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Symmetry of Solutions to the Optimal Sojourn Time Control Problem

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Abstract— **In this paper, we study the solutions to the optimal sojourn time control problem. For a control system whose state takes values in a multi-dimensional state space and whose state dynamics is subject to the perturbations of random noises, we try to find the state feedback control laws with a fixed cost that can keep the state inside a subset of the state space called the safe set for as long as possible. We show that for a radially symmetric safe set, the optimal feedback control law is also radially symmetric. This reduces the complexity of finding the problem solutions significantly. Furthermore, in the case when one can design the shape of the safe set as well, we show by using the isoperimetric inequality that, among all safe sets with a fixed volume, the radially symmetric one is the best in that it yields the best performing optimal sojourn time control law.**

I. Introduction

The problem of optimal control of systems under uncertainty has many practical applications, for example, in aircraft conflict detection and resolution [2], [4], formation fly of Unmanned Aerial Vehicles (UAVs) [8], [9], [11], automated highway system [10], robotics, etc. In many such applications, the system under study can be modeled as a dynamical system whose state dynamics is perturbed by random noises. The goal is then to find the control to optimize an objective function subject to certain constraints. However, due to the presence of noises, both the objective and the constraints can be *probabilistic*, i.e., they can be the expectations of some random variables or the probabilities of some random events whose outcomes depend on the realizations of the random noises. Finding the optimal control in this probabilistic setting thus becomes an important task.

In this paper, we focus on an instance of the optimal control problems of systems under uncertainty called the optimal sojourn time control problem. This problem is particularly relevant in those applications where safety is a primary concern. In these applications, the system state space can be partitioned into two subsets, a safe set and an unsafe set. The system is declared to be safe as long as its state is in the safe set. A state feedback control law is typically applied to contain the state within the safe set. However, due to the random perturbations in the state dynamics, the state trajectory may occasionally wander into the unsafe set; and whenever this occurs, some usually costly emergency procedures have to be evoked to bring the state back inside the safe set. Hence it is a meaningful problem to find a feedback control law with a reasonable cost that can keep the state within the safe set for the longest possible expected time. Or equivalently, one wishes to find a feedback control with the least cost that can keep the system safe for a long enough period of time on average. This is the optimal sojourn time control problem.

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The optimal sojourn time control problem is first proposed in [3]. Under the assumption that the solutions are symmetric in symmetric state spaces, the optimal control in the one dimensional state space is characterized analytically in [3]. In [7], the problem is extended to the general setting of stochastic hybrid systems, and a numerical solution is presented that reformulates the problem as a partial differential equation (PDE) constrained optimization problem, and uses the adjoint method to find its solutions.

One main contribution of this paper is the proof of the symmetry conjecture for the solutions to the optimal sojourn time control problem in arbitrary multi-dimensional state spaces; namely, we prove that if the state space is radially symmetric, then the solutions to the optimal sojourn time control problem are also radially symmetric. Another contribution is our proof that, among all safe sets with a fixed volume, the radially symmetric one is the best in the sense that the optimal feedback control on it has the best performance. Thus if one has the luxury of designing the shape of the safe set, he should choose the radially symmetric one in order to employ the most efficient optimal sojourn time control scheme. The technique used in the proofs, called the symmetrization method, is originally proposed in [6], and has been applied in the study of a variety of variational problems. This paper represents a highly nontrivial application of the symmetrization technique to the optimal control problems of systems under uncertainty.

The rest of the paper is organized as follows. In Section II, the problem of optimal sojourn time control is formulated. An equivalent and more convenient formulation of the problem, together with some properties of its solutions, is presented in Section III. In Section IV, we prove the main results of the paper through a series of technical lemmas.

II. Problem Formulation

Let Ω be a bounded open connected domain of $\mathbb{R}^n$ for some $n \geq 2$, and let $u : \Omega \to \mathbb{R}^n$ be a piecewise continuous vector field on Ω. Consider the following stochastic differential equation:

$$dX_t = u(X_t)dt + dB_t. \tag{1}$$

From a control system perspective, X_t can be thought of as the state trajectory of a system under the state feedback control law u, subject to the perturbations in the speed of X_t by the white noises dB_t/dt.

