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Constructive design of output feedback weakened anti-windup compensators for linear systems with additive/multiplicative perturbations

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Abstract— **A two-step algorithm for the synthesis of output feedback weakened anti-windup compensators is proposed for the case of additively perturbed systems. The first step determines a state feedback stabilizer guaranteeing a finite gain on a suitable nonlinear "mismatch" system. In the second step, a loop-shaping approach is adopted to design a linear filter which ensures overall robust stability, meanwhile minimizing the amount of anti-windup performance sacrificed. Both steps can be efficiently implemented using standard LMI software.**

I. Introduction

The ubiquitous presence of input saturation nonlinearities, and the consequent dramatic performance losses known as "windup" effects, have been the main reason for the development of a vast literature dealing with anti-windup compensation. Surveys of early schemes are presented in [1], [2], while more recent, advanced techniques, providing formal stability and performance guarantees, and often arising from optimality-based synthesis algorithms, can be found, *e.g.*, in [2], [3], [4], [5] and references therein.

As ubiquitous as saturation, uncertainty has also motivated a huge number of studies in which robustness is pursued through many different approaches. Among these, small gain results for both linear and nonlinear systems are among the most widely used tools for achieving robust stability and performance. However, as noticed in [6], there are few results dealing with robustness of anti-windup closed loop systems.

A rigorous definition of the anti-windup problem entails two requirements [7]:

$a)$ The closed loop trajectories must not be modified as long as saturation is inactive.

$b)$ Input/output stability between certain signals must be achieved.

In the presence of uncertainty on the controlled plant, a "natural" robust anti-windup problem can be directly defined by requiring $a)$ and $b)$ to hold for all the perturbations in a given family. Such an approach has been considered, *e.g.*, in [6], [8], [9], where both analysis and synthesis results are presented. However, it was shown in [10] that requirement $a)$ may impair the robust-in-the-large (*i.e.* for all the uncertainties in a prior given family) achievement of requirement $b)$. This motivated the definition of a *relaxed* anti-windup problem, in which a robust-in-the-large solution is pursued by requiring $a)$ only in nominal operating conditions, meanwhile requiring $b)$ robustly-in-the-large. Indeed, if the nominal model is an accurate description of the controlled plant in most operating conditions, but large uncertainties (possibly impairing the achievement of robust performance) may appear from time to time, achieving nominal performance and robust stability (the *weakened* approach) is enough to yield a high performance control system. For a more in-depth discussion on this topic, the reader is referred to [10].

A. Background

The state-feedback solution of the relaxed problem proposed in [10] is a modification of the $\mathcal{L}_2$ anti-windup compensator in [7], whose main drawback consists in its inherently state-feedback nature. In fact, its implementation by output feedback requires rather restrictive conditions [11].

The above drawback motivated the search for existence conditions of output feedback solutions of the relaxed anti-windup problem. It was shown in [12] that such solutions always exist at least for the class of additively perturbed systems (and consequently for input and output multiplicative perturbations, which can always be recast as additive ones).

The class of additively perturbed systems has been also considered in the recent paper [6], where the problem of robust non-weakened anti-windup compensation is addressed.

B. Paper contribution

The scope of this paper is multiple:

- To provide a ready-to-use algorithm for the design of weakened anti-windup compensators, by making explicit the approach summarized in [10] and [12].
- To highlight the two main tuning parameters (namely, K and $F(s)$) of the weakened anti-windup compensator, and to present algorithms for their selection.
- To stress that parameter K does not need to satisfy a small gain constraint, and hence a much more aggressive recovery after saturation than that achieved in [12] is possible.
- To specialize the weakened anti-windup approach [10] to a class of uncertain systems commonly encountered in applications.
- To highlight that the availability of frequency dependent bounds on the magnitude of the additive uncertainty can be exploited in the design of the anti-windup compensator in order to limit the performance loss due to off-nominal operating conditions.
- To allow a clear comparison of the results achievable, and a better understanding of the advantages offered by the weakened and the non-weakened anti-windup approaches in the presence of uncertainty, by considering the same class of perturbations as in [6].

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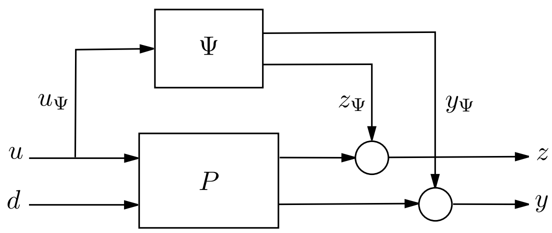

Fig. 1. The uncertain system P_Ψ.

A two-step synthesis approach involving the design of a state feedback stabilizer K and a filter $F(s)$ was suggested in both [10] and [12], but detailed design algorithms were not presented in those papers. The algorithm proposed in Sec. V can be used to design the filter $F(s)$ for both compensators in [10] and [12], whereas the design algorithm proposed to determine K in Sec. IV can only be applied to the compensator in [12]. In fact, for additively perturbed plants the proposed design of K is to be performed on a nonlinear (but not uncertain) system, and powerful tools are available for this. On the other hand, K must solve a robust stabilization problem for a nonlinear uncertain system when more general uncertainty structures (as in [10]) are considered.

