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Some ancestors of contraction analysis

Jérôme Jouffroy
Center for Ships and Ocean Structures (CeSOS)
Norwegian University of Science and Technology (NTNU)
NO-7491 Trondheim, Norway
e-mail: jouffroy@ntnu.no

***Abstract*— Contraction analysis is a recent tool for analyzing the convergence behavior of nonlinear systems in state-space form (see Lohmiller and Slotine [16] for the main reference). However, it seems that earlier results derived by mathematicians in the 1950s closely match some of the results of contraction analysis. In this paper, we review and place into perspective some references of this era, and relate them with contraction.**

I. INTRODUCTION

Contraction analysis, also called contraction theory or simply contraction, is a recent body of results for analyzing the convergence of nonlinear systems trajectories with respect to one another. Its specificities lie in the differential framework under which it is defined, as well as its close connection with Riemannian geometry.

A 1998 paper by Lohmiller and Slotine [16] is generally regarded as the main reference on contraction theory. Also available are the PhD thesis by Lohmiller [13] and the early developments of contraction analysis in the interesting papers by the same authors [14][15].

The concepts of stability and convergence for systems of ordinary differential equations are already relatively ancient, and the discovery of criteria aiming at determining the presence or the absence of such properties has been one of the goals of many researchers whose field of study includes ODEs. Also, as the tools of Riemannian geometry have been applied in many areas of science that involve dynamical systems, one might wonder whether contraction theory could be related in a concrete way with earlier studies.

The goal of this paper is to show that some of the results of contraction theory are older than it is generally assumed. To this end, we review several references that are closely related, in the formulation, the concept, and the results, to contraction theory as presented in Lohmiller and Slotine [16].

After recalling a few results of contraction analysis, we take the 1982 classical textbook on ODEs by Hartman [5] as a starting point and gradually go back in time by presenting the work of several mathematicians whose results are shortly presented and compared with contraction. The oldest reference dates back to 1949. Finally, we conclude this paper with a discussion in the light of this short historical perspective together with the results of contraction analysis.

Parts of the study were presented, albeit in french, in Jouffroy [7, section 2.3, pp. 48-54].

II. CONTRACTION ANALYSIS BY LOHMILLER AND SLOTINE, 1998

In the following, we consider systems described by general nonlinear deterministic differential equations of the form

$$\dot{x} = f(x,t) \tag{1}$$

where x is the n-dimensional vector corresponding the state of the system, t is time, and f is a nonlinear vector field. In addition, we make the further assumption that the system is smooth and that any solution $x(x_0,t)$ initialized in x_0 of (1) exists and is unique. One of the main features of contraction theory is to use the concept of virtual displacements δx of the state x which are infinitesimal displacements at fixed time.

From there, the so-called virtual dynamics are introduced

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x,t)\delta x \tag{2}$$

If now a state dependent local and virtual change of coordinates

$$\delta z = \Theta(x,t)\delta x$$

(where $\Theta(x,t)$ is a nonsingular transformation matrix) is performed on expression (2), the virtual dynamics can be expressed in δz-coordinates as

$$\delta\dot{z} = F(x,t)\delta z$$

where the *generalized Jacobian* F is given by

$$F = \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1}$$

We are now ready to state the main definition of [16]:

Definition 1: Given the system equations $\dot{x} = f(x,t)$, a region of the state space is called a contraction region with respect to a uniformly positive definite metric $M(x,t) = \Theta^T\Theta$, if there exists a strictly positive constant β_M such that

$$F = \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1} \leq -\beta_M\, I \tag{3}$$

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or equivalently

$$\frac{\partial f}{\partial x}^T M + \dot{M} + M\frac{\partial f}{\partial x} \leq -2\beta_M\, M$$

are verified in that region. □

From this definition, Theorem 2 in [16] is stated as follows.

