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Generalized pseudo-inverse based nested discrete-time LQ-control with indefinite cost

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Abstract— **A generalized pseudo-inverse approach to discrete-time finite horizon linear quadratic control with final state constraint is considered. Necessary and sufficient conditions for existence of a unique optimal solution are provided. Both the test for existence of a solution, as well as the computation of it, decompose, via a nesting procedure, into problems of successively smaller size, thus rendering computational feasibility for problems too large to be handled directly.**

I. INTRODUCTION

Recently a nested approach to a discrete-time finite horizon linear quadratic control problem with final state constraint was considered by Marro et al in [1] (see also [2]). This is extended here to the more general setting of an indefinite cost criterion. An indefinite cost criterion of special form appears naturally in H_∞-control, admitting the sup-optimal full-information H_∞-problem to be recast as a linear quadratic control problem [3]. However, in this paper a more general criterion, with arbitrary indefinite weights, is employed. Similar to [1] a pseudo-inverse, non-recursive approach is taken. Compared with standard linear quadratic control, the presence of an indefinite inner product introduces some substantial changes into the problem and its solution. In particular, the optimal solution does not always need to exist. Precise conditions for existence of a unique optimal solution are therefore derived.

Both the test for existence of a solution, and its computation, decompose into problems of successively smaller size, by application of a nesting procedure. This admits computation of the solution even when the original problem is to large to be handled directly. The idea is similar to that of [1], but the procedure is augmented with the decomposed existence test.

II. INDEFINITE LQ-CONTROL WITH FINAL STATE CONSTRAINT

Consider a discrete time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0 \\ e(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}_n$, $u(k) \in \mathbb{R}_p$ and $e(k) \in \mathbb{R}_q$. Introduce an indefinite cost criterion

$$\mathscr{J} = \sum_{k=0}^{N-1} e(k)^T J_\phi e(k) + x(N) Z^T J_\psi Z x(N) \tag{2}$$

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where Z is a given matrix with, say, r rows, and

$$J_\phi = \operatorname{diag}(I_{q_1}, -I_{q_2}), \qquad J_\psi = \operatorname{diag}(I_{r_1}, -I_{r_2})$$

are given signature matrices. Moreover, let

$$Gx(N) = y_f \tag{3}$$

be a constraint on the final state, where G is a given matrix and y_f a given vector. It is assumed that the constraint is feasible in the sense that at least for some choice of the control $u(k)$ the final state satisfies (3).

Problem 1: Find, if such exists, a control sequence $u(k)$, $k = 0, 1, \ldots, N-1$, such that, subject to the system dynamics (1), and the terminal state constraint (3), the indefinite cost (2) is minimized.

To facilitate treatment of Problem 1, it is convenient to put the dependence of the output $e(k)$ on the control and initial state on a more compact form. Let

$$e_N = \begin{bmatrix} e(0) \\ e(1) \\ \vdots \\ e(N-1) \\ Zx(N) \end{bmatrix}, \quad u_N = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \tag{4}$$

$$A_N = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \\ ZA \end{bmatrix} \tag{5}$$

$$B_N = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \cdots & D \\ ZA^{N-1}B & ZA^{N-2}B & \cdots & ZB \end{bmatrix} \tag{6}$$

It is then easy to see that the (augmented) output sequence e_N may be expressed in terms of the initial state x_0 and the input sequence u_N as

$$e_N = A_N x_0 + B_N u_N \tag{7}$$

Remark 1: If the terminal cost in (2) is zero, that is $Z = 0$, the bottom block rows of A_N, B_N and e_N, containing Z, are vacuous, and may be removed. In the sequel, whenever $Z = 0$, it is assumed that these truncated versions of A_N, B_N and e_N are used.

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III. MINIMIZATION OF INDEFINITE QUADRATIC FORMS

In order to derive conditions for existence of solutions to Problem 1, it is expedient to first examine minimization of indefinite quadratic forms in general. First some notations related to (finite dimensional) indefinite inner product spaces are introduced (for discourses on these spaces in general, see e.g. [4], [5].) Consider the special case of an indefinite inner product on, say, $\mathbb{R}_m$, defined by

