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A Polynomial Matrix Approach for the Solution of the Robust Morgan's Problem via Restricted state Feedback

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***Abstract*— The robust Morgan's problem, via restricted static state feedback and constant precompensator, is studied for nonsquare linear time-invariant systems with nonlinear uncertain structure, using a polynomial matrix approach. Sufficient solvability conditions are established. A class of independent from the uncertainties static controllers solving the problem is explicitly characterized and the respective decoupled closed loop system is analytically determined.**

I. INTRODUCTION

MORGANS problem, namely the nonsquare decoupling problem, has been widely investigated for the last decades (see f.e. [1]-[7], [12]-[16], [18]). Morgan's problem is of grate importance for both theoretical analysis and performance improvement of practical systems (f.e. interconnected systems, large industrial processes, flight control). Despite the numerous contributions, the non uncertain Morgan's problem has not as yet completely been solved via static controllers. Necessary and sufficient conditions have been established only for the case of restricted static state feedback ([6], [2]), i.e. the case where the feedback matrix is limited to belong to the image of the rectangular static precompensator. For systems involving uncertainties, the respective results are limited to those in [7] (linear uncertainties). In [11] the problem is defined in state space description and sufficient conditions for its solution have been derived using a pure state space approach.

The polynomial approach [17] or transfer function approach [19] appears to have advantages and disadvantages as compared to the state space approach. Indisputably, in many industrial applications the processes are modeled in the so called Rosenbrock's system matrix form [17]. The polynomial approach appears to facilitate the determination of the closed loop system transfer function and its stability properties. In this paper, motivated by these characteristics as well as the ambition to enlighten alternative sides of the Morgan's problem, we study the robust Morgan's problem using a polynomial matrix approach. A static and restricted state feedback controller is applied. Simple algebraic solvability conditions are derived. The feedback matrices and the diagonal elements of the respective decoupled closed loop system are directly derived.

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II. PROBLEM FORMULATION

Consider the controllable linear time-invariant system with non linear uncertain structure, described in state space form:

$$\dot{x}(t) = A(q)x(t) + B(q)u(t) \ , \ y(t) = C(q)x(t) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ $(m \geq p)$. The matrices $A(q) \in [\wp(q)]^{n\times n}$, $B(q) \in [\wp(q)]^{n\times m}$, $C(q) \in [\wp(q)]^{p\times n}$ are function matrices depending upon the uncertainty vector $q = [q_1 \ \cdots \ q_l] \in \mathbb{Q}$ ($\mathbb{Q}$ denotes the uncertain domain). The set $\wp(s,q)$ is the set of nonlinear functions of s and q. The uncertainties do not depend upon the time. With regard to the nonlinear structure of the system matrices $A(q), B(q)$, $C(q)$ no limitation or specification is assumed (f.e. boundness, continuity etc.). The system (1) can be described in Rosenbrock's system matrix form [17] as follows

$$P(s,q) = \begin{bmatrix} T(s,q) & I_m \\ -V(s,q) & 0 \end{bmatrix} \tag{2}$$

where $T(s,q) \in \wp[s,q]^{m\times m}$ is a column proper uncertain polynomial matrix having *i*-th column degree equal to the *i*-th controllability index, let $\lambda_i(q) (i = 1,2,\ldots,m)$ and where $V(s,q) \in \wp[s,q]^{p\times m}$ is an uncertain polynomial matrix having its *i*-th column degree strictly less than λ_i. To system (1) apply the restricted static state feedback law

$$u(t) = K\Phi x(t) + K\omega(t) \tag{3}$$

where $K \in \mathbb{R}^{m\times p}$ and $\Phi \in \mathbb{R}^{p\times n}$ are matrices independent from q while $\omega(t) \in \mathbb{R}^p$ is the external input vector. The design requirement, namely the Robust Morgan's problem, is the diagonalization of the transfer function of the closed loop system for all q. For the special category of static state feedback with regular static precompensator $(p = m)$, the problem has extensively been solved in [8]. Here, the problem is studied for $p < m$. Note that the uncertainties may not allow the existence of p inputs, corresponding to a $p \times p$ decouplable subsystem.

Definition 1. The robust Morgan's problem is solvable for the system (2), via the restricted static state feedback (3), if there exist matrices Φ and K (independent from q) such that the transfer function of the closed loop system is diagonal and invertible, for every $q \in \mathbb{Q}$. □

I. Transformation of the Problem

In order to solve the problem, some transformations of the feedback matrices are introduced. Consider an independent from q, $m\times(m-p)$ matrix K_0, having the property that the columns of the matrix are linear independent among themselves as well as linear independent with regard to the column of K. Clearly, such a matrix always exist, thus leading to an invertible matrix $G=[K \mid K_0]$. Apply the feedback law $u(t)=K\Phi x(t)+G\omega_g(t)$ to the open loop system (2) the transfer function matrix of the closed loop system can be expressed as $P_c(s,q)=\begin{bmatrix} T_c(s,q) & I_m \\ -V_c(s,q) & 0\end{bmatrix}$, where $T_c(s,q)=G^{-1}\{T(s,q)-K\Phi S(s,q)\}$, $V_c(s,q)=V(s,q)$ and $S(s,q)=\underset{\mathrm{i=1,...,m}}{\mathrm{diag}}\{s_i(s,q)\}$, $s_i(s,q)=\begin{bmatrix}1 & s & \cdots & s^{\lambda_i(q)-1}\end{bmatrix}^T$.

