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Unknown Input Observers for a class of distributed parameter systems

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Abstract— **The objective of this note is to extend the results of disturbance decoupling observers to a class of infinite dimensional systems. Using the fact that the operator describing the disturbance distribution is known, but with the disturbance being an unknown signal, an Unknown Input Observer is proposed for a class of infinite dimensional systems with bounded observation operators. The solvability of the problem hinges on the solution to a corresponding operator equation and which subsequently yields exponential convergence of the state observation error to zero. A numerical study of a 1-D parabolic system is included to provide insight into the implementation of UIO for infinite dimensional systems.**

I. INTRODUCTION

In this note we extend the finite dimensional results of Unknown Input Observer (UIO) to a class of infinite dimensional systems. In summary, the UIO attempts to decouple the disturbances from the observer and in the process guarantees exponential convergence of the state observation error to zero. The necessary and sufficient conditions for existence of an UIO which assumes the distribution matrix of the disturbance is known rely on solvability of a given matrix relationship.

The proposed extension of these results to the infinite dimensional cases are quite relevant and useful since it is often the case that in processes described by partial differential equations, disturbances entering the dynamics do so only in a portion of the spatial domain. Very often one has complete knowledge on the spatial distribution of the disturbance and/or unmodelled dynamics and uncertainties, but has no knowledge on the temporal variation (strength) of the disturbing signal.

The goal undertaken here is to offer an extension of the UIO results to a class of infinite dimensional systems and introduce the requisite conditions imposed on the spatial distributions of the input, output and disturbance operators.

In order to familiarize the reader with the conditions pertaining to the well-posedness of the UIO, we summarize in the next section the corresponding results for the finite dimensional case as taken from [2]. The proposed UIO for a class of infinite dimensional systems and the equivalent conditions for the existence of this UIO are given in Section III and results of a numerical study on a 1-D diffusion system with disturbance entering in a portion of the spatial domain are given in Section IV. Conclusions with future work follow in Section V.

II. FINITE DIMENSIONAL RESULTS

We summarize a scheme based on unknown input observers (UIOs) for finite dimensional systems as presented in [2]. The system under consideration is generally described by the state space model as

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t)+Ew(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $y(t)\in\mathbb{R}^m$ is the output vector, $u(t)\in\mathbb{R}^r$ is the known input vector and $w(t)\in\mathbb{R}^q$ is the unknown input (or disturbance) vector. The input distribution matrix E is assumed to be known and have full column rank [2]. Following [1], we have the following definition of a disturbance decoupling observer.

Definition 2.1: An observer for the system (1) is defined as an unknown input observer if its state estimation error approaches zero asymptotically regardless of the presence of the unknown input $w(t)$ in the system.

The full-order observer in this case is given by

$$\begin{aligned} \dot{z}(t) &= Fz(t)+TBu(t)+Ly(t) \\ \widehat{x}(t) &= z(t)+Hy(t) \end{aligned} \tag{2}$$

where $\widehat{x}\in\mathbb{R}^n$ is the *estimated state vector* and $z\in\mathbb{R}^n$ is the *state of the observer*. The matrices F, T, L, H are design matrices required for decoupling the unknown input. Using (1), (2) above, the state estimation error $e=x-\widehat{x}$ satisfies

$$\begin{aligned} \dot{e} &= (A-HCA-L_1C)e-[F-(A-HCA-L_1C)]\widehat{x} \\ &\quad +(FH-L_2)y+[I-HC-T]Bu+(I-HC)Ew \end{aligned} \tag{3}$$

If the following relations hold

$$\begin{aligned} (I-HC)E &= 0, \\ T &= I-HC, \\ F &= A-HCA-L_1C = TA-L_1C, \\ L_2 &= FH, \end{aligned} \tag{4}$$

with $L=L_1+L_2$, then the state estimation error satisfies

$$\dot{e}=Fe. \tag{5}$$

Using the above conditions, we present the following lemma from [1].