C. Notation

For a given convex set $\mathcal{U} \subset \mathbb{R}^p$ and a vector $u \in \mathbb{R}^p$, let $\mathrm{dist}_{\mathcal{U}}(u) \triangleq \inf_{w \in \mathcal{U}} |u - w|$, where $|\cdot|$ denotes the Euclidean norm. The stacking $[x' \; y']'$ of two vectors x and y is denoted by (x, y). The $\mathcal{L}_2$ norm of a signal $w(\cdot)$ is defined as $\|w\|_2 \triangleq \sqrt{\int_0^\infty |w(t)|^2 dt}$, and $w \in \mathcal{L}_2$ if $\|w\|_2 < \infty$. A system Σ with input u, state x and output y is said to have *finite ($\mathcal{L}_2$ induced) gain* $\gamma_{y,u}^{(\Sigma)} \in \mathbb{R}^+$ from u to y if, for any initial condition x_0 and any input $u(\cdot)$, it holds:

$$\|y(\cdot; x_0, u)\|_2 \leq \gamma_{y,u}^{(\Sigma)} \|u\|_2 + \beta(|x_0|),$$

where $\beta(\cdot)$ is a nondecreasing, nonnegative function, and $y(t; x_0, u)$ is the output response of the system at time t with initial condition x_0 and input $u(\cdot)$. If Σ is linear time-invariant (LTI) and asymptotically stable with transfer matrix $W(s)$, its $\mathcal{L}_2$ gain equals $\|W(s)\|_\infty \triangleq \sup_{\omega \in \mathbb{R}} \bar{\sigma}(W(j\omega))$, where $\bar{\sigma}(\cdot)$ denotes the maximum singular value of the argument. The trivial system (whose output is identically null for any input) is denoted by 0, and has zero gain.

The decoupled saturation function $\sigma : \mathbb{R}^p \to \mathbb{R}^p$ (such that $y = \sigma(u)$ means $y_i = \mathrm{sign}(u_i) \min\{|u_i|, u_{i,\mathrm{sat}}\}$ for all $i = 1, \ldots, p$) is considered in this paper. The extension to more general classes of saturation functions (as that considered in [7]) is quite straightforward. The deadzone function $\mathrm{dz}(\cdot)$ is defined by $\mathrm{dz}(u) \triangleq u - \sigma(u)$. Both $\sigma(\cdot)$ and $\mathrm{dz}(\cdot)$ satisfy a $[0, I]$ sector condition [13], namely:

$$[v - \sigma(v)]' \sigma(v) = [v - \mathrm{dz}(v)]' \mathrm{dz}(v) \geq 0, \; \forall v \in \mathbb{R}^p.$$

II. Problem Data and Definition

The considered uncertain systems P_Ψ are formed by connecting a nominal system P and a “perturbation” Ψ as shown in Fig. 1, and are described by the equations:

$$\dot{x} = Ax + B_2 u \quad (1a)$$
$$z = C_1 x + D_{11} d + D_{12} u + z_\Psi \quad (1b)$$
$$y = C_2 x + D_{21} d + D_{22} u + y_\Psi, \quad (1c)$$

where $y \in \mathbb{R}^q$ is the measured output, z is the performance output, $u \in \mathbb{R}^p$ is the control input, d is the exogenous disturbance, and y_Ψ and z_Ψ are the outputs of the perturbation system $\Psi \in \mathcal{S}$. Here $\mathcal{S}$ is a set of asymptotically stable LTI systems, with $\Psi \in \mathcal{S}$ described by:

$$\dot{x}_\Psi = A_\Psi x_\Psi + B_\Psi u \quad (2a)$$
$$z_\Psi = C_{z,\Psi} x_\Psi + D_{z,\Psi} u \quad (2b)$$
$$y_\Psi = C_{y,\Psi} x_\Psi + D_{y,\Psi} u, \quad (2c)$$

(different elements of $\mathcal{S}$ may have different state spaces). It is assumed that $0 \in \mathcal{S}$, so that $P_0 = P$. Note that there is no loss of generality, from the input/output and stability point of view, in considering the disturbance d as not affecting (1a), since (1) is linear and will be later assumed to be asymptotically stable for the sake of a global discussion. Under such conditions, the effects of a disturbance d_0 affecting all equations in (1) may be represented as the effect of a disturbance d only affecting (1b) and (1c), with d being a filtered version of d_0.