$$\langle x, y \rangle = x^T J y \tag{8}$$

where $J = \operatorname{diag}(I_{m_1}, -I_{m_2})$ is a signature matrix, with $m_1 + m_2 = m$. Two vectors $x, y \in \mathbb{R}^m$ are said to be orthogonal to each other w.r.t $\langle \cdot, \cdot \rangle$, if $\langle x, y \rangle = 0$. A vector $x \in \mathbb{R}^m$ is said to be orthogonal to $\mathscr{M} \subseteq \mathbb{R}^m$ w.r.t $\langle \cdot, \cdot \rangle$, denoted $x \perp \mathscr{M}$, if $\langle x, y \rangle = 0$ for all $y \in \mathscr{M}$. The subspace $\mathscr{M}^\perp := \{x \in \mathbb{R}^m \mid x \perp \mathscr{M}\}$ is called the orthogonal complement of $\mathscr{M} \subseteq \mathbb{R}^m$ w.r.t $\langle \cdot, \cdot \rangle$. Let $\mathscr{L}$ be a subspace of $\mathbb{R}^m$, and let $x \in \mathbb{R}^m$. If $x \in \mathbb{R}^m$ can be written as $x = \hat{x} + \tilde{x}$, with $\hat{x} \in \mathscr{L}$ and $\tilde{x} \in \mathscr{L}^\perp$, then $\hat{x}$ is called a projection of x unto $\mathscr{L}$ (w.r.t $\langle \cdot, \cdot \rangle$). Contrary to the definite case a projection does not always need to exist, nor does it need to be unique [4, Lemma 8.1, Theorem 8.3]

Let Θ be an $m \times n$ matrix, with $m > n$, and let $\beta \in \mathbb{R}_m$. Consider the problem of finding a $\mu \in \mathbb{R}_n$ minimizing the indefinite quadratic form

$$\begin{aligned} Q(\mu) &= \langle \beta - \Theta\mu, \beta - \Theta\mu \rangle \\ &= \langle \beta, \beta \rangle - \langle \beta, \Theta\mu \rangle - \langle \Theta\mu, \beta \rangle + \langle \Theta\mu, \Theta\mu \rangle \end{aligned} \tag{9}$$

The vector μ_o is said to be a stationary point of the quadratic form $Q(\mu)$ if

$$\left. \frac{\partial Q(\mu)}{\partial \mu} \right|_{\mu = \mu_0} = 0$$

or equivalently, by computing the derivative, μ_o is a stationary point of $Q(\mu)$ if an only if it is a solution to the normal equation

$$\Theta^T J \Theta \mu = \Theta^T J \beta \tag{10}$$

Clearly the quadratic form $Q(\mu)$ then has a unique stationary point, if and only if, $\Theta^T J \Theta$ is non-singular. This unique stationary point is given by

$$\mu_o = \Theta^\natural \beta \tag{11}$$

where

$$\Theta^\natural := (\Theta^T J \Theta)^{-1} \Theta^T J \tag{12}$$

The matrix $\Theta^\natural$ is a pseudo-inverse of Θ in the sense that $\Theta^\sharp \Theta = I$. However, it is not the Moore-Penrose pseudo-inverse, but one associated with the geometry induced by the indefinite inner product in question.

If a projection, with respect to $\langle \cdot, \cdot \rangle$, of β unto $\operatorname{im}\Theta$ exists, then it is unique, if and only if, $\Theta^T J \Theta$ is non-singular. In fact, given a projection $\hat{\beta}$, any other projection may be written $\hat{\beta} + \Theta\eta$, for some $\eta \in \mathbb{R}^n$, with $\Theta\eta \in \operatorname{im}\Theta^\perp$. However, $\Theta\eta \in \operatorname{im}\Theta^\perp$, if and only if, $\eta^T \Theta^T J \Theta \mu = 0$ for all $\mu \in \mathbb{R}^n$, or equivalently, $\eta^T \Theta^T J \Theta = 0$. Hence $\eta \neq 0$ is possible, if and only if, $\Theta^T J \Theta$ is singular.

On the other hand, non-singularity of $\Theta^T J \Theta$ is sufficient (but not necessary) for the projection $\hat{\beta}$, with respect to $\langle \cdot, \cdot \rangle$, of β unto $\operatorname{im}\Theta$, to exist [4, Theorem 8.5], and by the above it is then unique. Moreover it is given by

$$\hat{\beta} = \Theta\mu_0 = \Theta\Theta^\natural \beta$$

A stationary point μ_o is a minimum, if and only if, the Hessian of $Q(\mu)$ is positive semidefinite. Since the Hessian of $Q(\mu)$ is $2\Theta^T J \Theta$, $Q(\mu)$ has a minimum point, if and only if,

$$\Theta^T J \Theta \geq 0 \tag{13}$$

Moreover, the quadratic form $Q(\mu)$ has a unique minimum point μ_o, if and only if, the Hessian is positive definite, that is, if and only if,

$$\Theta^T J \Theta > 0 \tag{14}$$

This unique minimum point is, of course, given by (11).

The above constructions are recasts of expositions in [6], [7], [8], [4].