Theorem 1: Given the system equations $\dot{x} = f(x,t)$, any trajectory, which starts in a ball of constant radius with respect to the metric $M(x,t)$, centered at a given trajectory and contained at all times in a contraction region with respect to $M(x,t)$, remains in that ball and converges exponentially to that trajectory. □

Intuitively, the above result means that if the temporal evolution of a virtual displacement tends to zero as time goes to infinity, this being true for all state x and at all time, the whole flow will "shrink" to a point, hence the term "contraction".

The system is said to be *semi-contracting* in the metric if F is only negative semi-definite.

For the sake of comparison with the early publications that will follow in the next sections, let us also briefly mention a few other results.

For example, as stated in Lohmiller and Slotine [16, section 3.7 (vi)], the state of any time-invatiant contracting system driven by a periodic input $\omega(t)$ of period $T > 0$

$$\dot{x} = f(x, \omega(t)) \tag{4}$$

tends exponentially to a periodic signal with the same period.

Additionally, as proposed in Jouffroy [8][7], the explicit presence of the inputs can also be added into the framework of contraction. Indeed, consider now the following class of systems.

$$\dot{x} = f(x, u, t) \tag{5}$$

where u is an input signal. In this case, the virtual dynamics take the form

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x,u,t)\delta x + \frac{\partial f}{\partial u}(x,u,t)\delta u$$

Providing the system (5) is contracting with reference to a uniformly bounded metric M for all input u and that the input Jacobian $\frac{\partial f}{\partial u}$ is also bounded, one can deduce the following ISS-like inequality [7]

$$||\delta x|| \leq k||\delta x_0||e^{-\beta t} + \gamma||\delta u||_{\mathcal{L}_\infty} \tag{6}$$

where $|| \bullet ||_{\mathcal{L}_\infty}$ indicates the supremum norm, and where k, β, γ are three strictly positive constants.

In Wang and Slotine [27][26], the authors show that contraction theory can be used not only to infer convergence of the trajectories of a system, but also of two or more different systems.

Consider for example the two following coupled systems

$$\dot{x}_1 = f(x_1,t) + k(x_2,t) - k(x_1,t) \tag{7}$$

$$\dot{x}_2 = f(x_2,t) + k(x_1,t) - k(x_2,t) \tag{8}$$

where $k(x_i,t)$ represent the coupling forces. Assume that the auxiliary system

$$\dot{x} = f(x,t) - 2k(x,t) + k(x_1,t) + k(x_2,t)$$

is contracting. Then the particular solutions $x = x_1$ and $x = x_2$ converge exponentially to each other.

We are now ready to "go back in time" and introduce the references alluded to in the introduction. Note that in what follows, we purposely avoided too many technicalities to allow the reader to easily compare and relate the different results and concepts. Also, and in the same spirit, the notations adopted in the present paper were unified on the basis of the notations used in Lohmiller and Slotine [16].

III. A TIME-INVARIANT METRIC IN HARTMAN, 1961

In the references of the articles by Lohmiller and Slotine, the name of Hartman comes up on several occasions for his book on differential equations [5]. This work seems to be cited for its relationship to contraction analysis. Among the other publications of Hartman, one can single out an older article [4] addressing the problem of stability analysis of systems of differential equations in Riemann spaces using the formalism of tensors (see [2],[18] or [1]).

An interesting lemma of this article (see Hartman [4, Lemma 2, section 5]), after translation into the notations of contraction, is given below.

Theorem 2: If there exists a metric $M(x) = \Theta^T(x)\Theta(x)$ such that, for the equation $\dot{x} = f(x,t)$, the following inequality is verified

$$\Theta^T\frac{\partial\Theta}{\partial x}f + M\frac{\partial f}{\partial x} < 0 \tag{9}$$

then any solution $x(t)$ exists for large $t > 0$. Moreover, the distance between any couple of trajectories $x_1(t)$ and $x_2(t)$ is decreasing. □

Besides the use of a time-invariant metric tensor $M(x)$, which, in a sense, represents a restriction compared to what is presented in Lohmiller and Slotine [16], the result of Theorem 2 is very close to the definition and theorem of section II. Indeed, the negativity condition of expression (9) is another way of stating that it is actually the symmetrical part of this expression which is under study.