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Define $T = \inf\{t \geq 0 : X_t \notin \Omega\}$ as the first exit time (escape time) of X_t from Ω, which is also called the sojourn time of X_t in Ω. Then T is a random stopping time. The expected value of T is denoted by

$$V(x) \triangleq E^x[T]. \tag{2}$$

Here E^x means that the expectation is taken under the initial condition $X_0 = x$. Thus $V(x)$ is the expected time the state X_t with dynamics (1) will stay inside Ω before it first exits, given that it starts from x at time $t = 0$. The following lemma characterizes $V(x)$ as the solution to a PDE, and can be derived from the standard results of stochastic calculus [5].

Lemma 1: $V(x)$ is the unique solution to the PDE:

$$\frac{1}{2}\Delta V(x) + u(x) \cdot \nabla V(x) + 1 = 0, \quad x \in \Omega, \tag{3}$$

with the boundary condition $V|_{\partial\Omega} \equiv 0$. In particular, V is bounded and second order differentiable with piecewise continuous second order derivatives.

In Lemma 1, Δ is the Laplacian operator: $\Delta V = \sum_{i=1}^{n} \frac{\partial^2 V}{\partial x_i^2}$; $\nabla V = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})$ is the gradient of V (as a column vector); and $u(x) \cdot \nabla V(x)$ denotes the dot product of the two vectors $u(x)$ and $\nabla V(x)$. For simplicity, we often drop the variable x from the equations. Thus (3) is simplified as

$$\frac{1}{2}\Delta V + u \cdot \nabla V + 1 = 0, \quad V|_{\partial\Omega} \equiv 0. \tag{4}$$

We now formulate the problem to be studied in this paper. As suggested earlier, the SDE (1) defines the dynamics of a control system with state space Ω, where X_t is its state at time t and $u : \Omega \to \mathbb{R}^n$ is the state feedback control. A natural problem is to find the control u with the least energy expenditure that can keep the state X_t inside Ω for at least a certain amount of expected time. More precisely, for a control $u : \Omega \to \mathbb{R}^n$, its energy (cost) is defined as

$$J(u) \triangleq \int_\Omega \|u\|^2 \, dx. \tag{5}$$

The performance of u is measured by the quantity

$$W(u) \triangleq \int_\Omega w(V) \, dx, \tag{6}$$

where V is the solution to the PDE (4), and $w : \mathbb{R}_+ \to \mathbb{R}_+$ is an arbitrary monotonically increasing function on $\mathbb{R}_+ = [0, \infty)$ with $w(0) = 0$. As an example, one can choose $w(x) = x^k$, $\forall x \geq 0$, for some positive integer k.

Consider the following two problems.

Problem 1 (Optimal Sojourn Time Control): Find the piecewise continuous u that achieves the largest $W(u)$ with an energy $J(u) \leq J_0$ for some positive constant J_0.

Problem 2 (Dual Optimal Sojourn Time Control): Find the piecewise continuous u with the least energy $J(u)$ subject to $W(u) \geq W_0$ for some positive constant W_0.

The above two problems are dual to each other in the following sense. For each control u, the pair $(W(u), J(u))$ can be represented as a point on the plane. The set of all such pairs forms a planar region called the feasible region of $(W(u), J(u))$, and is plotted in Fig. 1 as the shaded region. In particular, it is bounded from the left and the right by the curves Γ_1 and Γ_2, respectively. The corner point corresponds to the zero control $u \equiv 0$, which has zero energy $J(0) = 0$ and positive performance index $W(0) > 0$ (intuitively, even if the control does nothing, it still takes time for the state to get outside the safe set if it starts from the interior). The curve Γ_1 corresponds to the solutions to the problem of finding the control u with at most energy J_0 that can achieve the least $W(u)$ for different choices of J_0. This problem is the exact opposite of Problem 1. On the other hand, the curve Γ_2 corresponds to the solutions to Problem 1 (or Problem 2) with different J_0 (or W_0). In fact, denote by $W_{max}(J_0) \triangleq \sup\{W(u) | \, J(u) \leq J_0\}$ and $J_{min}(W_0) \triangleq \inf\{J(u) | \, W(u) \geq W_0\}$ the solutions to Problem 1 and 2, respectively. Then $W_{max}(J_0)$, $J_0 \geq 0$, is a strictly increasing function in J_0; $J_{min}(W_0)$, $W_0 \geq W(0)$, is a strictly increasing function in W_0; they are inverse function to each other; and the graph of $J_{min}(W_0)$ for $W_0 \geq W(0)$ is exactly the curve Γ_2.

