) \\ \vdots \\ \tilde{a}_{n+r-\mu(q)-2}(q) \end{bmatrix}$, $\hat{a}_2(q) = \begin{bmatrix} \tilde{a}_{n+r-\mu(q)-1}(q) \\ \vdots \\ \tilde{a}_{n-1}(q) \end{bmatrix}$,

$$\hat{\omega}_1(q) = \begin{bmatrix} \tilde{\omega}_{0,1}(q) & \cdots & \tilde{\omega}_{0,n+r}(q) \\ \vdots & \cdots & \vdots \\ \tilde{\omega}_{\mu(q)-r,1}(q) & \cdots & \tilde{\omega}_{\mu(q)-r,n+r}(q) \end{bmatrix}$$

$$\hat{\omega}_2(q) = \begin{bmatrix} \tilde{\omega}_{\mu(q)-r+1,1}(q) & \cdots & \tilde{\omega}_{\mu(q)-r+1,n+r}(q) \\ \vdots & \cdots & \vdots \\ \tilde{\omega}_{\mu(q),1}(q) & \cdots & \tilde{\omega}_{\mu(q),n+r}(q) \end{bmatrix}$$

Based on the above definitions and the results in [7] and [8] the following theorem is presented.

Theorem 1. The problem of robust output command tracking for the uncertain system (1) via the controller (9) is solvable, if the following conditions are satisfied

(i) The elements of $A^{**}(q)$ are continuous functions of q for every $q \in \mathbb{Q}$

(ii) There exists $(n+r+1)-$ row submatrix of $A^{**}(q)$, let $A^{*}(q)$ which is positive antisymetric.

Proof: According to Lemma 1 the problem of robust output command tracking for the uncertain system (1) via the controller (9) is solvable if the closed loop polynomial of the extended case is robustly stable. According to the results in [7] and [8] the uncertain polynomial is Hurwitz invariant if conditions (i) and (ii) of Theorem 1 are satisfied. ■

Remark 1. The class of the systems that satisfy condition (ii) of Theorem 1, can be widen, if, instead of $A^{**}(q)$ the matrix $A^{**}(q)T$ is considered where T is an appropriate invertible and independent from q matrix.

For the definition of positive antisymetric matrices see [7-8]. An analytic algorithm for the computation of the vector f that preserve Hurwitz invariability can be found in [7-9].

The results of the present section can further be extended if one considers that the dimension of the system depends upon the uncertainties $n = n(q)$ using the results in [8] for uncertainty dependent system order.

Remark 2. For the case of a constant reference output $y_r(t)$ the augmented system with the tracking error dynamics is reduced to the following simple state space equations:

$$\frac{d}{dt}\tilde{x}(t) = \tilde{A}(q)\tilde{x} + \tilde{b}(q)\tilde{u}(t); \quad \tilde{u}(t) = \dot{u}(t) \quad \tilde{x}(t) = \begin{bmatrix} e(t) \\ z(t) \end{bmatrix},$$

$$z(t) = \dot{x}(t), \; \tilde{A}(q) = \begin{bmatrix} 0 & c(q) \\ 0 & A(q) \end{bmatrix}, \; \tilde{b}(q) = \begin{bmatrix} 0 \\ b(q) \end{bmatrix}.$$

The control law can also be rewritten as follows $u(t) = f_1 \int_{\tau=0}^{t} e(\tau)d\tau + f_2 x(t)$ where $f_1 \in \mathbb{R}$, $f_2 \in \mathbb{R}^{1\times n}$ and the resulting closed loop structure is shown in Figure 1.