Lemma 2.1 ((Lemma 3.1, [2], [1])): The equation $(HC-I)E=0$ is solvable iff

$$\mathrm{rank}(CE)=\mathrm{rank}(E), \tag{6}$$

and a special solution is

$$H^{*}=E[(CE)^{T}CE]^{-1}(CE)^{T}. \tag{7}$$

Conditions for the above observer to be a UIO are summarized below in a theorem due to [1].

Theorem 2.1 ((Theorem 3.1, [1])): Necessary and sufficient conditions for (2) above to be a UIO for the system (1) are

1) $\mathrm{rank}(CE)=\mathrm{rank}(E)$,
2) $(C,A-H^{*}CA)=(C,TA)$ is a detectable pair, where

$$TA=A-E[(CE)^{T}CE]^{-1}(CE)^{T}CA.$$

If the conditions above hold, the state estimation error $e(t)$, which satisfies $\dot{e}=Fe$, converges to zero asymptotically since F is Hurwitz.

III. EXTENSION TO INFINITE DIMENSIONAL SYSTEMS

We now consider the following class of systems

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)+Ew(t), \quad \text{in } \mathscr{X}\\ y(t)&=Cx(t)\end{aligned} \tag{8}$$

where A is the infinitesimal generator of the strongly continuous semigroup $\mathscr{T}(t)$ on a Hilbert space $\mathscr{X}$, B is a bounded linear operator from a Hilbert space U to $\mathscr{X}$ or $B\in\mathscr{L}(U,\mathscr{X})$, $E\in\mathscr{L}(W,\mathscr{X})$, $C\in\mathscr{L}(\mathscr{X},Y)$, where W and Y are appropriate Euclidean spaces for the disturbance and observation spaces ($Y=\mathbb{R}^{p}, W=\mathbb{R}^{q}$).

The proposed UIO for the above system is given by

$$\begin{aligned}\dot{z}(t)&=Fz(t)+TBu(t)+Ly(t), \quad \text{in } \mathscr{X}\\ \widehat{x}(t)&=z(t)+Hy(t)\end{aligned} \tag{9}$$

with $H\in\mathscr{L}(Y,\mathscr{X})$, $T\in\mathscr{L}(\mathscr{X},\mathscr{X})$ and $L=L_1+L_2\in\mathscr{L}(Y,\mathscr{X})$. Similar to the arguments given in the finite-dimensional case, one can show that the state estimation error $e=x-\widehat{x}$ satisfies

$$\begin{aligned}\dot{e}&=(A-HCA-L_1C)e-[F-(A-HCA-L_1C)]\widehat{x}\\ &+(FH-L2)y+[I-HC-T]Bu+(I-HC)Ew\end{aligned} \tag{10}$$

With $HC:\mathscr{X}\to\mathscr{X}$, then one may define the operator $T:\mathscr{X}\to\mathscr{X}$ by

$$T=(I-HC). \tag{11}$$

One then requires that an operator H can be found so that

$$TE=0 \quad \text{in } \mathscr{X}. \tag{12}$$

Additionally, the following relations should also hold

$$\begin{aligned}F&=A-HCA-L_1C=TA-L_1C,\\ L_2&=FH,\end{aligned} \tag{13}$$

with $L=L_1+L_2$. With all the conditions above, then the state estimation error satisfies

$$\dot{e}=Fe. \tag{14}$$

For the observer to be an UIO for the system (8), we have to show that F is the infinitesimal generator of an exponentially stable C_0-semigroup $S_F(t)$ on the Hilbert space $\mathscr{X}$. This means that there should exist positive constants M and α such that

$$\|S_F(t)x\|_{\mathscr{X}}\leq Me^{-\alpha t}\|x_0\|_{\mathscr{X}} \tag{15}$$

The corresponding conditions for the above to be UIO are given in the following theorem.

Theorem 3.1: Necessary and sufficient conditions for (9) to be a UIO for the system (8) are

1) $TE=(I-HC)E=0$ in $\mathscr{X}$ be solvable for $\mathscr{H}$,
2) the pair (C,TA) be exponentially detectable.