For $\rho > 0$, let $\mathcal{S}_\rho \triangleq \{\Psi \in \mathcal{S} : \gamma_{(z_\Psi, y_\Psi), u_\Psi}^{(\Psi)} < \rho\}$, *i.e.* $\mathcal{S}_\rho \subset \mathcal{S}$ contains only uncertainties with gain from $u_\Psi = u$ to (z_Ψ, y_Ψ) less than ρ. A property $\mathcal{P}$ (*e.g.*, $\mathcal{L}_2$ stability) that is enjoyed by a system Σ_Ψ parameterized by $\Psi \in \mathcal{S}$, is:

- *nominal*, if $\mathcal{P}$ is enjoyed by Σ_Ψ when $\Psi = 0$;
- *robust-in-the-small* (with respect to $\mathcal{S}$), if there exists $\rho > 0$ such that $\mathcal{P}$ is enjoyed by Σ_Ψ for all $\Psi \in \mathcal{S}_\rho$;
- *robust-in-the-large* (with respect to $\mathcal{S}$), if $\mathcal{P}$ is enjoyed by Σ_Ψ for all $\Psi \in \mathcal{S}$.

In the Anti-Windup (AW) problem, a controller K_M (that is here assumed to be linear, and described by the equations:

$$\dot{x}_c = A_c x_c + B_c u_c + E_c r \quad (3a)$$
$$y_c = C_c x_c + D_c u_c + F_c r, \quad (3b)$$

where r is the reference signal, $u_c \in \mathbb{R}^q$ is the feedback signal, and $y_c \in \mathbb{R}^p$ is the controller output) is supposed to be given and designed for system (1) based on the interconnection where $u_c = y$ and $u = y_c$. Since the control input u is in fact affected by saturation, the goal of AW synthesis is to design an add-on AW compensator K_{AW} which, suitably connected to P_Ψ and K_M (see Fig. 2), will guarantee some nice properties for the overall closed loop system also in the case of saturating input.

It will be useful to have shorthand notations for referring to different interconnections of P_Ψ, K_M and K_{AW}, with and without saturation. To this aim, the following Closed-Loop Systems (CLSs) are defined:

- (1)-(3) form the *unsaturated* CLS $\bar{\Sigma}_U$ when $u_c = y$ and $u = y_c$, and the *saturated* CLS $\hat{\Sigma}_S$ when $u_c = y$ and $u = \sigma(y_c)$;
- (1)-(3) and K_{AW} form the *unsaturated* AW CLS $\widetilde{\Sigma}_{UAW}$ when $u_c = y + v_2$ and $u = y_c + v_1$, and the *(saturated)* AW CLS $\check{\Sigma}_{SAW}$ when $u_c = y + v_2$ and $u = \sigma(y_c + v_1)$.

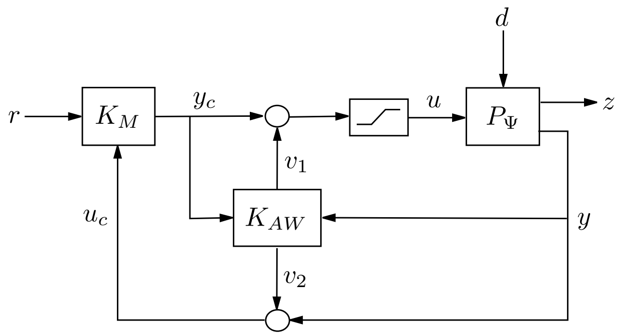

Fig. 2. The structure of the AW closed-loop system.

Note that different "hats" are used for clarity to denote a closed-loop system and its signals. For instance, the state of P as a subsystem of $\breve{\Sigma}_{SAW}$ is denoted by $\breve{x}$.

The following definition of the robust AW problem is considered in this paper. Let $\mathcal{U}$ be a strict, compact, and convex subset of $\{u \in \mathbb{R}^p : u = \sigma(u)\}$.

Definition 2.1: The *weakened global $\mathcal{L}_2$ AW problem for $\mathcal{U}$ with domain of robustness $\mathcal{S}$* is to find an AW compensator K_{AW} such that $\breve{\Sigma}_{SAW}$ is well-posed, and:

1) for $\Psi = 0$ and $d = 0$, $\exists x_{aw}^0$: if $x_{aw}(0) = x_{aw}^0$ and $\bar{u}(\cdot) \equiv \sigma(\bar{u}(\cdot))$, then $\breve{z}(\cdot) \equiv \bar{z}(\cdot)$;
2) $\tilde{\Sigma}_{UAW}$ is well-posed and internally stable, $\forall \Psi \in \mathcal{S}$;
3) if $\mathrm{dist}_{\mathcal{U}}(\tilde{u}(\cdot)) \in \mathcal{L}_2$, then $(\breve{z} - \tilde{z})(\cdot) \in \mathcal{L}_2, \forall \Psi \in \mathcal{S}$. □

For the weakened global $\mathcal{L}_2$ AW problem to make sense, two assumptions are needed.

Assumption 2.2: System $\bar{\Sigma}_U$ is well-posed and internally stable for $\Psi = 0$. □

Assumption 2.3: A is Hurwitz, and $\exists \rho > 0$: $\mathcal{S}_\rho = \mathcal{S}$. □

Assumption 2.2 requires $\bar{\Sigma}_U$ to be only nominally stable, so that K_M may be designed to maximize nominal performance, disregarding robust stability issues. Since no global result may be obtained for exponentially unstable controlled plants with bounded controls, Assumption 2.3 is quite mild in the present context. Moreover, Assumption 2.3 is coherent with the kind of nominal models and perturbation sets obtained by many identification approaches.