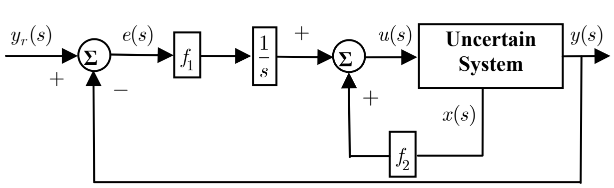


Fig. 1. Closed loop structure

PART B: AIRCRAFT APPLICATION

II. AIRCRAFT MODEL

The longitudinal dynamics of an aircraft are characterized by the phugoid (long period) and short period modes. Assuming that the aircraft horizontal velocity component U remains constant and dropping the pitch angle θ from the states the following non-linear short period dynamics of an AFTI/F-16 (see Fig. 2) are presented [10], [11]:

$$\dot{a} = \frac{-\left[L\cos(a) + D\sin(a)\right]}{mU}\cos^2(a) + q_a\cos^2(a)$$
$$\dot{q}_a = \frac{M}{I_{yy}} \quad , \dot{\delta}_e = -20\delta_e + 20u_e \tag{17}$$

where a, is the angle of attack, q_a is the pitch rate and δ_e is the elevator deflection angle and u_e is the elevator command. The aerodynamic forces D, L and the moment M are defined as follows:

$$D = \bar{q}S\left(C_{da}a + C_{d\delta e}\delta_e\right), \; L = \bar{q}S\left(C_{La}a + \frac{C_{Lq}c}{V_t}q_a + C_{L\delta e}\delta_e\right),$$
$$M = \bar{q}Sc\left(C_{ma}a + \frac{C_{mq}c}{2V_t}q_a + C_{m\delta e}\delta_e\right) \tag{18}$$

The parameters of the aircraft appearing in equations (17) and (18) are defined in Table I.

TABLE I
VARIABLE DEFINITION

Symbol	Definition	Values
V_t	aircraft velocity	$V_t = \sqrt{W^2 + U^2}\,(m/\sec)$
p	density of air	$0.65381 kg/m^3$ $1.26848\cdot 10^{-3} slug/ft^3$
$\bar{q}$	dynamic pressure	$\bar{q} = 0.5 p V_t^2 (Nt/m^2)$
S	surface	$27.87899 m^2 (300 ft^2)$
c	mean a/d chord	$3.450336 m (11.32 ft)$
m	mass	$9530.302 kg (652.7329 slug)$
g	gravity acceleration	$9.8 m/\sec^2 (32.2 ft/\sec^2)$
U	horizontal velocity	$284.4 m/\sec (933 ft/\sec)$
I_{yy}	moment of inertia	$73046.53 kg\cdot m^2 (53876 slug\cdot ft^2)$
C_{Lq}	a/d[a] force due to q	$3.162 (unitless)$
$C_{L\delta e}$	a/d[a] force due to δ_e	$0.55 (unitless)$
C_{ma}	a/d[a] moment due to a	$0.1146 (unitless)$
C_{mq}	a/d[a] moment due to q	$-2.382 (unitless)$
$C_{m\delta e}$	a/d[a] moment due to δ_e	$-0.6933 (unitless)$
C_{da}	a/d[a] force due to a	$0.151261 (unitless)$
$C_{d\delta e}$	a/d[a] force due to δ_e	$0.009912 (unitless)$

[a] a/d stands for "aerodynamic".

The parameter C_{La} is an uncertain parameter q, valued in the domain $q = C_{La} \in \left[C_{La_{\min}}, C_{La_{\max}}\right]$. The elevator deflection is constrained according to the inequalities $\left|\delta_e\right| < 25\deg$ and $\left|\dot{\delta}_e\right| < 60(\deg/\sec)$. The linearized equations of the AFTI F-16 can be written in the state space form as follows:

$$\dot{x}(t) = A(C_{La})x(t) + bu(t)\,,\ y(t) = cx(t)\,,\ x(0^-) = x_0 \quad (19)$$

where

$$x(t) = \begin{bmatrix} a(t) \\ q_a(t) \\ \delta_e(t) \end{bmatrix}, y(t) = a(t), u(t) = u_e(t), b = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix},$$

$c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and

$$A(C_{La}) = \begin{bmatrix} -\frac{0.5C_{La}pSU}{m} & 1-\frac{0.5cC_{Lq}pS}{m} & -\frac{0.5C_{L\delta e}pSU}{m} \\ \frac{0.5cC_{ma}pSU^2}{I_{yy}} & \frac{0.25c^2C_{mq}pSU}{I_{yy}} & \frac{0.5cC_{m\delta e}pSU^2}{I_{yy}} \\ 0 & 0 & -20 \end{bmatrix}$$