The first condition can be satisfied if we require that $\mathscr{N}(CE)=\{0\}$, with $p\geq q$, where $\mathscr{N}$ refers to the null space of an operator. This condition is equivalent to the operator CE being one-to-one, which implies that $(CE)^{*}(CE)$ is invertible [5], [8], [9]. The left-inverse operator $(CE)^{*}(CE))^{-1}(CE)^{*}$ is well-defined in this case, and it is called the pseudoinverse $(CE)^{\dagger}$.

The solution to the equation $TE=(I-HC)E=0$ can be given in the same fashion as the finite-dimensional case:

$$H=E[(CE)^{*}CE]^{-1}(CE)^{*}=E(CE)^{\dagger}. \tag{16}$$

The second condition means that there exists an operator L_1 such that $TA-L_1C$ is exponentially stable (in the sense defined in equation (15), see also [3]). Once these conditions are satisfied, one can easily show that in the state error equation, the operator $F=A-HCA-L_1C=TA-L_1C$ is exponentially stable, which in turn implies that the above observer is an UIO for the distributed parameter system defined in equation (8).

Remark 3.1: Both conditions of Theorem 3.1 can be used for sensor selection as it is unlikely that a chosen sensor location would yield an associated observation operator C that can solve $HCE=E$, and ensure that the resulting pair (C,TA) be exponentially detectable. Thus, for a given (known) operator representation E of a disturbance distribution, one seeks an observation operator C, via the sensor locations, that would satisfy the above two conditions. This solves the feasibility problem. If more than one such operator C exists, then one may choose the sensor locations that minimize the effects of the disturbance on the estimated state $\widehat{x}$ via the associated transfer function. Under additional conditions, this may be linked to the solution of an associated operator Lyapunov equation (Gramian). An alternative to the above which addresses issues related to implementation of computational schemes, would be to consider the minimization of the total energy of the error equation (14). The total energy is then defined by

$$\int_0^{\infty}E(t)\,dt=\int_0^{\infty}\|S_F(t)x\|_{\mathscr{X}}^{2}\,dt. \tag{17}$$

Such a criterion was utilized, under different design criteria, in [7] for the optimal damping design in structural systems. However, the basic idea is the same and hence one may utilize and adapt these results for the problem under consideration. In summary, one solves an associated operator Lyapunov equation for $\Pi \in \mathscr{L}(\mathrm{dom}(F), \mathrm{dom}(F^*))$ given by

$$(F^*\Pi + \Pi F + I)x = 0, \quad \text{for all } x \in \mathrm{dom}(F). \tag{18}$$

By considering a finite set of candidate sensor locations Θ, where for each element $\theta \in \Theta$ both conditions of Theorem 3.1 are satisfied

$$\begin{cases} HC(\theta)E = E & \text{and} \\ (C(\theta), T(\theta)A) & \text{approx. detectable} \end{cases} \qquad \forall \theta \in \Theta,$$

i.e. one defines

$$\Theta = \left\{ \theta : \begin{array}{l} HC(\theta)E = E, \quad \text{and} \\ (C(\theta), T(\theta)A) \text{ approx. detectable} \end{array} \right\},$$

one then considers the θ-parameterized Lyapunov solution $\Pi(\theta)$ and defines the corresponding stability criterion to be optimized. The above optimization is realized through the use of Datko's lemma [10], since the operator $F(\theta)$, parameterized by the sensor location, is exponentially stable on $\mathscr{X}$ and hence the minimum of the total energy is given in terms of the solution to a Lyapunov equation. Thus one considers

$$\int_0^\infty \|S_{F(\theta)}(t)x\|^2_{\mathscr{X}}\, dt = \langle \Pi(\theta)x_0, x_0 \rangle_{\mathscr{X}} \tag{19}$$

where $\Pi(\theta)$ satisfies the Lyapunov equation

$$\Big(F^*(\theta)\Pi(\theta) + \Pi(\theta)F(\theta) + I\Big)x = 0, \ \text{ for all } x \in \mathrm{dom}(F(\theta)),$$

and therefore the optimal sensor location is given via

$$\theta^{opt} = \arg\min_{\theta\in\Theta} \sup_{\|x_0\|=1} \langle \Pi(\theta)x_0, x_0 \rangle_{\mathscr{X}}.$$

To remove the dependence on the initial conditions in (19), one may consider, in the case of a compact $\Pi(\theta)$ [9], [6], the trace of the operator, and thus the above optimization measure becomes

$$\theta^{opt} = \arg\min_{\theta\in\Theta} \mathrm{trace}\,\Pi(\theta). \tag{20}$$

We now present the results of a numerical study for a 1D parabolic PDE and which show the effectiveness of the proposed UIO to handle and minimize the effects of the disturbance distribution on the state estimate.