The solution (11) remains the same if the elements of J are diagonally permuted. To see this, suppose that $J_\Pi = \Pi^T J \Pi$, where Π is any permutation matrix. Thus J_Π is obtained by permutation of the diagonal elements of J. This will be referred to as a general signature matrix. Consider now the problem of finding a $\mu \in \mathbb{R}_n$ minimizing the quadratic form

$$Q(\mu) = \langle \beta - \Theta\mu, \beta - \Theta\mu \rangle_\Pi \tag{15}$$

where $\langle x, y \rangle_\Pi := x^T J_\Pi y$. Clearly,

$$Q(\mu) = \langle \Pi\beta - \Pi\Theta\beta, \Pi\beta - \Pi\Theta\mu \rangle = \langle \beta_\Pi - \Theta_\Pi\mu, \beta_\Pi - \Theta_\Pi\mu \rangle$$

where $\beta_\Pi = \Pi\beta$ and $\Theta_\Pi = \Pi\Theta$. By (11) and (12) the minimum point is given by

$$\mu_0 = (\Theta_\Pi^T J \Theta_\Pi)^{-1} \Theta_\Pi^T J \beta_\Pi = (\Theta^T J_\Pi \Theta)^{-1} \Theta J_\Pi \beta \tag{16}$$

Summarizing the above results yields.

Lemma 1: Consider an indefinite quadratic form in the vector μ

$$\langle \beta - \Theta\mu, \beta - \Theta\mu \rangle \tag{17}$$

where Θ is a given matrix, β a given vector, and J a general signature matrix.

(i) The quadratic form (17) has unique stationary point, if and only if, $\Theta^T J \Theta$ is non-singular. In this case the unique stationary point is given by

$$\mu_o = \Theta^\natural \beta \tag{18}$$

and the value of $\beta - \Theta\mu$ at the stationary point is

$$(I - \Theta\Theta^\natural)\beta \tag{19}$$

(ii) The unique stationary point in (i) is a minimum point, if and only if, $K^T \Theta^T J \Theta K > 0$.

Proof: By the discussion preceding the lemma, and substitution of (11) into $\beta - \Theta\mu$ to get (19). □

A useful generalization of Lemma 1 is obtained by imposing a linear constraint on the argument μ. This is the content of the following lemma. In addition to the indefinite inner product induced pseudo-inverse $(\cdot)^\natural$ in (12), it also makes use of the Moore-Penrose pseudo-inverse, which will be denoted $(\cdot)^\dagger$.

Lemma 2: Consider an indefinite quadratic form in the vector μ

$$\langle \beta - \Theta\mu, \beta - \Theta\mu \rangle \tag{20}$$

where Θ is a given matrix, β a given vector, and J a general signature matrix. Let Γ be a given matrix, and γ a given vector in $\operatorname{im}\Gamma$. Introduce the following constraint on μ.

$$\Gamma\mu = \gamma \tag{21}$$

Let K be basis matrix for $\ker\Gamma$.

(i) Under the restriction (21) the quadratic form (20) has a unique stationary point, if and only if, $K^T\Theta^T J\Theta K$ is non-singular. In this case the unique stationary point is given by

$$\mu_o = (I - K(\Theta K)^\natural\Theta)\Gamma^\dagger\gamma + K(\Theta K)^\natural\beta \tag{22}$$

The value of $\beta - \Theta\mu$ at the stationary point μ_o is

$$(I - \Theta K(\Theta K)^\natural)\beta - (I - \Theta K(\Theta K)^\natural)\Theta\Gamma^\dagger\gamma \tag{23}$$

(ii) The unique stationary point in (i), of (20) restricted to (21), is a minimum point, if and only if,

$$K^T\Theta^T J\Theta K > 0 \tag{24}$$

Proof: The vector μ satisfies the constraint (21), if and only if,

$$\mu = \Gamma^\dagger\gamma + K\tilde{\mu} \tag{25}$$

where K is the basis of $\ker\Gamma$, and $\tilde{\mu}$ is a free parameter. Hence

$$\beta - \Theta\mu = \beta - \Theta\Gamma^\dagger\gamma - \Theta K\tilde{\mu} = \tilde{\beta} - \tilde{\Theta}\tilde{\mu} \tag{26}$$

where

$$\tilde{\beta} = \beta - \Theta\Gamma^\dagger\gamma, \qquad \tilde{\Theta} = \Theta K \tag{27}$$

The problem of minimizing (20) under the constraint (21) is therefore equivalent to the unconstrained problem of minimizing

$$\langle \tilde{\beta} - \tilde{\Theta}\tilde{\mu}, \tilde{\beta} - \tilde{\Theta}\tilde{\mu} \rangle \tag{28}$$

with respect to $\tilde{\mu}$.