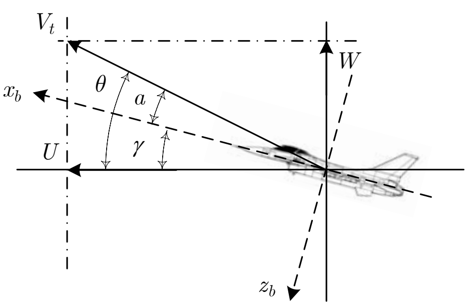


Fig. 2. Aircraft Schematic

III. ROBUST TRACKING CONTROLLER

In this section a robust tracking controller for asymptotic step tracking of the angle of attack, will be designed. According to (6) for $r = 1$, the following augmented state space description of the aircraft model is derived:

$$\dot{\tilde{x}} = \tilde{A}(C_{La})\tilde{x} + \tilde{b}\tilde{u} \quad (20)$$

where

$$\tilde{x} = \begin{bmatrix} e \\ z \end{bmatrix}, z = \dot{x} \text{ and } \tilde{A}(C_{La}) = \begin{bmatrix} 0 & c \\ 0 & A(C_{La}) \end{bmatrix}, \tilde{b} = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad (21)$$

To system (20) apply the following static state feedback law

$$f = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix} \quad ; \quad f_i \in \mathbb{R} \quad (22)$$

The augmented system closed loop characteristic uncertain polynomial is derived to be

$$\begin{aligned} p_{cl}(s, C_{La}, f) &= s^4 + \gamma_0(C_{La}, f)s^3 + \\ &+ \gamma_1(C_{La}, f)s^2 + \gamma_2(C_{La}, f)s + \gamma_3(C_{La}, f) \end{aligned} \quad (23)$$

where

$$\gamma_0(C_{La}, f) = \frac{20I_{yy}m - 20f_4I_{yy}m + 0.5C_{La}I_{yy}pSU - 0.25c^2C_{mq}mpSU}{mI_{yy}}$$

$$\gamma_1(C_{La}, f) = \frac{pSU}{mI_{yy}}\left[\gamma_{11}(C_{La}, f) + \gamma_{12}(C_{La}, f)\right]$$

$$\gamma_{11}(C_{La}, f) = 10C_{L\delta e}f_2I_{yy} + C_{La}\left(10(1-f_4)I_{yy} - 0.125c^2C_{mq}pSU\right)$$

$$\begin{aligned} \gamma_{12}(C_{La}, f) = c[&5(-0.1C_{ma} - 2C_{m\delta e}f_3)mU + \\ &+ c\left(5C_{mq}(-1+f_4)m + 0.25C_{Lq}C_{ma}pSU\right)] \end{aligned}$$

$$\gamma_2(C_{La}, f) = \frac{pSU}{mI_{yy}}\left[\gamma_{21}(C_{La}, f) + cU\left(\gamma_{22}(C_{La}, f) + \gamma_{23}(C_{La}, f)\right)\right]$$

$$\gamma_{21}(C_{La}, f) = C_{L\delta e}\left(10f_1I_{yy} + cpSU\left(-2.5cC_{mq}f_2 + 5C_{ma}f_3U\right)\right)$$

$$\begin{aligned} \gamma_{22}(C_{La}, f) = &2.5cC_{La}C_{mq}(-1+f_4)pS + \\ &+ C_{ma}\left(10(-1+f_4)m + 5cC_{Lq}(1-f_4)pS\right) \end{aligned}$$

$$\gamma_{23}(C_{La}, f) = 5C_{m\delta e}\left(-2f_2m + cC_{Lq}f_2pS - C_{La}f_3pSU\right)$$

$$\gamma_3(C_{La}, f) = \frac{5cf_1pS\left(-2C_{m\delta e}m + cC_{Lq}C_{m\delta e}pS - 0.5cC_{L\delta e}C_{mq}pS\right)U^2}{mI_{yy}}$$