IV. EXAMPLE AND NUMERICAL RESULTS

As an example, we consider the 1D diffusion equation given by

$$\frac{\partial}{\partial t}x(t,\xi) = 0.06\frac{\partial^2}{\partial \xi^2}x(t,\xi) + b(\xi)u(t) + e(\xi)w(t), \ 0 < \xi < \ell,$$

$$y(t) = \begin{bmatrix} \int_0^\ell c_1(\xi)x(t,\xi)\,d\xi \\ \int_0^\ell c_2(\xi)x(t,\xi)\,d\xi \end{bmatrix} = \begin{bmatrix} C_1x(t) \\ C_2x(t) \end{bmatrix},$$

with Dirichlet boundary conditions, $x(t,0) = 0 = x(t,\ell)$, and $x(0,\xi) = 5\sin(2\pi x/\ell)$. The control, disturbance and observation spatial distributions are given by

$$b(\xi) = \begin{cases} 1 & \text{if } \xi \in [0.765\ell, 0.815\ell] \\ 0 & \text{otherwise} \end{cases},$$

$$e(\xi) = \begin{cases} 1 & \text{if } \xi \in [0.50\ell, 0.70\ell] \\ 0 & \text{otherwise} \end{cases},$$

$$c_1(\xi) = \begin{cases} 1 & \text{if } \xi \in [0.573\ell, 0.573\ell] \\ 0 & \text{otherwise} \end{cases},$$

$$c_2(\xi) = \begin{cases} 1 & \text{if } \xi \in [0.881\ell, 0.901\ell] \\ 0 & \text{otherwise} \end{cases}.$$

The corresponding operators are given by

$$\langle A\phi, \psi \rangle = \int_0^\ell 0.06\phi''(\xi)\psi(\xi)\,d\xi, \text{ with } \mathscr{D}(A) = H_0^1(0,\ell),$$

$$\langle Bu, \phi \rangle = \int_0^\ell b(\xi)\phi(\xi)\,d\xi\, u = \int_{0.765\ell}^{0.815\ell} \phi(\xi)\,d\xi\, u$$

$$\langle Ew, \phi \rangle = \int_{0.50\ell}^{0.70\ell} \phi(\xi)\,d\xi\, w,$$

$$C_1\phi = \langle c_1, \phi \rangle = \int_0^\ell c_1(\xi)\phi(\xi)\,d\xi = \int_{0.573\ell}^{0.593\ell} \phi(\xi)\,d\xi,$$

$$C_2\phi = \int_{0.881\ell}^{0.901\ell} \phi(\xi)\,d\xi.$$

In order to simulate the above system, a Galerkin-based finite dimensional approximation is implemented by assuming

$$x(t,\xi) \approx \sum_{i=1}^N \alpha_i(t)\phi_i(\xi),$$

where $\phi_i(\xi)$ are the linear (spline) elements satisfying the boundary conditions, [12], [11]. That is, let $\{\phi_i^n\}$ be the standard B-splines on the spatial interval $[0,\ell]$ with respect to the uniform mesh $\{0, \frac{\ell}{n}, \frac{2\ell}{n}, \dots, \ell\}$, that is for $i = 0, 1, 2, \dots, n$

$$\phi_i^n(\xi) = \begin{cases} 1 - \left|\frac{n\xi}{\ell} - i\right| & \xi \in \left[\frac{(i-1)\ell}{n}, \frac{(i+1)\ell}{n}\right], \\ 0 & \text{otherwise, on } [0,\ell]. \end{cases}$$