Thus, after a simple computation, we get

$$\left(\Theta^T\frac{\partial\Theta}{\partial x}f + M\frac{\partial f}{\partial x}\right)^T + \Theta^T\frac{\partial\Theta}{\partial x}f + M\frac{\partial f}{\partial x}$$
$$= \frac{\partial f}{\partial x}^T M + \dot{M} + M\frac{\partial f}{\partial x}$$

which is the same expression as the second condition for contracting behavior in Theorem 1.

Another interesting point is that Hartman presents in the same lemma another result that can be considered as the first definition of *semi-contracting* systems. This result can be described by the following corollary.

Corollary 1: If inequality (9) is replaced by

$$\Theta^T \frac{\partial \Theta}{\partial x} f + M \frac{\partial f}{\partial x} \leq 0$$

then the property of existence of $x(t)$ is preserved, and the distance between $x_1(t)$ and $x_2(t)$ is nonincreasing. □

Finally, let us mention that another result of the same article (see Harman [4, Theorem I]) asserts that an autonomous and stable system under the condition (9) has one unique stable equilibrium point, which is a result that is also presented in Lohmiller and Slotine [16, section 3.7 (v)].

IV. The stability criterion of Opial, 1960

The paper by the Polish mathematician Z. Opial [20] represents in our opinion an important reference because it introduces some concepts that are closely connected to contraction in a relatively simple way compared to the other references quoted in this note.

Thus, when Opial defines asymptotic stability, it does not mention the presence of an equilibrium point (for example at the origin of state-space), but just states that in addition to the property of attractivity, two trajectories $x(x_{10}, t)$ and $x(x_{20}, t)$ of a stable system must check the condition

$$\lim_{t \to \infty} \|x(x_{10}, t) - x(x_{20}, t)\| = 0$$

condition which is recognized as an incremental form of convergence.

To address the incremental stability problem, Opial considers two-dimensional systems of the form $\dot{x} = f(x, t)$ for which the length of an arc $C(t)$ between two points $x_1(t)$ and $x_2(t)$ is defined as

$$L(t) = \int_{s=0}^{s=1} \left\| \frac{\partial x}{\partial s}(t, s) \right\| ds$$

where $x(t, s)$ is a parametric definition of $C(t)$, with $0 \leq s \leq 1$ and for $x(t, 0) = x_1(t)$ and $x(t, 1) = x_2(t)$.

Then, he studies the dynamical properties of $v(t, s) \triangleq \partial x(t, s)/\partial s$ that satisfy the following differential equation

$$\frac{dv}{dt} = \frac{\partial f}{\partial x} v$$

which is quite close in spirit to the virtual dynamics (2). As in the previous section, the stability criterion by Opial (see [20, Theorem 2]) consists in saying that, roughly speaking, if the time-derivative of the positive quadratic form $v^T M(x) v$ is uniformly negative definite, where $M(x)$ is a time-invariant metric tensor, then the system under consideration will be asymptotically stable. Thus, while deriving $v^T M(x) v$, it comes

$$v^T \left(\frac{\partial f}{\partial x}^T M + \dot{M} + M \frac{\partial f}{\partial x} \right) v$$

expression which the reader will immediatly recognize as one of the conditions allowing to check that a system is contracting.

V. Comparing different systems in Opial, 1959-1960

In [19] and [21], Opial uses his results from [20] (see previous section) not only to study the behavior of any couple of trajectories of a particular system but also applies them to the comparison of different systems.

In particular, he focuses his attention on the second order differential equation

$$\ddot{x} + F(\dot{x}) + g(x) = p(t) \tag{10}$$

where the functions $g(x)$, $F(y)$ and $p(t)$ are continuous and such that $\lim_{|x| \to \infty} g(x) sgn(x) = +\infty$, $\lim_{|y| \to \infty} F(y) sgn(y) = +\infty$ and $p(t)$ is uniformly bounded by a constant P (see Opial [19]).