According to the above transformations the robust Morgan's problem for system (1) is solvable via the feedback law $u(t)=K\Phi x(t)+G\omega_g(t)$ if the closed loop transfer function is:

$$V(s,q)\left[T_c(s,q)\right]^{-1}=\left[\underset{\mathrm{i=1,...,p}}{\mathrm{diag}}\{h_i(s,q)\} \;\middle|\; R(s,q)\right],\ \ \forall q\in\mathbb{Q} \quad (4)$$

with $h_i(s,q)\neq 0, \forall q\in\mathbb{Q}$ and where $R(s,q)$ is a strictly proper uncertain matrix of dimension $p\times(m-p)$. Defining $\begin{matrix} p\{ \\ m-p\{ \end{matrix}\begin{bmatrix}\Gamma_1\\ \Gamma_2\end{bmatrix}=\Gamma=G^{-1}$, equation (4) under the condition $\det\Gamma\neq 0$ can be rewritten equivalently as follows

$$\begin{aligned} V(s,q)[T(s,q)]^{-1} &= \underset{i=1,...,p}{\mathrm{diag}}\{h_i(s,q)\} \\ &\quad \left\{\Gamma_1-\Phi S(s,q)[T(s,q)]^{-1}\right\}+R(s,q)\Gamma_2 \end{aligned} \quad (5)$$

Define the matrix

$$G_a(q)=\begin{bmatrix} g_{a,1}(q)\\ \vdots \\ g_{a,p}(q)\end{bmatrix};\ g_{a,i}=\lim_{s\to\infty}\left\{s^{d_i(q)+1}V_i(s,q)\left[T(s,q)\right]^{-1}\right\}$$

where $V_i(s,q)$ is the i-th row of the $V(s,q)$ and where

$$d_i(q)=\begin{cases}\min\{j/\lim\limits_{s\to\infty}\left\{s^{j+1}V_i(s,q)\left[T(s,q)\right]^{-1}\right\}\neq 0 \quad \forall q\in\mathbb{Q} \ , \ j=0,...,n-1\\ n-1 \ if \lim\limits_{s\to\infty}\left\{s^{j+1}V_i(s,q)\left[T(s,q)\right]^{-1}\right\}=0 \quad \forall q\in\mathbb{Q} \ , \quad j\leq n\end{cases}$$

For the uncertain system case the rank properties of $G_a(q)$ vary upon the values of the uncertainties. So, without loss of generality we may divide the uncertainty domain $\mathbb{Q}$ in to two subdomains, let $\tilde{\mathbb{Q}}$ and $\hat{\mathbb{Q}}$, $\mathbb{Q}=(\tilde{\mathbb{Q}}\cup\hat{\mathbb{Q}})$. The elements of the first domain $\tilde{\mathbb{Q}}$ are characterized by the property:

$$\mathrm{rank}\left[G_a(q)\right]=p \ , \ \forall q\in\tilde{\mathbb{Q}} \quad (6)$$

The second domain $\hat{\mathbb{Q}}=\mathbb{Q}-\tilde{\mathbb{Q}}$ contains all the elements of $\mathbb{Q}$ for which the property $\mathrm{rank}\left[G_a(q)\right]=p$ falls. Due to the rank properties of $G_a(q)$ the uncertainties $q\in\tilde{\mathbb{Q}}$ are called regular uncertainties, while the uncertainties $q\in\hat{\mathbb{Q}}$ are called singular uncertainties.

II. Solution of the Robust Morgan's problem for Regular Uncertainties

For the domain of regular uncertainties ($q\in\tilde{\mathbb{Q}}$) define

$$\begin{aligned} j^*(i,q)&=\begin{cases}\min\left[j\in\{1,...,m\}/g_{a,i}^{(j)}(q)\neq 0\right] \\ \quad 0 \ if \ \ g_{a,i}^{(j)}(q)=0\end{cases}, q\in\tilde{\mathbb{Q}} \\ \upsilon_i(q)&=\begin{cases} g_{a,i}^{(j^*(i,q))}(q) & if \quad j^*(i,q)\neq 0 \\ 1 & if \quad j^*(i,q)=0\end{cases}, q\in\tilde{\mathbb{Q}}\end{aligned} \quad (7)$$

where $g_{a,i}^{(j)}(q)$ is the j-th element of $g_{a,i}(q)$. Assume that (6) holds. Choose the rational matrix $R(s,q)=0$. According to this choice the equation (5) takes on the form

$$\underset{i=1,...,p}{\mathrm{diag}}\left\{\left[h_i(s,q)\right]^{-1}\right\}V(s,q)[T(s,q)]^{-1}=\Gamma_1-\Phi S(s,q)[T(s,q)]^{-1} \quad (8)$$

Consider the i-th row of (8) ($i=1,...,p\,(q\in\tilde{\mathbb{Q}})$), i.e.

$$\tilde{p}_i(s,q)s^{d_i(q)+1}V_i(s,q)\left[T(s,q)\right]^{-1}=\gamma_i-\phi_i S(s,q)\left[T(s,q)\right]^{-1} \quad (9)$$

where $\tilde{p}_i(s,q)=\tilde{p}_{i,0}(q)+\tilde{p}_{i,1}(q)s^{-1}+\cdots=s^{-d_i(q)-1}\left[h_i(s,q)\right]^{-1}$ and where $\gamma_i\,(i=1,...,m)$ and $\phi_i\,(i=1,...,p)$ are the i-th rows of the matrices Γ and Φ, respectively. Application of the limit operator to both sides of (9), for $s\to\infty$, yields $\gamma_i=\tilde{p}_{i,0}(q)g_{a,i}(q)$. Based on the latter expression and definition (7), the following lemma, can be used.