(i): By Lemma 1 the quadratic form (28) has a unique stationary point $\tilde{\mu}_o$, if and only if, $\tilde{\Theta}^T J\tilde{\Theta}$ is nonsingular. In this case the unique stationary point is given by

$$\tilde{\mu}_o = \tilde{\Theta}^\natural\tilde{\beta} = (\Theta K)^\natural\beta - (\Theta K)^\natural\Theta\Gamma^\dagger\gamma \tag{29}$$

Consequently, by (25) the unique stationary point μ_o of (20), under the constraint (21), is given by

$$\begin{aligned}\mu_o &= \Gamma^\dagger\gamma + K\tilde{\mu} = \Gamma^\dagger\gamma + K(\Theta K)^\natural\beta - K(\Theta K)^\natural\Theta\Gamma^\dagger\gamma \\ &= (I - K(\Theta K)^\natural\Theta)\Gamma^\dagger\gamma + K(\Theta K)^\natural\beta \end{aligned}\tag{30}$$

Furthermore, by Lemma 1 the value of $\tilde{\beta} - \tilde{\Theta}\tilde{\mu}$ at the stationary point $\tilde{\mu}_o$ is given by

$$\begin{aligned}(\hat{\tilde{\beta}} - \tilde{\Theta}\tilde{\mu})_{|\tilde{\mu}=\tilde{\mu}_o} &= (I - \tilde{\Theta}\tilde{\Theta}^\natural)\tilde{\beta} \\ &= (I - \Theta K(\Theta K)^\natural)(\beta - \Theta\Gamma^\dagger\gamma) \\ &= (I - \Theta K(\Theta K)^\natural)\beta \\ &\quad - (I - \Theta K(\Theta K)^\natural)\Theta\Gamma^\dagger\gamma \end{aligned}\tag{31}$$

However, by (26) this then is also the value of $\beta - \Theta\mu$ at μ_o. This proves part (i).

(ii): By Lemma 1 $\tilde{\mu}_o$ is a the unique minimum point of the quadratic form (28), if and only if, $\tilde{\Theta}^T J\tilde{\Theta} = K^T\Theta^T J\Theta K > 0$. The statement then follows by observing that μ_o is the unique minimum point of (20) under the constraint (21), if and only if, $\tilde{\mu}_o$ is the unique minimum point of (28). □

IV. Pseudo-inverse solution of the indefinite LQ-problem

Define the general signature matrix

$$J_\sigma = \operatorname{diag}\left(J_\phi, \cdots, J_\phi, J_\psi\right)$$

The cost criterion (2) may then be written as the indefinite quadratic form

$$\mathscr{J} = \langle e_N, e_N \rangle = \langle A_N x_o + B_N u_N, A_N x_o + B_N u_N \rangle \tag{32}$$

The following theorem solves Problem 1 by giving conditions for existence of a control minimizing the indefinite quadratic form (32) under the constraint (3). It also gives explicit formulas for the solution. For simplicity only the case of a unique solution is considered. The theorem is a generalization to the indefinite case of [1, Theorem 1].

Theorem 3: Let

$$L_N = \begin{bmatrix} A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \tag{33}$$

Furthermore, let K be a matrix, the columns of which form a basis for the null space of GL_N. Subject to the system dynamics (1), and the terminal state constraint (3), a unique control sequence u_N minimizing the indefinite cost (2) then exists, if and only if,

$$K^T B_N^T J_\sigma B_N K > 0 \tag{34}$$

In this case the unique optimal control sequence is given by

$$u_N = T_N x_0 + V_N y_f \tag{35}$$

and the optimal (augmented) output sequence is given by

$$e_N^o = C_N x_0 + D_N y_f \tag{36}$$

where the matrices T_N, V_N, C_N and D_N are defined as

$$\begin{aligned} T_N &= -(I - K(B_N K)^\natural B_N)(GL_N)^\dagger GA^N - K(B_N K)^\natural A_N \\ V_N &= (I - K(B_N K)^\natural B_N)(GL_N)^\dagger \\ C_N &= (I - B_N K(B_N K)^\natural)(A_N - B_N(GL_N)^\dagger GA^N) \\ D_N &= (I - B_N K(B_N K)^\natural)B_N(GL_N)^\dagger \end{aligned}$$

Proof: By (7), $e_N = A_N x_0 + B_N u_N$. Note $x(N) = A^N x_0 + L_N u_N$. Thus the terminal constraint may be written $GL_N u_N = y_f - GA^N x_0$.