According to definitions (11) and (12) the following matrices are defined:

$$\tilde{a}(C_{La}) = \begin{bmatrix} 1 & \tilde{a}_0(C_{La}) & \tilde{a}_1(C_{La}) & \tilde{a}_2(C_{La}) & 0 \end{bmatrix} \quad (24)$$

$$\left[-\tilde{\Omega}(C_{La})\right]^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 \\ 0 & \tilde{\omega}_{1,2} & \tilde{\omega}_{1,3} & \tilde{\omega}_{1,4}(C_{La}) \\ \tilde{\omega}_{2,1} & \tilde{\omega}_{2,2} & \tilde{\omega}_{2,3}(C_{La}) & \tilde{\omega}_{2,4}(C_{La}) \\ \tilde{\omega}_{3,1} & 0 & 0 & 0 \end{bmatrix} \quad (25)$$

where $\tilde{a}_0(C_{La}) = 20 + \left(-\frac{0.25c^2C_{mq}}{I_{yy}} + \frac{0.5C_{La}}{m}\right)pSU$,

$$\begin{aligned} \tilde{a}_1(C_{La}) = \frac{pSU}{mI_{yy}}[&c\left(-5cC_{mq}m - 0.5C_{ma}mU + 0.25cC_{Lq}C_{ma}pSU\right) + \\ &+ C_{La}\left(10I_{yy} - 0.125c^2C_{mq}pSU\right)] \end{aligned},$$

$$\tilde{a}_2(C_{La}) = \frac{cpS}{mI_{yy}}\left(-10C_{ma}m + 5cC_{Lq}C_{ma}pS - 2.5cC_{La}C_{mq}pS\right)U^2,$$

$$\tilde{\omega}_{1,2} = \frac{10C_{L\delta e}pSU}{m}, \tilde{\omega}_{1,3} = \frac{-10cC_{m\delta e}pSU^2}{I_{yy}}, \tilde{\omega}_{2,1} = \frac{10C_{L\delta e}pSU}{m},$$

$$\tilde{\omega}_{1,4}(C_{La}) = \frac{5\left(c^2C_{mq}m - 2C_{La}I_{yy}\right)pSU}{mI_{yy}},$$

$$\tilde{\omega}_{2,2} = \frac{cpS\left(-10C_{m\delta e}m + 5cC_{Lq}C_{m\delta e}pS - 2.5cC_{L\delta e}C_{mq}pS\right)U^2}{mI_{yy}},$$

$$\tilde{\omega}_{2,3}(C_{La}) = \frac{5c\left(C_{L\delta e}C_{ma} - C_{La}C_{m\delta e}\right)p^2S^2U^3}{mI_{yy}},$$

$$\tilde{\omega}_{2,4}(C_{La}) = \frac{cpS\left(10C_{ma}m - 5cC_{Lq}C_{ma}pS + 2.5cC_{La}C_{mq}pS\right)U^2}{mI_{yy}},$$

$$\tilde{\omega}_{3,1} = \frac{cpS\left(-10C_{m\delta e}m + 5cC_{Lq}C_{m\delta e}pS - 2.5cC_{L\delta e}C_{mq}pS\right)U^2}{mI_{yy}}.$$

Theorem 2: The problem of robust output step command tracking for the short period model of the aircraft is always solvable.

Proof: According to definitions (24) and (25) the matrix

$A^{**}(C_{La})$ is constructed. The elements of $A^{**}(C_{La})$ are continuous functions of C_{La} for every $C_{La} \in [C_{La_{\min}}, C_{La_{\max}}]$. Choose the following 5×4 row submatrix of $A^{**}(C_{La})T$ (where $T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$)

$$A^*(C_{La}) = \frac{pSU}{mI_{yy}} \begin{bmatrix} mI_{yy}/(pSU) & 0 & 0 & 0 \\ \phi_1(C_{La}) & 20mI_{yy}/(pSU) & 0 & 0 \\ \phi_2(C_{La}) & \phi_3(C_{La}) & \phi_4 & 0 \\ \phi_5(C_{La}) & \phi_6(C_{La}) & \phi_7 & \phi_8 \\ 0 & \phi_9 & \phi_{10} & \phi_{11} \end{bmatrix} \quad (26)$$

where $\phi_1(C_{La}) = \frac{20mI_{yy}}{pSU} + 0.25\left(2I_{yy}C_{La} - mc^2C_{mq}\right)$,