The main difference between a usual robust AW problem and the weakened problem of interest in this paper consists in the fact that in the latter the overall closed loop response of $\breve{\Sigma}_{SAW}$ from (r, d) to $\breve{z}$ when $d \neq 0$ and/or $\Psi \neq 0$ is allowed to be different from the corresponding response of $\bar{\Sigma}_U$ in order to be able to robustify $\breve{\Sigma}_{SAW}$ with respect to a larger class of uncertainties. Indeed, as shown in [10], robust-in-the-large stability may be not achievable under the standard definition of the AW problem, whereas robust-in-the-large stable solutions of the weakened AW problem always exist. A thorough comparison with standard AW definitions, and a discussion concerning the implications of the weakened AW problem definition, can be found in [10].

III. A PARAMETERIZATION OF OUTPUT FEEDBACK WEAKENED AW COMPENSATORS

A parameterization of the output feedback weakened AW compensators has been proposed in [12], and is characterized by two parameters, namely a matrix gain K (determining the rate of recovery after saturation) and a transfer matrix $F(s) \in \mathcal{RH}_\infty$ (determining how close the responses of $\tilde{\Sigma}_{UAW}$ and $\bar{\Sigma}_U$ are). A major drawback of the result in [12] is that imposing the condition that $\bar{\sigma}(K)$ is small, severely limits the rate of recovery after saturation.

The main theoretical result of this paper is to show that, for the same parameterization as in [12], the small gain condition on K can be dropped, so that much more aggressive choices of K are allowed. In particular, K may be determined by solving a LMI problem. In the following theorem, (A_F, B_F, C_F, D_F) is a minimal realization of the $q-$input, $q-$output transfer matrix $F(s)$ with state $x_F \in \mathbb{R}^{n_F}$.

Theorem 3.1: Under Assumption 2.2 and Assumption 2.3, the AW compensator K_{AW} with state $x_{aw} \in \mathbb{R}^{2n+n_F}$, input $u_{aw} = (u, y, y_c)$, output $y_{aw} = (v_1, v_2)$, system matrices:

$$\left[\begin{array}{c|c} A_{aw} & B_{aw} \\ \hline C_{aw} & D_{aw} \end{array}\right] = $$
$$= \left[\begin{array}{ccc|ccc} A & 0 & 0 & B_2 & 0 & 0 \\ -B_F C_2 & A_F & 0 & -B_F D_{22} & B_F & 0 \\ 0 & 0 & A & 0 & 0 & B_2 \\ \hline -K & 0 & K & 0 & 0 & 0 \\ -D_F C_2 & C_F & C_2 & -D_F D_{22} & D_F - I & D_{22} \end{array}\right]$$

and parameters K and $F(s)$ determined as in Sec. IV and Sec. V, respectively, solves the problem in Definition 2.1. □

The AW closed-loop system obtained by inserting the AW compensator K_{AW} of Theorem 3.1 in Fig. 2, is shown in Fig. 3, where systems P_M and P_S are described by the following equations:

$$P_S : \begin{cases} \dot{x}_S &= A x_S + B_2 u \\ y_S &= C_2 x_S + D_{22} u, \end{cases} \tag{4}$$

$$P_M : \begin{cases} \dot{x}_M &= A x_M + B_2 y_c \\ y_M &= C_2 x_M + D_{22} y_c. \end{cases} \tag{5}$$

Motivations for the use of the signal $u = \sigma(y_c + v_1)$ as input of K_{AW} are twofold. First, it allows to write the AW compensator K_{AW} of Theorem 3.1 as a LTI system. Second, the saturation nonlinearity is here assumed to be known, and hence the signal $\sigma(y_c + v_1)$ can be anyway reproduced inside the AW compensator. Note that such an assumption is not restrictive in most implementations, where artificial saturations of the control inputs are introduced in order to avoid damage to actuators.

Remark 3.2: Additive uncertainty allows to solve the weakened AW problem via output feedback, by using the open-loop observer P_S for the nominal model P as in [12]. In fact, the exponentially convergent state estimation error $x - x_S$ is a $\mathcal{L}_2$ disturbance, which does not create stability problems, since the overall AW closed-loop system is $\mathcal{L}_2$ stable. For other kinds of uncertainty the output feedback solvability of the weakened AW problem is still an open issue (as shown in [11], the solution in [10] is inherently state feedback). □

A two-step optimization-based procedure is proposed to determine K and $F(s)$ in Sec. IV and Sec. V, respectively. Following the proof of Theorem 3.1 (omitted due to lack of space), first K is chosen to guarantee: a) finite $\mathcal{L}_2$ gain