Let $\beta = A_N x_0$, $\Theta = -B_N$, $\Gamma = GL_N$, $\gamma = y_f - GA^N x_0$ and $\mu = u_N$, and apply Lemma 2 (i). This yields

$$\begin{aligned} u_N^o &= \mu_o = (I - K(\Theta K)^\natural\Theta)\Gamma^\dagger\gamma + K(\Theta K)^\natural\beta \\ &= (I - K(B_N K)^\natural B_N)(GL_N)^\dagger(y_f - GA^N x_0) \\ &\quad - K(B_N K)^\natural A_N x_0 \\ &= -((I - K(B_N K)^\natural B_N)(GL_N)^\dagger GA^N - K(B_N K)^\natural A_N)x_0 \\ &\quad + (I - K(B_N K)^\natural B_N)(GL_N)^\dagger y_f = T_N x_0 + V_N y_f \end{aligned}\tag{37}$$

where T_N and V_N are as defined in the theorem. Similarly, by (23)

$$\begin{aligned} e_N^o &= \beta - \Theta\mu_o \\ &= (I - \Theta K(\Theta K)^\natural)\beta - (I - \Theta K(\Theta K)^\natural)\Theta\Gamma^\dagger\gamma \\ &= (I - B_N K(B_N K)^\natural) A_N x_0 \\ &\quad + (I - B_N K(B_N K)^\natural) B_N (GL_N)^\dagger (y_f - GA^N x_0) \\ &= (I - B_N K(B_N K)^\natural)(A_N - B_N (GL_N)^\dagger GA^N) x_0 \\ &\quad + (I - B_N K(B_N K)^\natural) B_N (GL_N)^\dagger y_f = C_N x_0 + D_N y_f \end{aligned}$$

where C_N and D_N are as defined in the theorem. □

Remark 2: In case of a definite cost a sufficient, but not necessary, condition for (34) to hold is that $D^T D > 0$. This corresponds to the common case of positive definite control weight [9, Sec. 6.1]. Rendering condition (34) in elementary terms will not be explored here.

V. Example

Consider the control of an DC-motor, with its discrete-time model (taken from [10]) given by

$$A = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix}, \quad B = \begin{bmatrix} 0.00484 & 0.0952 \end{bmatrix}$$

with state vector $x = [x_1 \;\; x_2]^T$, where x_1 and x_2 are the angular position the angular velocity of the shaft, respectively.The initial state is assumed to be $x_0 = [1 \;\; 0]^T$. Let the matrices defining the instantaneous cost term be

$$C = \begin{bmatrix} 0.95 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad J_\phi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and let the terminal cost be given by

$$Z = \mathrm{diag}(0, 20), \quad J_\psi = \mathrm{diag}(1, 1)$$

The constraint on the terminal state is assumed to be given by

$$G = \mathrm{diag}(1,\, 0), \quad y_f = [0.5 \;\; 0]^T$$

The constraint implies that the final angular position is to be exactly 0.5 rad. There is no constraint on the angular speed, but the relatively large terminal weight on the speed variable will likely bring it rather close to zero.

For a horizon of $N = 50$ the matrix $K^T B_N^T J_\sigma B_N K$ is positive definite (the smallest eigenvalue equals 0.2906). Hence the criterion has a unique minimum, which by application of Theorem 3 is seen to be $\min \mathscr{J} = -77.8348$. The states, and control trajectories are shown in Figure 1. Note the temporary excursion of the position, enabled by a corresponding negative entry in the signature matrix J_ϕ of the instantaneous cost term. However, owing to the terminal constraint the desired final angular position 0.5 is perfectly archived.

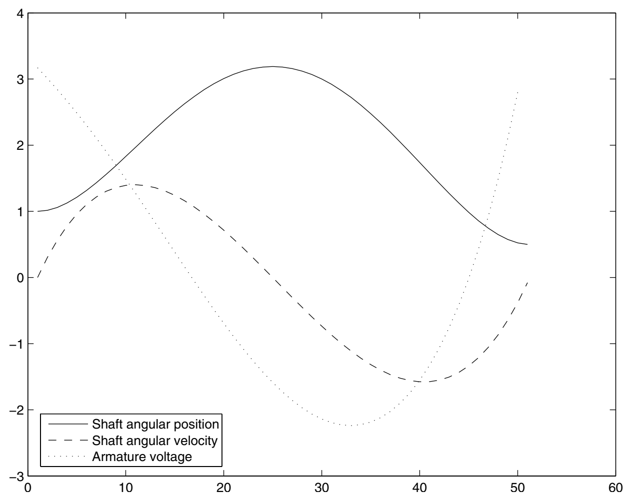

Fig. 1. State and control trajectories

VI. Nested decomposition

For sufficiently large N, solution of the original problem by the technique of Theorem 3 may become computationally unfeasible. However, the idea of [1] to decompose the original problem into smaller ones to achieve computational tractability, may be carried over to the indefinite case. This is a major motivation behind the nesting procedure to be outlined in the sequel. It is analogous to that of [1], but owing to the indefinite cost, existence of an optimal solution must be ensured. The procedure conveniently provides for this by a corresponding decomposition of the existence test for the original problem to existence tests for the subproblems, and for an overlying problem. If the overlying problem still is not computational tractability, a further decomposition may be carried out, and so forth.