$\phi_2(C_{La}) = c\left(-5cC_{mq}m - 0.5C_{ma}mU + 0.25cC_{Lq}C_{ma}pSU\right) + C_{La}\left(10I_{yy} - 0.125c^2C_{mq}pSU\right)$,

$\phi_3(C_{La}) = 5\left(2I_{yy}C_{La} - mc^2C_{mq}\right)$, $\phi_4 = 10C_{L\delta e}I_{yy}$, $\phi_8 = 10C_{L\delta e}I_{yy}$,

$\phi_5(C_{La}) = 2.5c\left(-4C_{ma}m + \left(2cC_{Lq}C_{ma} - cC_{La}C_{mq}\right)pS\right)U$,

$\phi_6(C_{La}) = -10C_{L\delta e}I_{yy} + 2.5c\left(-4mC_{ma} + \left(2cC_{Lq}C_{ma} - cC_{La}C_{mq}\right)pS\right)U$,

$\phi_7 = 5\left(2C_{L\delta e}I_{yy} + c\left(-2mC_{m\delta e} + \left(cC_{Lq}C_{m\delta e} - 0.5cC_{L\delta e}C_{mq}\right)pS\right)U\right)$,

$\phi_9 = -2.5c\left(-4mC_{m\delta e} + \left(2cC_{Lq}C_{m\delta e} - cC_{L\delta e}C_{mq}\right)pS\right)U$,

$\phi_{10} = 2.5c\left(-4mC_{m\delta e} + \left(2cC_{Lq}C_{m\delta e} - cC_{L\delta e}C_{mq}\right)pS\right)U$,

$\phi_{11} = 2.5c\left(-4mC_{m\delta e} + \left(2cC_{Lq}C_{m\delta e} - cC_{L\delta e}C_{mq}\right)pS\right)U$. The matrix $A^*(C_{La})$ is positive antisymetric, since it can be constructed using the three positive up augmentations: $\Phi_1(C_{La}) \to \Phi_2(C_{La}) \to \Phi_3(C_{La}) \to A^*(C_{La})$, where $\Phi_1(C_{La}) = \frac{pSU}{mI_{yy}}\begin{bmatrix} \phi_8 \\ \phi_{11} \end{bmatrix}$,

$$\Phi_2(C_{La}) = \frac{pSU}{mI_{yy}}\begin{bmatrix} \phi_4 & 0 \\ \phi_7 & \phi_8 \\ \phi_{10} & \phi_{11} \end{bmatrix}, \Phi_3(C_{La}) = \frac{pSU}{mI_{yy}}\begin{bmatrix} \frac{20mI_{yy}}{pSU} & 0 & 0 \\ \phi_3(C_{La}) & \phi_4 & 0 \\ \phi_6(C_{La}) & \phi_7 & \phi_8 \\ \phi_9 & \phi_{10} & \phi_{11} \end{bmatrix}.$$

The vector $\bar{c}(C_{La}) = \Phi_1(C_{La})$ is a Hurwitz invariant core since the associate polynomial of $\bar{c}(C_{La})$ ($\left[\bar{c}(C_{La})\right]^T [s \quad 1]^T$) is positive Hurwitz invariant. Hence, condition (ii) of Theorem 1 is also satisfied. ■

IV. Computation of the Controller Parameters

The controller $f = [f_1 \quad f_2 \quad f_3 \quad f_4]$ will be computed using the following algorithm [7]-[9]:

Step 1 (Construction of the augmentation matrices): The core of $A^*(C_{La})$ is $\bar{c}(C_{La}) = \Phi_1(C_{La})$. From $\Phi_1(C_{La})$ using an up positive augmentation the matrix $\Phi_2(C_{La})$ is constructed. Using another up positive augmentation the matrix $\Phi_3(C_{La})$ is constructed. Using another up augmentation the matrix $\Phi_4(C_{La}) = A^*(C_{La})$ is constructed. Let $\tau_1 = 1$.

Step 2 (Determination of the region of $\varepsilon_1 > 0$ for which $\Phi_2(C_{La})[\varepsilon_1 \quad 1]^T$ is positive Hurwitz invariant): According to the form of the associated polynomial it is observed that robust stability is guaranteed $\forall \varepsilon_1 > 0$. Hence, the region of ε_1 is $(0,+\infty)$. Let $\tau_2 = [\varepsilon_1 \quad 1]$ and choose $\varepsilon_1 = 0.9$.