The resulting finite dimensional representation of the 1D diffusion equation with $n = 160$ is given by

$$\begin{aligned} M^n\dot{\alpha}^n(t) &= K^n\alpha^n(t) + B^nu(t) + E^nw(t) \\ y(t) &= C\alpha^n(t), \end{aligned}$$

where the finite dimensional representation of the state is $\alpha^n(t) = [\ \alpha_1^n(t) \quad \alpha_2^n(t) \quad \dots \quad \alpha_n^n(t)\ ]$, and the *mass matrix* M^n and *stiffness matrix* K^n are given by

$$\left.\begin{aligned} [M^n]_{ij} &= \int_0^\ell \phi_i^n(\xi)\phi_j^n(\xi)\,d\xi, \\ [K^n]_{ij} &= -0.06\int_0^\ell (\phi_i^n)'(\xi)(\phi_j^n)'(\xi)\,d\xi, \end{aligned}\right. \qquad i,j = 1,\dots,n.$$

The representation for the control input and output matrices is given by

$$B^n = \begin{bmatrix} \int_{0.765\ell}^{0.815\ell} \phi_1^n(\xi)\,d\xi \\ \vdots \\ \int_{0.765\ell}^{0.815\ell} \phi_n^n(\xi)\,d\xi \end{bmatrix}$$

and

$$C^n = \begin{bmatrix} \int_{0.573\ell}^{0.593\ell} \phi_1^n(\xi)\,d\xi & \dots & \int_{0.573\ell}^{0.593\ell} \phi_n^n(\xi)\,d\xi \\ \int_{0.881\ell}^{0.901\ell} \phi_1^n(\xi)\,d\xi & \dots & \int_{0.881\ell}^{0.901\ell} \phi_n^n(\xi)\,d\xi \end{bmatrix},$$

respectively. The spatial distribution of the disturbance function is similarly given by

$$E^n = \begin{bmatrix} \int_{0.50\ell}^{0.70\ell} \phi_1^n(\xi)\,d\xi \\ \vdots \\ \int_{0.50\ell}^{0.70\ell} \phi_n^n(\xi)\,d\xi \end{bmatrix}.$$

The initial condition for the estimator state was taken as $\widehat{\alpha}^n(0) = -1.5(\alpha(0) - H^n C^n \alpha(0))$. The vector representation of the initial condition then becomes $\alpha^n(0) = 5M^{-1}\overline{\alpha}^n(0)$ with

$$[\overline{\alpha}^n(0)]_i = \int_0^\ell \sin(2\pi\xi/ell)\phi_i^n(\xi)\,d\xi,\ i = 1,2,\dots,n.$$

A sinusoidal signal is chosen as the disturbance, $w(t) = 2 + 0.01\sin(2\pi t)$. The resulting finite dimensional system of ordinary differential equations (24) was simulated for $t = 2$ seconds using the stiff ODE solver from the Matlab® ODE library, routine `ode23s` based on a fourth order Runge-Kutta scheme.

Following the proposed UIO in (9), we simulated the finite representation of the plant (8)

$$\begin{aligned} \dot{\alpha}^n(t) &= \Big((M^n)^{-1}K^n\Big)\alpha^n(t) \\ &\quad + \Big((M^n)^{-1}B^n\Big)u(t) + \Big((M^n)^{-1}E^n\Big)w(t) \\ &= A^n\alpha^n(t) + \widetilde{B}^n u(t) + \widetilde{E}^n w(t) \end{aligned}$$

$$y(t) = C^n\alpha^n(t),$$

and the finite dimensional representation of the UIO

$$\dot{\zeta}^n(t) = F^n\zeta^n(t) + T^n\widetilde{B}^n u(t) + L^n\Big(C^n\alpha^n(t)\Big)$$

$$\widehat{\alpha}^n(t) = \zeta^n(t) + H\Big(C^n\alpha^n(t)\Big).$$

In addition to the UIO, we also simulated an H_∞ filter given by

$$\dot{\widehat{\alpha}}_\infty^n(t) = \Big(A^n - L_\infty^n C^n\Big)\widehat{\alpha}_\infty^n(t) + L_\infty^n y(t) + \widetilde{B}^n u(t),$$

case	error norm
UIO	3.5153
H_∞ filter	3.9818

TABLE I

$L_2\big([0,2];\mathscr{X}\big)$ NORM OF $e(t,\xi)$ FOR UIO AND H_∞ FILTER.