Divide the time interval $0, 1, \ldots, N$ into N_2 parts, each having a length of N_1 steps, that is, $N = N_2 N_1$.

$$\begin{aligned} \tilde{x}(j+1) &= \tilde{A}\tilde{x}(j) + \tilde{B}\alpha(j), \quad \tilde{x}(0) = x_0 \\ \tilde{e}(j) &= \tilde{C}\tilde{x}(j) + \tilde{D}\alpha(j) \end{aligned} \tag{38}$$

where

$$\tilde{A} = A^{N_1}, \qquad \tilde{C} = C_{N_1} + D_{N_1} A^{N_1}, \qquad \tilde{D} = D_{N_1}\tilde{B} \tag{39}$$

and where $\tilde{B}$ is a basis matrix of $\mathrm{im} L_{N_1}$, with L_{N_1} defined according to (33). Define an associated cost criterion

$$\tilde{J} = \sum_{k=0}^{N_2-1} \tilde{e}^T(k)^T J_\varsigma \tilde{e}(k) + \tilde{x}(N_2)^T Z^T J_\psi Z \tilde{x}(N_2) \tag{40}$$

where $J_\varsigma = \mathrm{diag}\left(J_\phi, \cdots, J_\phi\right)$ is the signature matrix of the running cost term, with N_1 number of J_ϕ terms, and where $\tilde{x}(N_2) = x(N)$. Let the final state constraint be as in Problem 1, that is

$$G\tilde{x}(N_2) = y_f \tag{41}$$

Consider the following problem.

Problem 2 (Overlying problem): Find, if such exists, an input sequence $\alpha(j)$, $j = 0, 1, \ldots, N_2 - 1$, such that, subject to the system dynamics (38) and the terminal constraint (41), the indefinite cost criterion (40) is minimized.

For $j = 0, 2, \ldots, N_2 - 1$, consider also the following subproblems related to the original problem.

Problem 3 (Subproblem j): Find, if such exists, an input sequence $u(jN_1), u(jN_1+1), \ldots, u((j+1)N_1 - 1)$ such that subject to the system dynamics (1), and given boundary conditions $x(jN_1) = x_{jN_1}$ and $x((j+1)N_1) = x_{(j+1)N_1}$, the indefinite cost

$$\mathscr{J}_j = \sum_{k=jN_1}^{(j+1)N_1 - 1} e(k)^T J_\phi e(k) \tag{42}$$

is minimized.

Lemma 4: Suppose that the original problem, Problem 1, admits a unique optimal solution. Then the following holds for $j = 0, 1, \ldots, N_2 - 1$.

(i) The j-th subproblem, Problem 3, admits a unique optimal control, no matter what the boundary conditions $x(jN_1) = x_{jN_1}$ and $x((j+1)N_1) = x_{(j+1)N_1}$ of the subproblem are.

(ii) Suppose that $\{x^o(k)\}$ and $\{u^o(k)\}$ are the unique optimal state and control trajectories of Problem 1. Let the boundary conditions of the j-th subproblem be $x_{jN_1} = x^o(jN1)$ and $x_{jN_1} = x^o(jN1)$, respectively. Then the unique optimal state and control trajectory of the j-th subproblem agrees with the restriction of $\{x^o(k)\}$ and $\{u^o(k)\}$ to the time intervals $jN_1, jN_1+1, \ldots, (j+1)N_1$ and $jN_1, jN_1+1, \ldots, (j+1)N_1 - 1$, respectively.

Proof: *(i):* Assume that the original problem, Problem 1, has a unique optimal control $\{u^o(k)\}$, with optimal state trajectory $\{x^o(k)\}$. Consider a restriction of Problem 1 to the time interval $jN_1, jN_1+1, \ldots, jN_1+N_1$, with initial state $x^o(jN_1)$ and final state $x^o((j+1)N_1)$. Then (42) is the original cost criterion restricted to this time interval. By Bellman's principle of optimality [11], $u^o_{jN_1}, u^o_{jN_1+1}, \ldots, u^o_{jN_1+N_1-1}$ then is the unique control minimizing (42). Consequently, subproblem j with boundary conditions $x(jN_1) = (x^o(jN_1)$ and $x((j+1)N_1) = x^o((j+1)N_1)$ admits a unique optimal solution, and this may therefore be computed by application of Theorem 3. Since the final state of the subproblem is assigned sharply, the terminal state constraint may be set up with $G = I$. Thus no weight Z on the terminal states needs to be included in the cost criterion (42). Therefore, the matrices T_{N_1}, V_{N_1}, C_{N_1}, D_{N_1} and K obtained by Theorem 3 do not depend on the boundary conditions, and are hence common to all subproblems, that is, they do not depend on j. Consequently, each subproblem j admits a unique optimal solution, no matter what the boundary conditions are.