Step 3 (Determination of the region of $\varepsilon_2 > 0$ such that $\Phi_3(C_{La})[\varepsilon_2 \quad \tau_2]^T$ is positive Hurwitz invariant): The respective associated polynomial is robustly stable $\forall \varepsilon_2 \in (0, 0.124639) \cup (1.15601, 1.9)$. Let $\tau_3 = [\varepsilon_2 \quad \varepsilon_1 \quad 1]$ and choose $\varepsilon_2 = 0.12$.

Step 4 (Determination of the region of $\varepsilon_3 > 0$ such that $\Phi_4(C_{La})[\varepsilon_3 \quad \tau_2]^T$ is positive Hurwitz invariant): The respective associate polynomial is robustly stable $\forall \varepsilon_3 \in (0, 0.000446414)$. Choose $\varepsilon_3 = 0.0004$.

Step 5 (Derivation of the gain vector): The gain vector $\tilde{f}$ that robustly stabilizes the associated polynomial of $A^{**}(C_{La})T\tilde{f}^T$ is $\tilde{f} = [\varepsilon_3 \quad \varepsilon_2 \quad \varepsilon_1 \quad 1 \quad 0]$ and consequently the gain vector that robustly stabilizes the associate polynomial of $A^{**}(C_{La})$ is

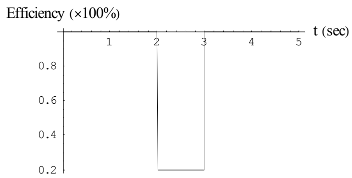
$$[1 \quad f] = T\tilde{f}^T = \left[1 \quad \left(\frac{1}{e_3} + \frac{e_1}{e_3} - \frac{e_2}{e_3}\right) \quad \frac{e_1}{e_3} \quad 0 \quad -\frac{e_2}{e_3}\right].$$

According to (9) the control signal is defined as

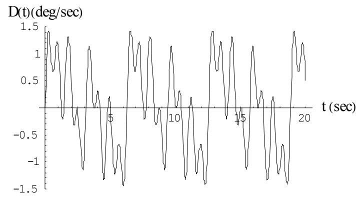
$$u(t) = \left(\frac{1}{e_3} + \frac{e_1}{e_3} - \frac{e_2}{e_3}\right)\int_0^t e(\tau)d\tau + \left[\frac{e_1}{e_3} \quad 0 \quad -\frac{e_2}{e_3}\right]x(t) \quad (27)$$

V. Simulation Results

The parameters' values, for an aircraft flying at 20.000 *ft* and at 0.9 M, are defined in Table I. Let the desired manoeuvre be a loop with a steady angle of attack of 0.05 radians. Suppose that the aircraft is flying through a windy and gusty environment as shown in Fig. 3(b) [12], [13]. Additionally, for a time period $t_e \in [2,3](\text{sec})$ the elevator actuator presents up to 80% reduced efficiency as shown in Fig. 3(a). Applying the control law (27) to the non linear aircraft model, the state responses are quite satisfactory as illustrated in Fig. 4(a) – Fig. 4(c), for the minimum, maximum and nominal values of the uncertain parameter C_{La}. The angular velocity of the actuator is shown in Fig. 4(d), while the tracking error is shown in Fig. 4(e).

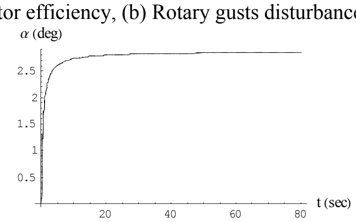


(a)

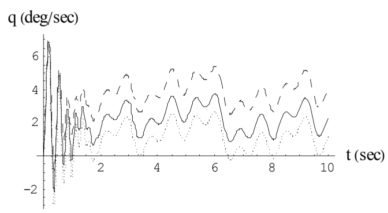


(b)

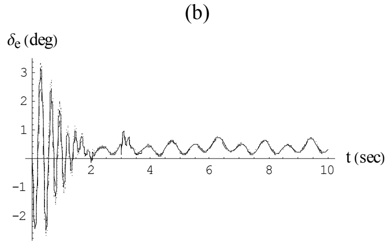
Fig. 3. (a) Elevator efficiency, (b) Rotary gusts disturbance