Lemma 1. The vector $\gamma_i=\tilde{p}_{i,0}(q)g_{a,i}(q)$ is independent from $q\,(q\in\tilde{\mathbb{Q}})$, via an appropriate $\tilde{p}_{i,0}(q)$, if and only if there exist an independent from q vector $m_i\in\mathbb{R}^{1\times m}$ such that $g_{a,i}(q)=\upsilon_i(q)m_i\ (\forall q\in\tilde{\mathbb{Q}})$. □

From (7) it holds that $\upsilon_i(q)\neq 0\ (\forall q\in\tilde{\mathbb{Q}})$. Choosing $\tilde{p}_{i,0}(q)=p_i^*\left[\upsilon_i(q)\right]^{-1}$ with $p_i^*\in\mathbb{R}-\{0\}$ and using Lemma 1, an independent from the uncertainties matrix Γ_1 is:

$$\Gamma_1=\Lambda_0 M\ ;\ \Lambda_0=\underset{i=1,...,p}{\mathrm{diag}}\{p_i^*\},M=\begin{bmatrix}m_1\\ \vdots\\ m_p\end{bmatrix}, m_i=\left[\upsilon_i(q)\right]^{-1}g_{a,i}(q) \quad (10)$$

Substitute the solution for Γ_1 derived in (10) to (9), yields

$$s^{d_i(q)+1}\left(p_i^*\right)^{-1}\tilde{p}_i(s,q)V_i(s,q)=m_iT(s,q)-\overline{\phi}_iS(s,q) \quad (11)$$

where $\bar{\phi}_i = (p_i^*)^{-1}\phi_i$. Define the invertible matrix

$$V_i^+(s,q) = \left[\left[V_i(s,q)\right]^T \left\{ V_i(s,q)\left[V_i(s,q)\right]^T \right\}^{-1} \,\middle|\, \left[V_i(s,q)\right]^\perp \right] \quad (12)$$

where $\left[V_i(s,q)\right]^\perp$ is the right orthogonal of $V_i(s,q)$. The matrix $\left[V_i(s,q)\right]^\perp$ is chosen to be polynomial. Clearly, such a polynomial matrix can always be constructed with order less or equal to $\lambda_i(q)-1$. Premultipling both sides of (11) by $V_i^+(s,q)$ equation (11) brakes to the following equations

$$s^{d_i(q)+1}\left(p_i^*\right)^{-1}\tilde{p}_i(s,q) = \left[m_i T(s,q) - \bar{\phi}_i S(s,q)\right] \times \left[V_i(s,q)\right]^T \left\{ V_i(s,q)\left[V_i(s,q)\right]^T \right\}^{-1} \quad (13)$$

$$\left[m_i T(s,q) - \bar{\phi}_i S(s,q)\right]\left[V_i(s,q)\right]^\perp = 0 \quad (14)$$

Expanding $S(s,q)\left[V_i(s,q)\right]^\perp$ and $m_i T(s,q)\left[V_i(s,q)\right]^\perp$ in Laurent series as follows

$$W_i(s,q) = S(s,q)\left[V_i(s,q)\right]^\perp = W_{i,1}(q)s^{\rho_i(q)-1} + W_{i,2}(q)s^{\rho_i(q)-2} + \cdots + W_{i,\rho_i(q)}(q)$$

$$\Psi_i(s,q) = m_i T(s,q)\left[V_i(s,q)\right]^\perp = \Psi_{i,0}(q)s^{\rho_i(q)} + \Psi_{i,1}(q)s^{\rho_i(q)-1} + \cdots + \Psi_{i,\rho_i(q)}(q)$$

where $\rho_i(q) \le \max_{i=1,\ldots,m}\left[\lambda_i(q)\right] + \lambda_i(q) - 1$ and equating like power of s in (14), the following, equivalent to (14), algebraic set of equations is derived:

$$\Psi_{i,0}(q) = 0 \quad (15a)$$

$$\bar{\phi}_i W_i(q) = \Psi_i(q) \quad \left(i = 1,\ldots,p\right) \quad (15b)$$

where

$W_i(q) = \left[W_{i,1}(q) \ \cdots \ W_{i,\rho_i(q)}(q)\right]$, $\Psi_i(q) = \left[\Psi_{i,1}(q) \ \cdots \ \Psi_{i,\rho_i(q)}(q)\right]$

According to (6) and (10) equation (15a) is always satisfied.

Defining the operator $[\cdot]_{\mathbb{R}}^\perp$ denoting an independent from q matrix which is orthogonal to the argument matrix, the operator $\mathrm{rank}_{\mathbb{R}}[\cdot]$ denoting the rank of an uncertain matrix on the field of real numbers and the operator $\langle \cdot \ \backslash \ \cdot \rangle_{\mathbb{R}}$ denoting the projection (in the field of real numbers) of an uncertain vector to the subspace defined by the rows of the uncertain matrix (for more details see [8]-[10]) the following lemmas are is presented.