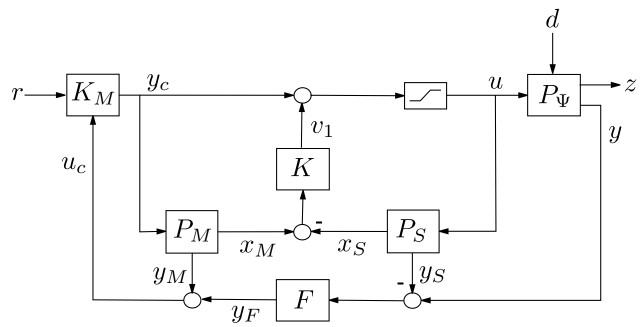

Fig. 3. Block diagram of the AW closed-loop system, where the internal structure of the AW compensator K_{AW} of Theorem 3.1 is disclosed.

for a suitable nonlinear system depending on the nominal dynamics P and the saturation, and b) quick convergence of its state to zero when the input is 0. Note that a) is needed in order for the design of $F(s)$ to be possible, while b) ensures fast recovery after saturation. Then, $F(s)$ is chosen to guarantee: c) a suitable small gain condition, and d) $\|I-F\|_\infty$ as small as possible, where c) ensures robust-in-the-large stability for $\breve{\Sigma}_{SAW}$, and d) tries to recover the natural AW requirements, since the responses of $\bar{\Sigma}_U$ and $\widetilde{\Sigma}_{UAW}$ coincide for $F(s)=I$.

IV. On the choice of K

Let w_1, w_2 be any pair of measurable signals such that $\breve{w} \triangleq (w_1 - w_2) \in \mathcal{L}_2$. Let $\dot{x}_1 = Ax_1 + B_2\sigma(-Kx_1 + w_1)$ and $\dot{x}_2 = Ax_2 + B_2(-Kx_2 + w_2)$ be the state dynamics (1a) of P under the saturated feedback $u = \sigma(-Kx + w_1)$ and the unconstrained feedback $u = -Kx + w_2$, respectively. The property required to K in the proof of Theorem 3.1 is to ensure a finite (and possibly small) $\mathcal{L}_2$ gain from $(\breve{w}, \mathrm{dz}(e_2))$ to $\breve{e} = \sigma(e_1) - e_2$, where $e_1 \triangleq -Kx_1 + w_1$, $e_2 \triangleq -Kx_2 + w_2$.

Since A is Hurwitz by Assumption 2.3, $K = 0$ is always a feasible choice yielding a unit gain from $\breve{w}$ to $\breve{e}$, and actually corresponds (under the additional choice $F(s) = I$) to the IMC AW solution. However, it is a well known fact that the IMC AW solution may have a very sluggish recovery after saturation, which would not be made quicker by choosing a different $F(s)$. Consequently, it is desirable to consider alternative solutions which, though possibly yielding a gain greater than 1, will result in faster recovery after saturation. Among these solutions, the small gain selection used in [10], [12] usually yields only marginal advantages with respect to the IMC choice. As in [8], a better solution derives from LMIs based on a sector condition and a quadratic storage function.

Let $\delta_e \triangleq e_1 - e_2$. The dynamics of $\breve{x} \triangleq x_1 - x_2$ is:

$$\dot{\breve{x}} = (A - B_2K)\breve{x} + B_2\breve{w} - B_2\,\mathrm{dz}(e_2) - B_2\varphi(\delta_e, e_2), \quad (6)$$

where $\varphi(\delta_e, e_2) \triangleq \mathrm{dz}(e_2 + \delta_e) - \mathrm{dz}(e_2)$ is such that:

$$[\delta_e - \varphi(\delta_e, e_2)]'\varphi(\delta_e, e_2) \geq 0. \quad (7)$$

Consider the quadratic function $V(\breve{x}) \triangleq \breve{x}'M\breve{x}$, where $M = M' \in \mathbb{R}^{n\times n}$ is positive definite. $V(\breve{x})$ establishes asymptotic stability and $\mathcal{L}_2$ gain less than γ for (6) if $\exists \varepsilon > 0$ such that:

$$\dot{V}(\breve{x}) < -2\varepsilon\breve{x}'M^2\breve{x} - \breve{u}'\breve{u} + \gamma^2\breve{w}'\breve{w} + \gamma^2\,\mathrm{dz}(e_2)'\,\mathrm{dz}(e_2). \quad (8)$$

By the S-procedure and the fact that (as in [8]) the inequality (7) is strict for suitable choices of $(\breve{x}, \breve{w}, \mathrm{dz}(e_2))$, (7) and (8) are satisfied for any choice of the free variables if and only if $\exists \tau \in \mathbb{R}$ such that, for any choice of the free variables:

$$\dot{V}(\breve{x}) < -2\varepsilon\breve{x}'M^2\breve{x} - \breve{u}'\breve{u} + \gamma^2\breve{w}'\breve{w} + \gamma^2\,\mathrm{dz}(e_2)'\,\mathrm{dz}(e_2) - 2\tau[\breve{e} - \varphi(\delta_e, e_2)]'\varphi(\delta_e, e_2). \quad (9)$$