The goal here is to analyze the behavior of system (10) when subject to different initial conditions and possibly different input signals $p(t)$. To do so, write the two equations

$$\begin{aligned} \ddot{x}_1 + F(\dot{x}_1) + g(x_1) &= p_1(t) \quad (11) \\ \ddot{x}_2 + F(\dot{x}_2) + g(x_2) &= p_2(t) \quad (12) \end{aligned}$$

where $p_1(t)$ and $p_2(t)$ are two inputs which difference is noted $\Delta p(t) = p_2(t) - p_1(t)$.

To compare (11) with (12), Opial introduces what he refers to as an *auxiliary* system

$$\ddot{x}(t, s) + F(\dot{x}(t, s)) + g(x(t, s)) = p_1(t) + s \Delta p(t)$$

where, as seen previously, $x(t, s)$ is the value of the state at coordinate $0 \leq s \leq 1$ along an arc joining the particular solutions $x(t, 0) = x_1(t)$ and $x(t, 1) = x_2(t)$. Note that in this case, we find respectively systems (11) and (12). The reader will certainly notice the ressemblance of this auxiliary system technique with the one used to show synchronization of the two systems (7) and (8) in section II (see also [9] in relation with the present discussion).

This technique, along with Opial's criterion of stability of [20], is then used to prove the following theorem (from [19, Theorem 1]).

Theorem 3: Assume that the functions $g(x)$, $F(y)$, $p_1(t)$ and $p_2(t)$ are such that

$$F'(y) > 0 \text{ for } |y| \leq B_2$$

$$g'(x) > 0 \text{ for } |x| \leq B_1$$

$$2\min_{|y|\leq B_2}\frac{F'(y)}{|y|} > \max_{|x|\leq B_1}\frac{|g''(x)|}{g'(x)}$$

where the superscript $'$ and $''$ stand for the first and second derivative of the functions under consideration, respectively. Assume that

$$|\Delta p(t)| \leq \varepsilon \tag{13}$$

at all time. Then there exists a constant $M > 0$ such that for any $x_1(t)$ and $x_2(t)$, we have

$$\lim_{t\to\infty}\sup|x_1(t) - x_2(t)| \leq M\varepsilon \tag{14}$$

and

$$\lim_{t\to\infty}\sup|\dot{x}_1(t) - \dot{x}_2(t)| \leq M\varepsilon \tag{15}$$

□

Note from the description of this theorem that it can be seen as being quite close in spirit to the consideration of inputs for contracting systems (see (5) in section II and Jouffroy [8]). Indeed, (14) and (15) stand for the ultimate boundedness induced by the bound on the input signal difference (13) which is reminiscent of the ISS-like inequality (6).

VI. Boundary dynamics in Seifert, 1958

Since Opial's work in [20] seems to be mainly inspired by the 1958 paper of G. Seifert [23], it is no surprise that we find in this paper the same elements as those exposed in section IV.

An interesting specificity, however, is that unlike all the other references reviewed in this short historical perspective of contraction analysis, Seifert does not examin the contraction of the length of a curve between two trajectories $x_1(t)$ and $x_2(t)$ of the system $\dot{x} = f(x,t)$. Rather, he studies the convergence of trajectories in a region R by considering the evolution in time of the length of the closed curve $\Gamma(t)$ materializing the boundary of the region R. He also replaces his approach in the context of an intuitive vision of Riemannian geometry by picturing the closed curve $\Gamma(t)$ as evolving on a surface embedded in the cartesian space $\mathbb{R}^3$. In this case, assuming that the surface is represented by the functions $\Psi(x)$ taking values in $\mathbb{R}^3$, and where $x \in \mathbb{R}^2$ are the coordinates of a point on the surface, the metric tensor $M(x)$ can be written

$$M(x) = \frac{\partial\Psi}{\partial x}^T\frac{\partial\Psi}{\partial x}$$

which will certainly be well-known to readers familiar with Riemannian geometry.