Lemma 2. The robust Morgan's problem for regular uncertainties, under the conditions (6), is solvable if

$$\mathrm{rank}_{\mathbb{R}}\left[g_{a,i}(q)\right]^T = 1, \ i = 1,\ldots,p \quad \left(q \in \tilde{\mathbb{Q}}\right) \quad (16a)$$

$$\mathrm{rank}_{\mathbb{R}}\begin{bmatrix} W_i(q) \\ \Psi_i(q) \end{bmatrix} = \mathrm{rank}_{\mathbb{R}}\left[W_i(q)\right], \ i = 1,\ldots,p \quad \left(q \in \tilde{\mathbb{Q}}\right) \quad (16b)$$

Proof: According to Lemma 1 the matrix Γ_1 that satisfies equation (8) is independent from q if condition (16a) is satisfied (see [8]-[10]). According to (15b) and in order to find an independent from the uncertainties feedback matrix Φ condition (16b) must be satisfied (see [8]-[10]). ■

Lemma 3. If conditions of Lemma 2 are satisfied then the general solution of the compensator matrices K and Φ are

$$K = M^+\Lambda_0^{-1} \quad , \quad \Phi = \Lambda_0\left(H + \Lambda H^*\right) \quad (17)$$

where

$$H = \begin{bmatrix} \left\langle \Psi_1(q) \ \backslash \ W_1(q) \right\rangle_{\mathbb{R}} \\ \vdots \\ \left\langle \Psi_p(q) \ \backslash \ W_p(q) \right\rangle_{\mathbb{R}} \end{bmatrix}, \ H^* = \begin{bmatrix} \left[W_1(q)\right]_{\mathbb{R}}^\perp \\ \vdots \\ \left[W_p(q)\right]_{\mathbb{R}}^\perp \end{bmatrix},$$

$$M^+ = M^T\left(MM^T\right)^{-1}, \ \Lambda_0 = \mathrm{diag}\left\{p_i^*\right\},$$

$$\Lambda = \mathrm{diag}\left\{\lambda_i\right\}\left(\lambda_i \in \mathbb{R}^{1\times l_i}\right), i = 1,\ldots,p, \ l_i = n - \mathrm{rank}_{\mathbb{R}}\left[W_i(q)\right]$$

Proof: According to [8] the general solution of the equation (15b) for independent from q vector $\bar{\phi}_i$, is

$$\bar{\phi}_i = \lambda_i\left[W_i(q)\right]_{\mathbb{R}}^\perp + \left\langle \Psi_i(q) \ \backslash \ W_i(q) \right\rangle_{\mathbb{R}} \quad (18)$$

where $\lambda_i = \left[\lambda_{i,1} \ \cdots \ \lambda_{i,l_i}\right]$ are arbitrary. Substitution of (18) in $\phi_i = p_i^*\bar{\phi}_i$ and collecting the p rows of ϕ_i, the matrix Φ is derived as in (17). For the condition $\det\Gamma \neq 0$ to be satisfied, it is necessary that $\det\Lambda_0 \neq 0$. Since K_0 has been chosen to be orthogonal to the matrix K it can readily be proven that the matrix Γ_2 belongs to the subspace being orthogonal to Γ_1, i.e. that $\Gamma_1\Gamma_2^T = 0$ and consequently that

$$\Gamma_2 = \Lambda_2\left(M^\perp\right)^+ \quad (19)$$

where the $M^\perp$ is the right orthogonal of the independent from q matrix M, $\left(M^\perp\right)^+ = \left[\left(M^\perp\right)^T M^\perp\right]^{-1}\left(M^\perp\right)^T$ and Λ_2 is arbitrary. For $\det\Gamma \neq 0$ to be satisfied it is necessary that $\det\Lambda_2 \neq 0$. According to (10), (19) the matrix G is

$$G = \begin{bmatrix} \Lambda_0 M \\ \Lambda_2\left(M^\perp\right)^+ \end{bmatrix}^{-1} = \left[M^+\Lambda_0^{-1} \ \middle| \ M^\perp\Lambda_2^{-1}\right]$$

and consequently the matrix K is derived as in (17) ■

According to (13) and (17) the diagonally decoupled closed loop system for regular uncertainties is derived to be

$$h_i(s,q) = p_i^*\left\{\left[\left[\upsilon_i(q)\right]^{-1} g_{a,i}(q)T(s,q) - \left(\eta_i + \lambda_i\eta_i^*\right)S(s,q)\right] \times \left[V_i(s,q)\right]^T\left\{V_i(s,q)\left[V_i(s,q)\right]^T\right\}^{-1}\right\}^{-1}$$

where η_i, η_i^* are the *i*-th rows of H and H^*, respectively.