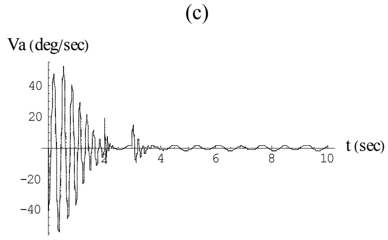
(a)



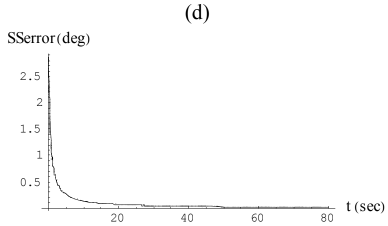
(b)



(c)



(d)



(e)

Fig. 4. Approximate command following for the angle of attack (a), responses of pitch rate (b), elevator deflection (c), elevator angular velocity (d), tracking error (e). *(Dotted line, Plain line, Dashed line is for the minimum, the nominal, and the maximum value of* C_{La} *, respectively)*

VI. Conclusion

In this paper the problem of robust output command tracking for polynomials reference signals has been solved for the first time for the case of linear systems with nonlinear uncertain structure. The problem has been solved using am independent of the uncertainties controller that includes a static state feedback, a unity output feedback and integral action. Sufficient conditions for the problem to have a solution are established. The above results have been applied to control the short period longitudinal motion of an aircraft Simulation results to the nonlinear aircraft model, have illustrated the effectiveness of the controller over a wide range of system uncertainty, wind disturbances and actuator failure.

Before closing it is important to mention that based on the results of the present paper further results regarding more general reference signals are currently under investigation.

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References

[1] C. T. Chen, *Linear System Theory and Design.* Holt, Rinehart and Winston, New York, 1984.

[2] I. M. Horowitz, *Synthesis Feedback Systems.* New York Academic, 1963.

[3] R. C. Dorf, and R. H. Bishop, *Modern Control Systems,* 9th ed., prentice Hall, 2001.

[4] M. J. Corless, G. Leitmann, and E. P. Ryan, "Tracking in the presence of bounded uncertainties," presented at the 4th Int. Conf. Control Theory, Cambridge, U.K., Sept, 1984.

[5] K. Takaba, "Robust preview tracking control for polytopic uncertain systems", in *Proc. 37th IEEE Conf. Decision Contr.,* Tampa, FL, 1998, pp. 1765-1770

[6] I. Yaesh and U. Shaked, "Two degrees of freedom H_∞ optimization of multivariale feedback systems", *IEEE Transactions on Automatic Control*, vol. 36, pp. 1272-1276, 1991.

[7] K Wei, B. and R. Barmish, " Making a polynomial Hurwitz invariant by choice of feedback gain", *Int. J. Contr.* Vol 50, pp 1025-1038,1989.

[8] F. N. Koumboulis and M. G. Skarpetis, "Input -Output decoupling for linear systems with non-linear uncertain structure", *J. of the Franklin Institute*, vol. 333(B), pp. 593-624, 1996.

[9] F. N. Koumboulis and M. G. Skarpetis, "Robust Triangular Decoupling with Application to 4WS Cars", *IEEE Transactions on Automatic Control*, vol. 45, pp. 344-352, 2000.

[10] A. W. Lee and J. K. Hendrick, "Application of Approximate I/O Linearization to Aircraft Flight Control," *Journal of Dynamic Systems, Measurements and Control*, Vol. 116, September 1994, pp. 429–436.

[11] M. G. Skarpetis and F. N. Koumboulis "Decoupling of the longitudinal modes of advanced aircraft ", *AIAA J. of Guidance, Control, and Dynamics*., vol. 19, pp. 1184-1186, 1996.

[12] M. G. Skarpetis, "Disturbance rejection for an aircraft flying in atmospheric disturbances", *Dynamics and Control*, vol. 5, pp. 389-400, 1995.

[13] R. C. Nelson, *Flight Stability and Automatic Control,* McGRAW-HILL, International Editions, 1990.