(ii): This is a direct consequence of the first part of the proof of (i). □

Lemma 5: Suppose that Problem 1 admits a unique minimizing control sequence $\{u(k)\}$. Then Problem 2 also admits a unique minimizing sequence $\{\alpha(j)\}$. For this it holds that $\min \tilde{\mathscr{J}} = \min \mathscr{J}$, where $\min \tilde{\mathscr{J}}$ is the minimal value of the cost criterion of Problem 2, and $\min \mathscr{J}$ is the minimal value of the cost criterion of Problem 1.

Proof: Suppose that Problem 1 admits a unique minimizing control sequence $\{u^o(k)\}$, with corresponding state sequence $\{x^o(k)\}$ and output sequence $\{e^o(k)\}$. By Lemma 4 (i) each subproblem j, with boundary conditions $x(jN_1) = x^o(jN_1)$ and $x((j+1)N_1) = x^o((j+1)N_1)$, then admits a unique optimal solution. Since the solution exists, Theorem 3 may be invoked to obtain the matrices C_{N_1} and D_{N_1} describing the optimal output of the j-th subproblem as

$$\tilde{e}(j) = C_{N_1}\tilde{x}(j) + D_{N_1}\tilde{x}(j+1) \tag{43}$$

where $\tilde{x}(j) = x^o(jN_1)$ is its initial state, and $\tilde{x}(j+1) = x^o((j+1)N_1)$ is its sharply assigned final state. However, by Lemma 4 (ii) the optimal output (43) equals the segment

$$[e^o(jN_1)^T, e^o(jN_1+1)^T, \ldots, e^o((j+1)N_1)^T]^T \tag{44}$$

of the optimal output of the original problem, Problem 1.

Let

$$u^o_{N_1}(j) = [u^o(jN_1)^T, u^o(jN_1+1)^T, \ldots, u^o((j+1)N_1-1)^T]^T$$

There exists then a unique vector $\alpha^o(j)$ such that the equality $L_{N_1}u^o_{N_1}(j) = \tilde{B}\alpha^o(j)$ holds. Using the definitions (39) it then follows that (43) is equivalent to $\tilde{e}(j) = \tilde{C}\tilde{x}(j) + \tilde{D}\alpha(j)$, and that $\tilde{x}(j+1) = \tilde{A}\tilde{x}(j) + \tilde{B}\alpha(j)$, which are the system equations (38). Hence sequence $\{\alpha^o(j)\}$ applied as input to the system (38) yields the vectors (44) as outputs, that is the optimal output of the original problem, Problem 1. Hence the optimal cost of Problem 2 must be less or equal to the optimal cost of Problem 1.

Suppose that there exists an input sequence $\{\alpha^*(j)\}$ for the system (38), such that the corresponding cost $\tilde{\mathscr{J}}^*$ satisfies $\tilde{\mathscr{J}}^* < \min \mathscr{J}$. Then there exists a sequence $\{u^*_{N_1}(j)\}$ such that $L_{N_1}u^*_{N_1}(j)) = \tilde{B}\alpha^*(j)$. However, taken as a control sequence for Problem 1

$$u^*_N = [u^*_{N_1}(1)^T, u^*_{N_1}(2)^T, \ldots, u^*_{N_1}(N_2)^T]^T$$

would then yield $\mathscr{J}^* < \min \mathscr{J}$, which is impossible. Thus $\{\alpha^o(j)\}$ must be an optimal input sequence of Problem 2, and the optimal cost of Problem 2 is equal to that of Problem 1. □

Confirmation that Problem 1 admits an optimal solution may be obtained by directly testing positive definiteness of the matrix (34). However, for large scale problems this may be computationally demanding. Another, more attractive way, that fits nicely to the scheme of decomposition into subproblems and an overlying problem, is given by the following proposition, which may be regarded a converse of Lemma 4 combined with Lemma 5.

Proposition 6: Suppose that Problem 2 and the subproblems, Problem 3, have unique optimal solutions. Then the original problem, Problem 1, also has a unique optimal solution.

Proof: See [12]. □

Irrespectively of the existence of a solution to the original problem, the subproblems may be formulated. Hence Lemma 4, Lemma 5, Proposition 6 and Theorem 3 lead to the following nesting procedure. It is similar to that of [1], except that it also provides a decomposition of the test for existence of solution of the original problem, to the subproblems and the overlaying problem, both of which are smaller in size.