where the filter gain L_∞^n is given via the solution to the associated filter Algebraic Riccati Equation (ARE)

$$\begin{aligned} &\Sigma_\infty^n(A^n)^T + A^n\Sigma_\infty^n + \widetilde{E}^n(\widetilde{E}^n)^T \\ &\qquad -\Sigma_\infty^n\Big((C^n)^T C^n - \frac{1}{\gamma^2}I\Big)\Sigma_\infty^n = 0, \end{aligned} \tag{21}$$

by $L_\infty^n = \Sigma_\infty^n(C^n)^T$. The ARE above was solved for the value of $\gamma = 6$ as a lower value of γ did not provide a positive definite solution to (21).

In Figure 1 we plot the $L_2(0,\ell)$ norm of the state observation error $e(t,\xi) = x(t,\xi) - \widehat{x}(t,\xi)$ using the proposed UIO and the H_∞ filter above. One can observe the convergence of the error norms to zero, with the UIO error norm converging to zero faster than the H_∞ error norm.

The distribution of the three functions $x(t,\xi)$, $\widehat{x}(t,\xi)$, $e(t,\xi)$ are plotted in Figure 2 against the spatial variable ξ at three time instances, $t = 0$, $t = 1$ and $t = 2$. It is observed that the estimation error converges to zero pointwise in space. The evolution of the spatial distribution of the observation error $e(t,\xi)$ for the UIO design is depicted in Figure 3, where is it once again observed that the error asymptotically converges to zero.

Finally, the spatial distribution of the state, the state estimate and the estimation error are plotted again in Figure 4 at the final time $t = 2$ and compared to the same quantities against the H_∞ filter. Similar results are also summarized in Table I, where one can observe that when the spatial distribution E is not considered in the observer design, the spatial error is large around the region where the disturbance is large.

To appreciate the effects of the sensor location on the efficacy of the UIO, we also consider the optimization in (20), where we now choose

$$e(\xi) = \begin{cases} 1 & \text{if } \xi \in [0.30\ell, 0.70\ell] \\ 0 & \text{otherwise} \end{cases},$$

in the previous example. A single sensor is considered and whose position is parameterized in the spatial interval $[0,\ell]$. The candidate position were evenly distributed in the spatial domain and 100 such points were considered $\theta_i = 0.005\ell + i\ell/100$ with

$$c(\xi,\theta_i) = \begin{cases} 1 & \text{if } \xi \in [\theta_i - 0.005\ell, \theta_i + 0.005\ell] \\ 0 & \text{otherwise} \end{cases}.$$

The set of admissible locations is subsequently found by discarding the candidate locations that did not satisfy the conditions of Theorem 3.1. To provide an appreciation for

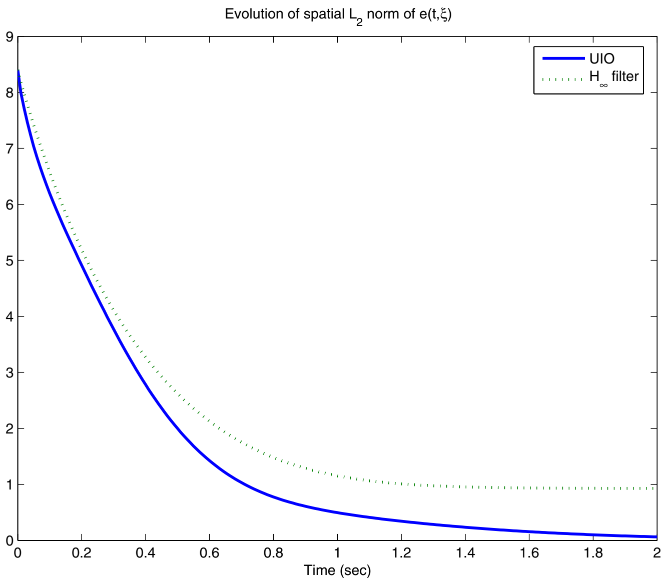

Fig. 1. Evolution of spatial L_2 error norm using UIO and H_∞ filter designs.