III. Solution of the Robust Morgan's problem for Singular Uncertainties

In this subsection, the robust Morgan's problem is studied for $q \in \hat{\mathbb{Q}}$, under the assumption that it is already solvable for regular uncertainties. Clearly, for both cases (regular and singular uncertainties) the robust Morgan's problem must be solvable via the same static and independent from q restricted state feedback law. Consider the equation (5) describing also the robust Morgan's problem at $q \in \hat{\mathbb{Q}}$. The matrices Γ_1, Φ and Γ_2 are according to (10), (17) and (19). Thus, for the singular uncertainties (5) takes on the form

$$V(s,q)[T(s,q)]^{-1} = \operatorname*{diag}_{i=1,\dots,p}\{h_i(s,q)\}\times \left[\Lambda_0 M - \Lambda_0\left(H+\Lambda H^*\right)S(s,q)[T(s,q)]^{-1}\right] + R(s,q)\Lambda_2\left(M^{\perp}\right)^{+} \quad (20)$$

Post multiplying (20) by the invertible matrix $\left[M^{+} \mid M^{\perp}\right]$ the following equations are derived

$$V(s,q)[T(s,q)]^{-1}M^{+} = \operatorname*{diag}_{i=1,\dots,p}\{h_i(s,q)\}\times \times\left[\Lambda_0 - \Lambda_0\left(H+\Lambda H^*\right)S(s,q)[T(s,q)]^{-1}M^{+}\right] \quad (21)$$

$$R(s,q) = V(s,q)[T(s,q)]^{-1}M^{\perp}(\Lambda_2)^{-1} + \operatorname*{diag}_{i=1,\dots,p}\{h_i(s,q)\}\Lambda_0\left(H+\Lambda H^*\right)S(s,q)[T(s,q)]^{-1}M^{\perp}(\Lambda_2)^{-1} \quad (22)$$

The problem has now been reduced to that of finding Λ_0, Λ, $\operatorname*{diag}_{i=1,\dots,p}\{h_i(s,q)\}$ that solve (21), while the $R(s,q)$ is given by (22) and Λ_2 is arbitrary. Brake (21) in to rows to yield

$$E_i(s,q) = h_i(s,q)p_i^*\left[e_i - \left(\eta_i + \lambda_i\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\right)\bar{E}(s,q)\right] \quad (23)$$

where e_i is the unity vector having the unit at its *i*-th position and where

$$E_i(s,q) = V_i(s,q)[T(s,q)]^{-1}M^{+} = E_{i,0}(q)s^{-1} + E_{i,1}(q)s^{-2} + \cdots$$

$$\bar{E}(s,q) = S(s,q)[T(s,q)]^{-1}M^{+} = \bar{E}_0(q)s^{-1} + \bar{E}_1(q)s^{-2} + \cdots$$

Define

$$\varepsilon_i(q) = \begin{cases} \min\{j / E_{i,j}(q) \neq 0 \quad \forall q \in \hat{\mathbb{Q}} \quad , \quad j = 0,\dots,n-1 \\ n-1 \text{ if } E_{i,j}(q) = 0 \quad \forall q \in \hat{\mathbb{Q}} \quad , \quad j \le n \end{cases} \quad (24)$$

According to definition (24) and equation (23) the relative degree of $h_i(s,q)$ is $\varepsilon_i(q)+1$. Define

$$h_i(s,q) = h_{i,0}(q)s^{-\varepsilon_i(q)-1} + h_{i,1}(q)s^{-\varepsilon_i(q)-2} + \cdots ; h_{i,0}(q) \neq 0 \quad (25)$$

$$\hat{p}_i(s,q) = s^{-\varepsilon_i(q)-1}\left[h_i(s,q)\right]^{-1} \quad (26)$$

Clearly, $\hat{p}_i(s,q)$ must be proper i.e. it must hold that $\hat{p}_i(s,q) = \hat{p}_{i,0}(q) + \hat{p}_{i,1}(q)s^{-1} + \cdots \left(\hat{p}_{i,0}(q) \neq 0, \forall q \in \hat{\mathbb{Q}}\right)$.

Expanding (23) in negative power of s and equating the first $2n+1$ like power of s the following algebraic equations are derived

$$\hat{p}_{i,0}(q)E_{i,\varepsilon_i(q)}(q) = p_i^* e_i \quad (27)$$

$$\left[\hat{p}_{i,0}(q) \mid \hat{p}_{i,1}(q) \ \cdots \ \hat{p}_{i,2n}(q)\right]\left[\frac{\Pi_i(q)}{\hat{N}_i(q)}\right] = = p_i^*\left(\eta_i + \lambda_i\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\right)\Pi(q) \quad (28)$$

where $\Pi_i(q) = \left[E_{i,\varepsilon_i(q)+1}(q) \quad E_{i,\varepsilon_i(q)+2}(q) \quad \cdots \quad E_{i,\varepsilon_i(q)+2n}(q)\right]$, $\Pi(q) = \left[\bar{E}_0(q) \quad \bar{E}_1(q) \quad \cdots \quad \bar{E}_{2n-1}(q)\right]$ and where

$$\hat{N}_i(q) = \begin{bmatrix} E_{i,\varepsilon_i(q)}(q) & E_{i,\varepsilon_i(q)+1}(q) & \cdots & E_{i,\varepsilon_i(q)+2n-1}(q) \\ 0 & E_{i,\varepsilon_i(q)}(q) & \cdots & E_{i,\varepsilon_i(q)+2n-2}(q) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{i,\varepsilon_i(q)}(q) \end{bmatrix}$$

Defining $\hat{\phi}_i(q) = \hat{p}_{i,0}(q)\Pi_i(q) - p_i^*\left(\eta_i + \lambda_i\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\right)\Pi(q)$ equation (28) may be rewritten as follows

$$\hat{\phi}_i(q) + \left[\hat{p}_{i,1}(q) \ \cdots \ \hat{p}_{i,2n}(q)\right]\hat{N}_i(q) = 0 \quad (29)$$