Using Schur complements, (9) becomes:

$$\mathrm{He}\begin{bmatrix} Z_1 & MB_2 & -MB_2 & -MB_2 & 0 \\ 0 & -\frac{\gamma^2}{2}I & 0 & -\tau I & -I \\ 0 & 0 & -\frac{\gamma^2}{2}I & 0 & I \\ -\tau K & 0 & 0 & -\tau I & I \\ K & 0 & 0 & 0 & -\frac{1}{2}I \end{bmatrix} < 0,$$

where $Z_1 \triangleq M(A - B_2K) + \frac{\varepsilon}{2}M^2$ and $\mathrm{He}(L) \triangleq L + L'$. Under the congruence transformation given by pre- and post-multiplication by $H \triangleq \mathrm{diag}(Q, I, I, \eta, I) = H'$ with $Q \triangleq M^{-1}$, $\eta \triangleq (\tau\sqrt{2})^{-1}$, and the definitions $G \triangleq KQ^{-1}$, $\gamma_0 \triangleq \gamma^2$, the previous condition becomes the LMI:

$$\begin{bmatrix} Z_2 & B_2 & -B_2 & -(B_2 + G') & G' \\ B_2' & -\gamma_0 I & 0 & -I & -I \\ -B_2' & 0 & -\gamma_0 I & 0 & I \\ -(G + B_2') & -I & 0 & -\eta I & I \\ G & -I & I & I & -I \end{bmatrix} < 0, \quad (10)$$

where $Z_2 \triangleq AQ + QA - B_2G - G'B_2' + \varepsilon I$. The problem of determining K can then be solved by finding a solution of (10), *i.e.* $Q = Q' \in \mathbb{R}^{n\times n}$ positive definite, $G \in \mathbb{R}^{p\times n}$, and positive γ_0, η, ε. Simple manipulations and the fact that A is Hurwitz show that (10) is feasible for any $\gamma_0 > 1$ and $\varepsilon > 0$. K can then be chosen by fixing some maximum level for γ (by imposing the constraint $1 < \gamma_0 < \gamma_{max}^2$), meanwhile maximizing ε in order to achieve a faster convergence after saturation.

Remark 4.1: Since neither (6) nor (8) depend on the uncertainty, K can be chosen in a very effective way. In this respect, the situation is much more favorable than in [10]. This also shows that the small gain choice of K suggested in [10], [12] is overly conservative when additive uncertainty is considered. □

V. On the choice of $F(s)$

Under Assumption 2.3, it is easy to see that there exist weighting matrices $W_1(s), W_2(s) \in \mathcal{RH}_\infty$ such that the transfer matrix from u_Ψ to y_Ψ of any $\Psi \in \mathcal{S}$ can be factorized in the form $W_2(s)\Delta(s)W_1(s)$, with $\Delta(s) \in \mathcal{RH}_\infty$, $\|\Delta(s)\|_\infty < 1$. Let $K \in \mathbb{R}^{p\times n}$ and $\gamma \geq 1$ be obtained through the procedure described in Sec. IV. The filter $F(s)$ can then be determined as the solution of the following optimization problem:

$$\begin{aligned} \min_{F(s)\in\mathcal{RH}_\infty} \quad & \|I - F(s)\|_\infty \\ s.t. \quad & \\ & \|W_1(s)T(s)F(s)W_2(s)\|_\infty \leq \gamma^{-1}, \end{aligned} \quad (11)$$

where $T(s)$ is the (open-loop) transfer matrix from y_F to $y_T \triangleq y_c + Kx_M$ in Fig. 3. Note that (11) expresses the minimization of the AW performance losses under a small gain constraint ensuring robust stability. For a more compact notation, let $\Gamma(s) = W_1(s)T(s)$. The problem in (11) can be easily rewritten as:

$$\begin{array}{ll} \min\limits_{\substack{F(s) \in \mathcal{RH}_\infty \\ \varepsilon \geq 0}} & \varepsilon \\ s.t. & \\ \bar{\sigma}\left(I - F(\jmath\omega)\right) \leq \varepsilon, \ \forall \omega & \\ \bar{\sigma}\left(\Gamma(\jmath\omega)F(\jmath\omega)W_2(\jmath\omega)\right) \leq \gamma^{-1}, \ \forall \omega. & \end{array} \tag{12}$$

In order to tackle the solution of (12), one may resort to frequency-by-frequency minimization, *i.e.* at each frequency ω one solves the following problem:

$$\begin{array}{ll} \min\limits_{\varepsilon_\omega \geq 0, \, F_\omega \in \mathbb{C}^{q \times q}} & \varepsilon_\omega \\ s.t. & \\ \bar{\sigma}\left(I - F_\omega\right) \leq \varepsilon_\omega & \\ \bar{\sigma}\left(\Gamma(\jmath\omega)F_\omega W_2(\jmath\omega)\right) \leq \gamma^{-1}. & \end{array} \tag{13}$$

Doing so, the single optimization problem in (12) is decomposed into an infinite number of optimization problems, each corresponding to a different frequency ω. Indeed, solving (13) at each frequency ω is not the same as solving (12), since the former approach corresponds to minimizing ε_ω at all ω's, whereas the problem in (12) minimizes $\varepsilon = \sup_\omega \varepsilon_\omega$. However, minimizing ε_ω at each frequency ω guarantees that ε in (12) is also minimized (the vice-versa is not true). In addition, there is nothing detrimental in achieving smaller bounds ε_ω at each frequency ω than what is possible over all frequencies, namely ε.