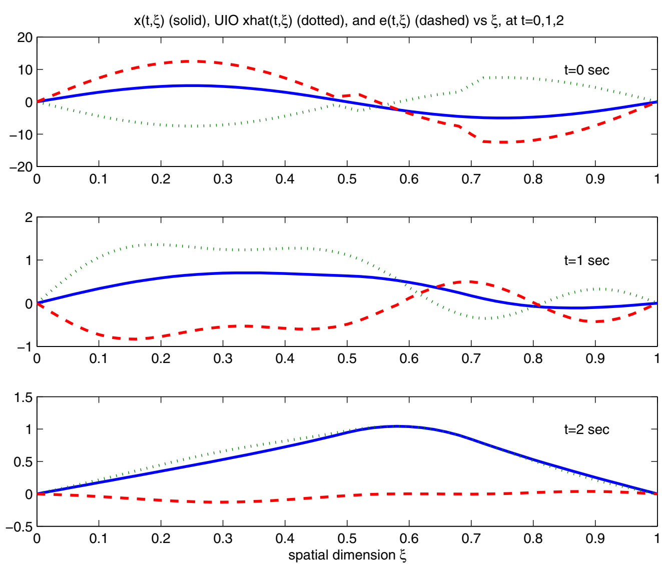

Fig. 2. Spatial distribution of $x(t,\xi)$, $\widehat{x}(t,\xi)$ and $e(t,\xi)$ at $t=0$, $t=1$ and $t=2$.

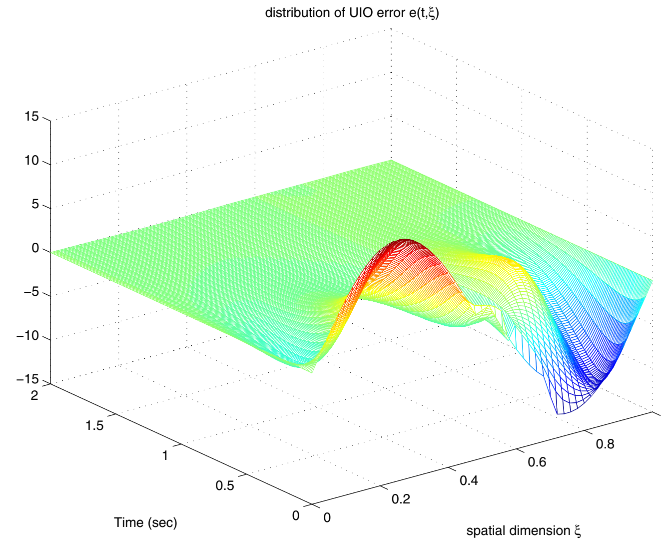

Fig. 3. Evolution of distribution $e(t,\xi)$ based on UIO design.

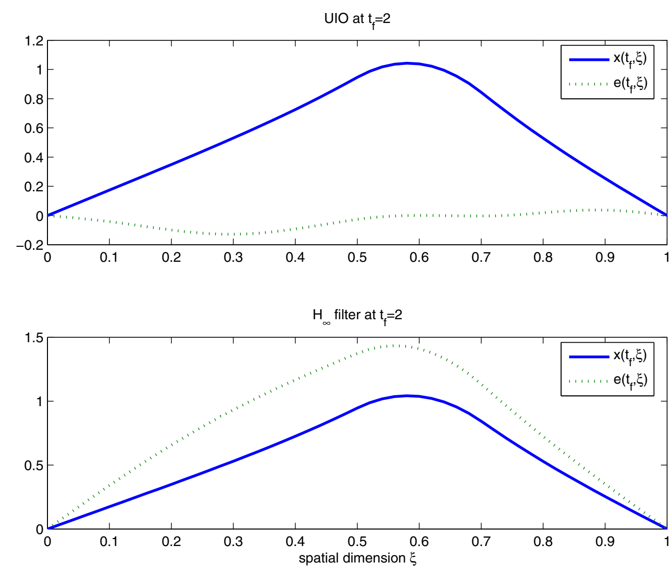

Fig. 4. Comparison of UIO and H_∞ observers; spatial distribution of $x(t,\xi)$, $\widehat{x}(t,\xi)$ and $e(t,\xi)$ at $t=2$.