To this end for the domain of singular uncertainties define

$$\hat{N}(q) = \begin{bmatrix} Z_0(q) & Z_1(q) & \cdots & Z_{2n-1}(q) \\ 0 & Z_0(q) & \cdots & Z_{2n-2}(q) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z_0(q) \end{bmatrix};$$

$$Z_j(q) = \begin{bmatrix} E_{1,\varepsilon_1(q)+j}(q) \\ \vdots \\ E_{p,\varepsilon_p(q)+j}(q) \end{bmatrix} \left(j = 0,1,\dots,2n-1\right)$$

According to (27) it holds that $\det\left[Z_0(q)\right] \neq 0$ and $\hat{N}_i(q)\left[\hat{N}(q)\right]^{-1} = \operatorname*{diag}_{i=1,\dots,2n}\left[e_i\right]$. Also define

$$k^*(i,q) = \begin{cases} \min\left[j \in \{1,\dots,p\} : E_{i,\varepsilon_i(q)}^{(j)} \neq 0\right] \\ 0 \ \text{if} \ E_{i,\varepsilon_i(q)}^{(j)} = 0 \end{cases}; \quad (30a)$$

$E_{i,\varepsilon_i(q)}^{(j)} : j-th$ element of $E_{i,\varepsilon_i(q)}$

$$f_i(i,q) = \begin{cases} E_{i,\varepsilon_i(q)}^{(k^*(i,q))} & \text{if} \quad k^*(i,q) \neq 0 \\ 1 & \text{if} \quad k^*(i,q) = 0 \end{cases} \quad (30b)$$

Post multiplying both sides of (29) by $\left[\hat{N}(q)\right]^{-1}$, yields

$$\hat{p}_{i,0}(q)\Theta_i(q) = p_i^*\left(\eta_i + \lambda_i\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\right)\Theta(q) \quad (31)$$

$$\begin{bmatrix} \hat{p}_{i,1}(q) & \cdots & \hat{p}_{i,2n}(q) \end{bmatrix} = \\ = p_i^*\left(\eta_i + \lambda_i\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\right)\bar{\Delta}_i(q) - \hat{p}_{i,0}(q)\Delta_i(q) \tag{32}$$

where

$$\Theta_i(q) = \left[\bar{\Theta}_{i,1}(q) \mid \bar{\Theta}_{i,2}(q) \mid \cdots \mid \bar{\Theta}_{i,2n}(q)\right]$$

$$\bar{\Theta}_{i,j}(q) = \left[\theta_{i,j}^{(1)}(q) \mid \theta_{i,j}^{(2)}(q) \mid \cdots \mid \theta_{i,j}^{(j-1)}(q) \mid 0 \mid \theta_{i,j}^{(j+1)}(q) \mid \cdots \mid \theta_{i,j}^{(p)}(q)\right]$$

$$\Theta(q) = \left[\Theta_1^*(q) \mid \Theta_2^*(q) \mid \cdots \mid \Theta_{2n}^*(q)\right]$$

$$\Theta_j^*(q) = \left[\bar{\Pi}_j^{(1)}(q) \mid \bar{\Pi}_j^{(2)}(q) \mid \cdots \mid \bar{\Pi}_j^{(j-1)}(q) \mid 0 \mid \bar{\Pi}_j^{(j+q)}(q) \mid \cdots \mid \bar{\Pi}_j^{(p)}(q)\right]$$

$$\Delta_i(q) = \left[\theta_{i,1}^{(i)}(q) \quad \theta_{i,2}^{(i)}(q) \quad \cdots \quad \theta_{i,2n}^{(i)}(q)\right]$$

$$\bar{\Delta}_i(q) = \left[\bar{\Pi}_1^{(i)}(q) \quad \bar{\Pi}_2^{(i)}(q) \quad \cdots \quad \bar{\Pi}_{2n}^{(i)}(q)\right]$$

with $\theta_{i,j}^{(k)}(q)\,(k = 1,...,p)$ being the *k*-th element of the *j*-th block of $\Pi_i(q)\left[\hat{N}(q)\right]^{-1}$ and with $\bar{\Pi}_j^{(k)}(q)\ (k = 1,...,p)$ being the *i*-th columns of $\Pi(q)\left[\hat{N}(q)\right]^{-1}$, respectively.

Lemma 4. The robust Morgan's problem for $q \in \hat{\mathbb{Q}}$ is solvable if conditions of Lemma 2 are satisfied and

$$E_{i,\varepsilon_i(q)}(q) = f_i(q)e_i \quad , \quad i = 1,...,p \tag{33}$$

$$\operatorname{rank}_{\mathbb{R}}\begin{bmatrix} \left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q) \\ \left[f_i(q)\right]^{-1}\Theta_i(q) - \eta_i\Theta(q) \end{bmatrix} = \\ = \operatorname{rank}_{\mathbb{R}}\left[\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q)\right],\ i = 1,...,p \tag{34}$$

Proof: According to [8] the equation (27) is satisfied with p_i^* independent from q $\left(\hat{p}_{i,0}(q) \neq 0, \forall q \in \hat{\mathbb{Q}}\right)$ if and only if condition (33) is satisfied. For (27) to be solvable for p_i^* independent from q, it is necessary for $\hat{p}_{i,0}(q)$ to be of the form $\hat{p}_{i,0}(q) = \left[f_i(q)\right]^{-1}\tau_{0,i}$ while the solution for p_i^* is $p_i^* = \tau_{0,i}$, where $\tau_{0,i} \in \mathbb{R} - \{0\}$ is arbitrary and independent from q. Substitution of $\hat{p}_{i,0}(q) = \left[f_i(q)\right]^{-1}\tau_{0,i}$ in (31) yields

$$\lambda_i\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q) = \left[f_i(q)\right]^{-1}\Theta_i(q) - \eta_i\Theta(q) \tag{35}$$