VII. Bounds on distances by Lewis, 1951

An earlier paper by Lewis [12] also presents some interesting results. His study is defined on a Finsler space, on which what is called a Finsler metric, which is more general than a Riemann metric (*i.e.* the quadratic constraint on the metric is relaxed), is defined. In the following, we will use the notation $M_f(x,\dot{x},t)$ for the Finsler metric.

Surprisingly, this paper is more dedicated to finding bounds (upper and lower) on the distance between two trajectories than to the search for a criterion for asymptotic stability. Indeed, consider the following theorem, adapted from Lewis [12, Theorem 2].

Theorem 4: If there exist two functions $\alpha(t)$ and $\beta(t)$ such that

$$\begin{aligned}\alpha(t) \quad \leq \quad & \frac{\partial M_f}{\partial x}(x,\lambda,t)f + \\ & \frac{\partial M_f}{\partial \dot{x}}(x,\lambda,t)\frac{\partial f}{\partial x}\lambda + \frac{\partial M_f}{\partial t}(x,\lambda,t) \leq \beta(t)\end{aligned}$$

is verified for any vector λ such that $M_f(x,\lambda,t) = 1$, where $M_f(x,\dot{x},t)$ is a nonstationary Finsler metric, then the distance $L(t)$ between two particles will be bounded by

$$L(0)e^{\int_0^t\alpha(\tau)d\tau} \leq L(t) \leq L(0)e^{\int_0^t\beta(\tau)d\tau}$$

□

In this theorem, the distance $L(t)$ represents the length of an arc joining two differently initialized particles. This length can be evaluated by calculating the integral

$$L(t) = \int_{s=0}^{s=1} M_f\left(x(t,s), \frac{\partial x}{\partial s}(t,s), t\right)ds$$

where s represents the curvilinear coordinate along the arc.

In order to be able to have the most accurate estimate for these bounds, which is of interest for convergence issues, it may be of help to consider the arcs which lengths are the shortest in the Finsler space. These arcs are referred to as *geodesics*, and are also mentioned in some papers on contraction [17]. To compute the expressions of such arcs, it is sufficient to solve either analytically, or numerically the Euler equations

$$\begin{aligned}&\frac{\partial M_f}{\partial x}\left(x(t,s), \frac{\partial x}{\partial s}(t,s), t\right) \\ &\qquad - \frac{d}{ds}\frac{\partial M_f}{\partial \dot{x}}\left(x(t,s), \frac{\partial x}{\partial s}(t,s), t\right) = 0\end{aligned}$$

which are well-known in the field of the calculus of variations [6][10].

Note that the above theorem is in a sense more general than a result on exponential convergence of trajectories since a particularization of the function $\beta(t)$ would allow to verify this latter property.

This article also includes a result (see Lewis [12, section 4]) which can be important from the practical point-of-view since, by using the previous theorem, it addresses the problem of calculating bounds on the approximation error that is done when linearizing a system around a specific working point in the state-space. Thus, consider that the approximation of $\dot{x} = f(x,t)$ can be put in the form

$$\dot{x} = A(x,t)$$

By gathering the two systems in a global system as follows

$$\dot{x}_G = \begin{pmatrix} \dot{x} \\ \dot{x}_S \end{pmatrix} = \begin{pmatrix} x_S A(x,t) + (1-x_S) f(x,t) \\ 0 \end{pmatrix}$$

one is able to give an estimate of the error made between A and f when these systems are identically initialized.

VIII. Stability and Finsler metric according to Lewis, 1949

Going a little further back in time, one can find a paper [11] in which Lewis studied the notion of stability using the same framework as what we have just seen in the previous section.