the effects of the sensor location on the UIO, we consider the optimal sensor location at $\xi = 0.505\ell$ (in the sense of (20)) and a non-optimal location at $\xi = 0.315\ell$. Both UIO were simulated for the above system having the same initial conditions for the observer state $z(t)$. The evolution of the L_2 norm of the error $e(t,\xi)$ is depicted in Figure 5, where one may observe the effects of the sensor location on the UIO performance.

V. CONCLUSIONS AND FUTURE WORK

In this note we considered the extension of the finite dimensional results on a disturbance decoupling observer to a class of infinite dimensional systems. The benefits of the disturbance decoupling observer are more prominent in the case of distributed parameter systems, as the effects of the spatial distribution of disturbances has spatial effects on the state estimation error. Comparison with a standard H_∞ filter reveals the inability to deal with spatial effects of disturbances. Thus, when one has knowledge of the spatial distribution of disturbances, this must be considered in the observer design via the application of a disturbance decoupling observer. However, an H_∞ along with a sensor optimization scheme might provide a comparable, or even better performance of the filter. The problem with this implementation lies in the numerical solution to the filter ARE (21). On a theoretical level, the H_∞ observer design can provide upper bound of the norm of an associated transfer

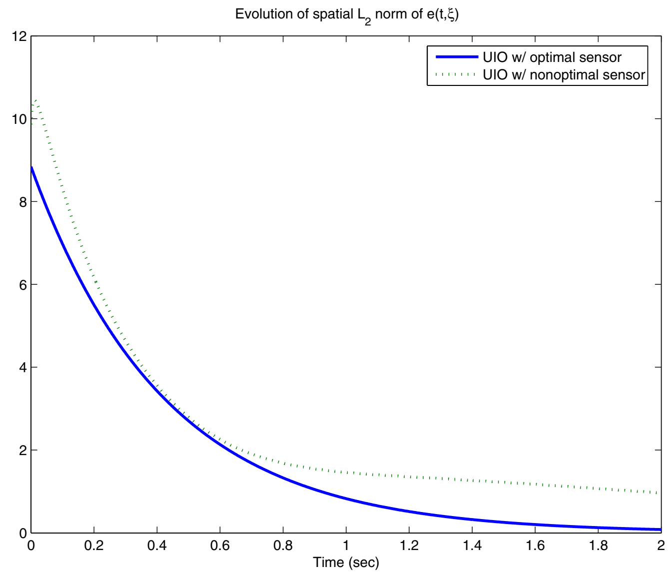

Fig. 5. Evolution of L_2 error norm of UIO with optimal and nonoptimal sensor locations.

function (from disturbance to observation error), but the numerical solution to (21) might be impossible to obtain for low values of γ. On the other hand, the conditions of Theorem 3.1 are easier to test numerically.

A direct extension is to consider a wider class of distributed parameter systems and integrate the problem of sensor placement, as summarized in Remark 3.1, such that (i) the solution to the operator equality $(I - HC)E = 0$ becomes feasible and (ii) the resulting sensor location yields a certain optimality, in the sense that it minimizes the effects of the disturbance on the state estimate via the transfer function $T_{\hat{x},w}(s)$ from the disturbance w to $\hat{x}$, using (9) and (8). Thus the energy for the error system (14) will no longer be used, but instead the effects of the disturbance (via the disturbance distribution operator E) on $\hat{x}$ will be minimized. Such an approach was used for the sensor placement in distributed parameter systems in which one had access to the disturbance distribution operator E and proceeded with a standard filter design whereby the associated H_2 norm of the transfer function from the disturbance to the state observation error e was optimized with respect to the sensor location [4].

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