PROCEDURE FOR NESTED OPTIMIZATION:

STEP 1 Formulate the subproblems, and test whether they admit a unique solution (eg. by (34). If so, continue with Step 2; otherwise stop, since no unique solution to the original problem exists either.

STEP 2 Apply Theorem 3 to any of the subproblems, Problem 3, to compute the matrices $T_{N_1}, V_{N_1}, C_{N_1}, D_{N_1}$ mapping the boundary conditions (initial and terminal states) to its optimal control and optimal output. Note that the boundary condition need not be specified at this stage, and that the matrices $T_{N_1}, V_{N_1}, C_{N_1}, D_{N_1}$ are common to all subproblems.

STEP 3 Form the system (38) of the overlying problem, Problem 2, using C_{N_1}, D_{N_1} and the original system matrix A. Test whether the overlying problem admits a unique solution (eg. by (34). If so, continue with Step 4; otherwise stop, since no unique solution to the original problem exists either.

STEP 4 Apply Theorem 3 to the overlying problem to obtain its unique optimal state trajectory $\{\tilde{x}^o(j)\}$. By Lemma 5, $\tilde{x}^o(j)$ is then the optimal state of original problem at time instance jN_1, that is, $\tilde{x}^o(j) = x^o(jN_1)$, $j = 0, 1, \ldots, N_2$.

STEP 5 For the j-th subproblem, Problem 3, set the initial state equal to $\{\tilde{x}^o(j)\}$ and the final state sharply assigned to $\{\tilde{x}^o(j+1)\}$ (the boundary conditions). By Theorem 3 the optimal control of the j-th subproblem is then

$$u^o_{N_1}(j) = T_{N_1}\tilde{x}^o(j) + V_{N_1}\tilde{x}^o(j+1) \tag{45}$$

Compute (45) for $j = 0, 1, \ldots, N_2 - 1$.

STEP 6 Obtain the optimal control of the original problem as

$$u^o_N = [u^o_{N_1}(0)^T, u^o_{N_1}(1)^T, \ldots, u^o_{N_1}(N_2-1)^T]^T$$

Similar to [1] the decomposition may be continued, in the following manner. Denote by $ol(1)$ and $sb(1,:)$ the overlying problem, respectively the subproblems, obtained by decomposition of the original problem. Assume that the number of time-steps of the subproblems $sb(1,:)$ is small enough to make them computationally feasible.

If $ol(1)$ is too large to be computationally feasible, a further decomposition of $ol(1)$ may be carried out. More precisely, interpret $ol(1)$ as the original problem, and decompose it into a new overlying problem $ol(2)$, and corresponding subproblems $sb(2,:)$, where the number of time-steps of the subproblems $sb(2,:)$ is chosen small enough to make them computationally feasible. If the new overlying problem $ol(2)$ is computationally feasible, then by the above nesting procedure the solution of $ol(2)$ yields the solution of $ol(1)$, which again by a use of steps 5-6 of nesting procedure yields the solution of the original problem.

On the other hand, if $ol(2)$ is too large to be computationally feasible, the process may be repeated with $ol(2)$ taking the role of the original problem. This yields a new overlying problem $ol(3)$, and corresponding subproblems $sb(3,:)$, where the number of time-steps of the subproblems $sb(3,:)$ again is chosen small enough to make them computationally feasible. If the new overlying problem $ol(3)$ is computationally feasible, then by an application of the nesting procedure the solution of $ol(3)$ yields the solution of $ol(2)$, which again by a use of steps 5-6 of the nesting procedure yields the solution of $ol(1)$; which by a further use of steps 5-6 of the nesting procedure yields the solution of the original problem. In this manner the nesting may be continued until a computationally feasible overlying problem is obtained.

Remark 3: To further ease the computation, the matrices C_{N_1} and D_{N_1} of row dimension qN_1 may be replaced by corresponding matrices of row dimension equal to $\mathrm{rank}[C_{N_1}\ \ D_{N_1}] \le 2n$ (see [1]).

VII. CONCLUSION

A pseudo-inverse approach to discrete-time finite horizon LQ-optimal control with indefinite cost has been proposed. Contrary to the definite case, an optimal solution does not always exist. Precise conditions for the existence of a unique optimal solution are derived. Both testing of existence of the solution, and its computation, decompose into problems of successively smaller size, by application of a nesting procedure. The nesting procedure is similar to that of [1], but is augmented with the decomposed existence test.

Only the case of a unique optimal solution is treated. A study of the general situation of non-unique solutions and their parametrization, is in progress.

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