A difference however, is that in Lewis [11], the Finsler metric is not allowed to change directly as a function of time, the metric being written $M_f(x, \dot{x})$. The first results of the paper are, in addition to a simplification due to the stationnarity of the Finsler metric, identical to those of the theorem in the previous section. Note however one of the remarks of Lewis, which in our opinion anticipates in a sense on what will be done later on for contraction analysis, which states that the existence of constant and negative bounds $\bar{\alpha}$ and $\bar{\beta}$ on the previously-mentioned functions $\alpha(t)$ and $\beta(t)$, that is to say

$$L(0)e^{\bar{\alpha}t} \leq L(t) \leq L(0)e^{\bar{\beta}t}$$

implies an exponential convergence.

In the last section of his paper, Lewis discusses what is often referred to as "qualitative integration of differential equations" when addressing stability issues in ODEs. Once simplified, the first theorem [11, Theorem 9] of this section can be stated as follows.

Theorem 5: Let the constant $\bar{\beta}$ such that the following inequality is verified in a region R of the state-space

$$\frac{\partial M_f}{\partial x}(x,\lambda) f + \frac{\partial M_f}{\partial \dot{x}}(x,\lambda) \frac{\partial f}{\partial x} \lambda \leq \bar{\beta} \tag{16}$$

Then, if $\bar{\beta}$ is strictly negative, any two solutions $x_1(t)$ and $x_2(t)$ must approach each other asymptotically. □

Clearly, this theorem prefigures some results of contraction analysis, in particular the concept of convergence of two different trajectories rather than with respect to a specific attractor. Moreover, the Finsler metric being more general than the Riemann metric which is used in contraction, it is very simple to show that using a minor restriction, one finds results which are very similar to those of contraction.

The following theorem (see Lewis [11, Theorem 10]) shows that the behavior of a stable periodic system and periodic has also been considered.

Theorem 6: Assume that the system $\dot{x} = f(x,t)$ depends periodically on time with period T and that it verifies inequality (16). Then there is a unique periodic solution toward which any trajectory resulting from a different initial condition will converge to asymptotically. □

Remark that this last result presents a rather strong resemblance with the result in Lohmiller and Slotine [16] which states that any autonomous contracting system forced by a periodic input has a converging behavior which is itself periodic (see (4), section II). Moreover, by scanning through the proof of the above theorem, one realizes that it is based on a Cauchy sequence [22, p.52], as it is done in Lohmiller and Slotine [16] for its contracting counterpart.

IX. Discussion

From the short review that was presented above, it seems that most of the results of contraction analysis were available in the 1949 paper by Lewis, the subsequent papers being mainly refinements, precisions or additional applications of Lewis' results (although examining the different ways used by the authors to described similar ideas is quite interesting in its own right).

One might wonder why the following authors needed to somehow redefine what Lewis had done. Interestingly, it seems that, with the notable exception of Hartman, neither Seifert nor Opial were aware of Lewis' papers when submitting theirs since they do not appear in their respective reference list. Furthermore, in the issue following the publication in the Annals of Mathematics of Seifert's paper, the latter published a short note acknowledging the anteriority of Lewis' contributions (see Seifert [24]), a note which was maybe not noticed by Opial.

Hence one can also wonder what differentiates these earlier studies from contraction analysis. First, let us remark that the common point of these earlier works seems to lie in the statement of criteria based on differential quantities, together with proofs of convergence based on path integration of an arc linking two points in the state-space.

Thanks to its framework based on virtual displacements and inspired from fluid mechanics, contraction analysis, however, can also be related to Linear Time-Varying systems analysis since, on the contrary to previous studies, the emphasis is put more on the virtual dynamics behavior analysis than on the subsequent path integration that is only alluded to (*i.e.* as opposed to explicitly defined in the early papers). Following this LTV perspective, which is beyond the scope of the present paper, the reader is referred to Wolovich [28] (or for example [3] for a more recent reference) where an interesting comparison with the results of [16, section 4.3] can be made.

Finally, an important aspect of these approaches that could be grouped under the term "differential stability" and that is especially emphasized in the work of Lohmiller and Slotine, is the conceptual advantage given by a differential analysis as opposed to a more usual error dynamics approach, which was revealed in [16] to be particularly useful in an observer context.

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