The two constraints in (13) can be transformed into Linear Matrix Inequalities (LMIs) through application of the Schur complement. Given $Z \in \mathbb{C}^{q \times p}$, the condition $\bar{\sigma}(Z) \leq \lambda$, with $\lambda > 0$, is equivalent to $Z^*Z - \lambda I \preceq 0$ ($H \preceq 0$ denotes semi-negative definiteness of the Hermitian matrix H), which in turn is equivalent to:

$$\begin{bmatrix} -I & Z \\ Z^* & -\lambda I \end{bmatrix} \preceq 0.$$

Hence, the problem in (13) can be rewritten as the following minimization of a linear objective under LMI constraints:

$$\begin{array}{l} \min\limits_{\varepsilon_\omega \geq 0, \, F_\omega \in \mathbb{C}^{q \times q}} \quad \varepsilon_\omega \\ s.t. \\ \begin{bmatrix} -I & I - F_\omega \\ I - F_\omega^* & -\varepsilon_\omega I \end{bmatrix} \preceq 0 \\ \begin{bmatrix} -I & \Gamma(\jmath\omega)F_\omega W_2(\jmath\omega) \\ W_2^*(\jmath\omega)F_\omega^*\Gamma^*(\jmath\omega) & -\gamma^{-1} I \end{bmatrix} \preceq 0, \end{array} \tag{14}$$

which can be solved using available LMI routines (see the subsequent Remark 5.2, addressing implementation issues).

The filter $F(s)$ can be finally obtained by fitting a stable transfer matrix to the samples F_ω along the imaginary axis, *i.e.* in such a way that $F(\jmath\omega) = F_\omega, \forall\omega$. Model reduction techniques may be possibly applied to reduce the order of the filter.

Remark 5.1: From the computational point of view, it is obviously not possible to solve the optimization problem in (14) at all ω's. Hence, the user has to define a finite grid of frequency values where the samples $F(\jmath\omega)$ are computed by solving (14). A good rule is to start with an equally spaced grid, and then to refine the grid in those frequency intervals where higher accuracy is required (*e.g.*, where $\bar{\sigma}(F(\jmath\omega))$ shows faster variability). □

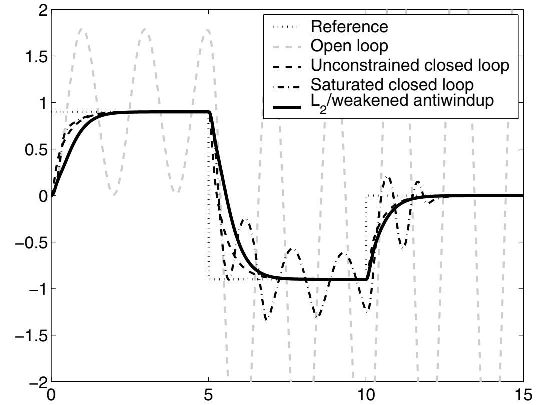

Fig. 4. Nominal ($\psi = 0$) output responses. The $\mathcal{L}_2$ AW and the weakened AW outputs coincide.

Remark 5.2: Most available LMI solvers are written for real-valued matrices and cannot directly handle LMI problems like (14), involving complex-valued matrices. However, complex-valued LMIs can be turned into real-valued LMIs by observing that a Hermitian matrix H satisfies $H \preceq 0$ if and only if:

$$\begin{bmatrix} \mathrm{Re}\,(H) & \mathrm{Im}\,(H) \\ -\mathrm{Im}\,(H) & \mathrm{Re}\,(H) \end{bmatrix} \preceq 0.$$

See, *e.g*, [14] for a systematic procedure to turn complex LMIs into real ones. □

VI. NUMERICAL EXAMPLE

Consider the nominal mass-spring-damper system:

$$\begin{aligned} \dot{x} &= Ax + B_2 u = \begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \\ y &= C_2 x = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \end{aligned}$$

for which $m = 0.1$, $k = 1$, $f = 0.06$, and the state x is formed by the position q and the velocity $\dot{q}$ of the body attached to the spring. The unconstrained, two-degree-of-freedom controller $y_c(s) = C_{fb}(s)\big(C_{ff}(s)r(s) - u_c(s)\big)$ is a priori given for this system, with $C_{fb}(s) = 200\frac{(s+5)^2}{s(s+80)}$ and $C_{ff}(s) = \frac{5}{2s+5}$. On the nominal model, this controller induces a quickly convergent response, asymptotic tracking of step references and rejection of step disturbances, despite the presence of very underdamped poles in the open loop system.

If input saturation and uncertainties are considered, the saturated closed-loop response undergoes remarkable deterioration. In nominal conditions, the responses of the open-loop, unconstrained closed-loop, and saturated closed-loop systems are shown in Fig. 4. The performance recovery achievable in nominal conditions by using either the weakened AW compensator or the $\mathcal{L}_2$ AW compensator (the two responses coincide in such a case, as explained in [10]) is also evident in Fig. 4.