Equation (35) is solvable for λ_i independent from q, if and only if (34) is satisfied (see [8]). ■

According to [8] the general solution of the equation (35) is

$$\lambda_i = \tau_i\left[\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q)\right]_{\mathbb{R}}^{\perp} + \zeta_i \quad (\lambda_i \in \mathbb{R}^{1\times l_i}) \\ ;\ \zeta_i = \left\langle\left\{\left[f_i(q)\right]^{-1}\Theta_i(q) - \eta_i\Theta(q)\right\} \quad \backslash \quad \left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q)\right\rangle_{\mathbb{R}} \tag{36}$$

where $\pi_i = l_i - \operatorname{rank}_{\mathbb{R}}\left[\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q)\right]$. The vectors $\tau_i = \left[\tau_{i,1} \ \cdots \ \tau_{i,\pi_i}\right]$ are arbitrary. Hence, if the robust Morgan's problem is solvable for $q \in \tilde{\mathbb{Q}}$ according to (36) and the relation $p_i^* = \tau_{0,i}$ the following lemma is presented

Lemma 5. The general analytical expression of the independent from the uncertainties matrices Λ_0, Λ_2 and Λ for singular uncertainties are

$$\Lambda_0 = \operatorname{diag}\{\tau_{0,i}\}\ ,\ \Lambda = T_0Z^* + Z \quad ,\ i = 1,...,p, \tag{37}$$

where

$$T_0 = \operatorname{diag}\{\tau_i\}\left(\tau_i \in \mathbb{R}^{1\times\pi_i}\right),\ \Lambda_2 \text{ arbitrary}\left(\det\left[\Lambda_2\right] \neq 0\right)$$

$$Z = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_p \end{bmatrix}, Z^* = \begin{bmatrix} \left[\left[W_1(q)\right]_{\mathbb{R}}^{\perp}\Pi_1(q)\right]_{\mathbb{R}}^{\perp} \\ \vdots \\ \left[\left[W_p(q)\right]_{\mathbb{R}}^{\perp}\Pi_p(q)\right]_{\mathbb{R}}^{\perp} \end{bmatrix}$$

■

According (36) and the definitions $p_i^* = \tau_{0,i}$ and $\hat{p}_{i,0}(q) = \left[f_i(q)\right]^{-1}\tau_{0,i}$, equation (32) takes on the form

$$\left[\hat{p}_{i,1}(q) \ \cdots \ \hat{p}_{i,2n}(q)\right] = \\ = \tau_{0,i}\left(\eta_i + \left\{\tau_i\left[\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q)\right]_{\mathbb{R}}^{\perp} + \zeta_i\right\}\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\right)\bar{\Delta}_i(q) \\ - \tau_{0,i}\left[f_i(q)\right]^{-1}\Delta_i(q) \tag{38}$$

Defining $\chi_i(q) = \left(\eta_i + \zeta_i\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\right)\bar{\Delta}_i(q) - \left[f_i(q)\right]^{-1}\Delta_i(q)$ and $\left[R_i(q)\right]_j$ to be the *j*-th row of $\left[\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q)\right]_{\mathbb{R}}^{\perp}\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\bar{\Delta}_i(q)$, (38) takes on the form

$$\left[\hat{p}_{i,1}(q) \ \cdots \ \hat{p}_{i,2n}(q)\right] = \tau_{0,i}\sum_{j=1}^{\pi_i}\left\{\tau_{i,j}\left[R_i(q)\right]_j\right\} + \tau_{0,i}\ \chi_i(q) \tag{39}$$

where $\tau_{i,j}$ is the *j*-th element of τ_i. For every $q \in \hat{\mathbb{Q}}$, there exist a unique bilateral correspondence between a strictly proper rational function (depending nonlinearly upon q, with order less than n and the vector involving the first $2n$ coefficients of the negative power series of s expansion of the rational function. Applying this property to (39) the following general form for $\hat{p}_i(s,q)$ is derived:

$$\hat{p}_i(s,q) = \left[f_i(q)\right]^{-1}\tau_{0,i}\left[1 + \sum_{j=1}^{\pi_i}\left\{\tau_{i,j}r_{i,j}(s,q)\right\} + \eta_i(s,q)\right] \tag{40}$$