When a real actuator with transfer function $V(s) = \frac{a}{s+a}$ is used, robustness issues appear. The parameter a usually

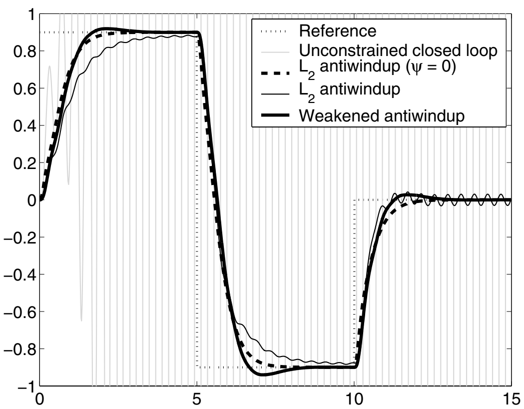

Fig. 5. Perturbed ($a = 10$) output responses. The weakened AW output is close to the nominal $\mathcal{L}_2$ AW output, while the other responses are unstable.

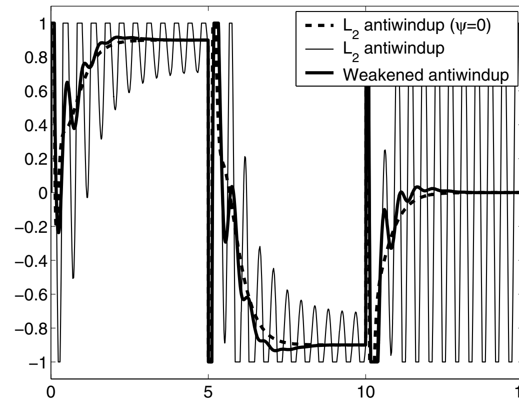

Fig. 6. Plant input for $a = 10$. Limits cycles show up if non-weakened AW is applied.

belongs to the set $[100, \infty)$, but may occasionally drop in the set $a \in [10, 100)$ due to particularly unfavorable operating conditions. Since the ideal actuator is modeled as a unity gain, *i.e.* $V_0(s) = 1$, the set of input multiplicative perturbations $\mathcal{M} \triangleq \{\mu(s) : \mu(s) = \frac{-s}{s+a}, a \in [10, \infty)\} \cup \{0\}$ is considered. For $a < a_{min} \approx 17.4$, the unconstrained closed-loop system $\bar{\Sigma}_U$ becomes unstable, so that for such values of a any non-weakened form of anti-windup is not applicable.

The set $\mathcal{M}$ can be recast as the set of additive perturbations $\mathcal{S}_a \triangleq \{\Psi(s) : \Psi(s) = P_0(s)\frac{-s}{s+a}, a \in [10, \infty)\} \cup \{0\}$, where $P_0(s) \triangleq C_2(sI - A)^{-1}B_2$. Such a set is contained in the larger set:

$$\mathcal{S} \triangleq \big\{\Psi(s) = W(s)\Delta(s) : \Delta(s) \in \mathcal{RH}_\infty, \|\Delta(s)\|_\infty < 1\big\}, \tag{15}$$

by choosing $W(s) := P_0(s)\frac{-s}{s+a}$ (a single frequency weight $W(s)$ can be considered since the uncertainties are scalar transfer functions). The design procedure described in this paper can then be easily applied to the nominal plant and this uncertainty set.

The gain $K = \begin{bmatrix} 0.3843 & -0.5771 \end{bmatrix}$ is obtained by solving the LMI derived in Sec. IV, and corresponds to a value of the $\mathcal{L}_2$ gain γ for the nonlinear mismatch system such that $\gamma - 1 < 10^{-4}$.

The filter $F(s)$ is designed by solving the frequency-by-frequency LMI optimization problem described in Sec. V over a logarithmically spaced grid of frequencies centered around the frequency range where $W(j\omega)T(j\omega)$ shows faster variations. Standard routines are then used to fit the samples F_ω obtained by this procedure with a stable transfer function. If needed, the order of the transfer function obtained via the fitting process is lowered by applying standard order reduction routines.

Fig. 5 shows the response of the weakened AW closed-loop system in presence of the severest perturbation in $\mathcal{S}$ ($a = 10$). Note that this response is very close to the nominal $\mathcal{L}_2$ AW response in Fig. 5, thus showing that the AW performance loss due to the weakened formulation of the AW problem is rather mild in this example. Fig. 5 also confirms that the unconstrained closed-loop system is unstable for $a = 10$, and shows that the response obtained with the $\mathcal{L}_2$ AW compensator for $a = 10$ is characterized by persistent oscillations, especially evident in the last 5 seconds of simulation in Fig. 5 (output plot) and Fig. 6 (input plot).

VII. Conclusions

A constructive, LMI-based approach to the design of output feedback weakened anti-windup compensators for additively perturbed plants has been described, and its effectiveness has been shown in simulation.

Future work will focus on optimal design and order reduction for weakened anti-windup compensators.

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