where $r_{i,j}(s,q)$ and $\eta_i(s,q)$ be the rational vector functions corresponding to the vectors $f_i(q)\left[R_i(q)\right]_j$ and $f_i(q)\chi_i(q)$ respectively. The rational vector functions $r_{i,j}(s,q)$ and $\eta_i(s,q)$ can be expressed as the ratio of two polynomials as follows $r_{i,j}(s,q) = \mu_{i,j}(s,q)/\alpha_i(s,q)\left(j = 1,...,\pi_i\right)$ and $\eta_i(s,q) = \mu_{i,0}(s,q)/\alpha_i(s,q)$ where $\mu_{i,j}(s,q)$ and $\alpha_i(s,q)$ are prime between themselves. For each $q \in \mathbb{Q}$ the polynomial $\alpha_i(s,q) = s^{\sigma_i} + \alpha_{i,\sigma_i-1}(q)s^{\sigma_i-1} + \cdots + \alpha_{i,0}(q)$ is the least common multiplier of the denominators of $r_{i,j}(s,q)$ and $\eta_i(s,q)$ $\left(j=1,...,\pi_i\right)$. The polynomials $\mu_{i,j}(s,q)$ are of the form $\mu_{i,j}(s,q) = \left(\mu_{i,j}\right)_{\sigma_i-1} s^{\sigma_i-1} + \cdots + \left(\mu_{i,j}\right)_0$ where $\left(\mu_{i,j}\right)_\zeta = \left[\left(\mu_{i,j}\right)_{\zeta,1} \ \cdots \ \left(\mu_{i,j}\right)_{\zeta,i}\right]$ $\left(\zeta = 0,...,\sigma_i - 1\right)$. Based upon the above definitions and (40) the closed loop structure for singular uncertainties is derived to be

$$h_i(s,q) = \left[\tau_{0,i}\right]^{-1} f_i(q) \frac{\alpha_i(s,q)}{\beta_i(s,q)s^{\varepsilon_i(q)+1}} =$$

$$\frac{s^{\sigma_i} + \alpha_{i,\sigma_i-1}(q)s^{\sigma_i-1} + \cdots + \alpha_{i,0}(q)}{\left[s^{\sigma_i} + \beta_{i,\sigma_i-1}(q)s^{\sigma_i-1} + \cdots + \beta_{i,0}(q)\right]s^{\varepsilon_i+1}}, \ \forall q \in \hat{\mathbb{Q}}\left(i=1,...,p\right)$$

$$\beta_{i,k} = \alpha_{i,k} + \left(\mu_{i,0}\right)_k + \sum_{j=1}^{\pi_i} \tau_{i,j}\left(\mu_{i,j}\right)_k \left(k = 0,...,\sigma_i - 1\right).$$

IV. Main Result

According to Lemma 2 and 4 the main Theorem is presented

Theorem 1. The robust Morgan's problem is solvable, via an independent from the uncertainties restricted static state feedback law, if

$$\text{rank}_{\mathbb{R}}\left[g_{a,i}(q)\right]^T = 1, \ i=1,...,p \ \left(\text{for all } q \in \tilde{\mathbb{Q}}\right)$$

$$\text{rank}_{\mathbb{R}}\begin{bmatrix} W_i(q) \\ \Psi_i(q) \end{bmatrix} = \text{rank}_{\mathbb{R}}\left[W_i(q)\right], \ i=1,...,p \ \left(\text{for all } q \in \tilde{\mathbb{Q}}\right)$$

$$E_{i,\varepsilon_i(q)}(q) = f_i(q)e_i, \ i=1,...,p \ \left(\text{for } q \in \hat{\mathbb{Q}}\right)$$

$$\text{rank}_{\mathbb{R}}\begin{bmatrix} \left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q) \\ \left[f_i(q)\right]^{-1}\Theta_i(q) - \eta_i\Theta(q) \end{bmatrix} = $$

$$= \text{rank}_{\mathbb{R}}\left[\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q)\right], \ i=1,...,p \ \left(\text{for } q \in \hat{\mathbb{Q}}\right)$$

■

According to Lemma 3 and 5 the following Theorem is presented

Theorem 2. If conditions of Theorem 1 are satisfied then a class of controller matrices K and Φ solving the robust Morgan's problem $\left(\forall q \in \mathbb{Q}\right)$ is given by $K = M^+\left[\Lambda_0\right]^{-1}$, $\Phi = \Lambda_0\left(H + T_0Z^*H^* + ZH^*\right)$ ■

The resulting form of the robustly decoupled closed loop system for $q \in \mathbb{Q}$ is analytically determined to be

$$h_i(s,q) =$$
$$\tau_{0,i}\left\{\left[\left[\upsilon_i(q)\right]^{-1} g_{a,i}(q)T(s,q) - \left(\eta_i + \left[\tau_i\left[\left[W_i(q)\right]_{\mathbb{R}}^{\perp}\Theta(q)\right]_{\mathbb{R}}^{\perp} + \zeta_i\right]\eta_i^*\right)S(s,q)\right]\right.$$
$$\left.\times\left[V_i(s,q)\right]^T\left\{V_i(s,q)\left[V_i(s,q)\right]^T\right\}^{-1}\right\}^{-1}, \ q \in \tilde{\mathbb{Q}}$$

$$h_i(s,q) = \left[\tau_{0,i}\right]^{-1} f_i(q)\frac{\alpha_i(s,q)}{\beta_i(s,q)s^{\varepsilon_i(q)+1}}, \ q \in \hat{\mathbb{Q}}.$$

Following the results in [11] the robust Morgan's problem with simultaneous robust stability, can easily be solved.

V. Conclusions

The robust Morgan's problem, via restricted static state feedback and constant precompensator, has been studied for nonsquare linear time-invariant systems with nonlinear uncertain structure, using a polynomial matrix approach. Sufficient solvability conditions are established. A class of independent from the uncertainties static controllers solving the problem has explicitly been characterized. Finally the decoupled closed loop system has analytically been determined. Before closing it is important to mention that the results of the paper, namely the sufficient conditions and the class of the controllers and the closed loop system, have been derived through pure algebraic manipulations thus offering their selves for computer implementation.

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