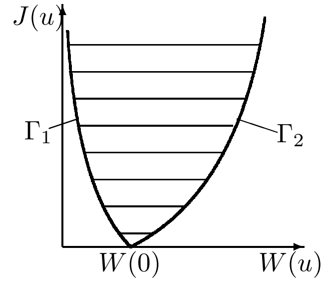

Fig. 1. Feasible region of $(W(u), J(u))$ shown as the shaded region.

Remark 1: The solution to Problem 2 is $u \equiv 0$ if $W_0 \leq W(0)$. Thus, we assume $W_0 > W(0)$ in the following.

III. Formulation in Terms of V

We now present an alternative formulation of the problem more convenient for proving the results in Section IV.

We first study the set of feasible V, i.e., the set of all functions $V : \Omega \to \mathbb{R}$ that is the expected sojourn time corresponding to some control u on Ω. A feasible V should satisfy certain conditions, for example, it should be nonnegative, has piecewise continuous second order derivatives, and, by Lemma 1, is a solution to the PDE (4). In particular, since $1 + \frac{1}{2}\Delta V = -u \cdot \nabla V$, $\Delta V = -2$ whenever $\nabla V = 0$. If in addition u is an optimal solution to Problem 1 or 2, then

Lemma 2: Let u be a solution to Problem 1 or 2, and V be the corresponding expected sojourn time. Then

$$u = \begin{cases} -(1 + \frac{1}{2}\Delta V)\frac{\nabla V}{\|\nabla V\|^2} & \text{if } \nabla V \neq 0; \\ 0 & \text{if } \nabla V = 0, \end{cases} \tag{7}$$

for almost every $x \in \Omega$. Moreover, $1 + \frac{1}{2}\Delta V \leq 0$ a.e. on Ω.

Proof: Suppose that u is a solution to Problem 2, and that (7) does not hold on a subset Ω_0 of Ω of nonzero measure. We shall prove that a solution $\hat{u}$ better than u can be constructed, contradicting the assumption that u is optimal.

By Lemma 1, we have $\frac{1}{2}\Delta V + u\cdot\nabla V + 1 = 0$, i.e., $u\cdot\nabla V = -(\frac{1}{2}\Delta V + 1)$, on Ω. Partition Ω into two subsets: $\Omega_1 = \{x\in\Omega : \nabla V = 0\}$ and $\Omega_2 = \{x\in\Omega : \nabla V \neq 0\}$. Thus Ω_0 is partitioned correspondingly into $\Omega_0\cap\Omega_1$ and $\Omega_0\cap\Omega_2$. Define a new control $\hat{u}$ as follows.

$$\hat{u} = \begin{cases} 0, & x\in\Omega_0\cap\Omega_1;\\ -(1+\frac{1}{2}\Delta V)\frac{\nabla V}{\|\nabla V\|^2}, & x\in\Omega_0\cap\Omega_2;\\ u, & \text{otherwise.}\end{cases}$$

On $\Omega_0\cap\Omega_1$, since $\hat{u}=0$, $\nabla V = 0$ and $u\neq 0$, we have $\|\hat{u}\| < \|u\|$ and $\hat{u}\cdot\nabla V = u\cdot\nabla V$. On $\Omega_0\cap\Omega_2$, since $\nabla V\neq 0$ and $u\neq\hat{u} = -(1+\frac{1}{2}\Delta V)\frac{\nabla V}{\|\nabla V\|^2}$, we have

$$\hat{u}\cdot\nabla V = -(1+\frac{1}{2}\Delta V)\frac{\nabla V}{\|\nabla V\|^2}\cdot\nabla V = -(1+\frac{1}{2}\Delta V) = u\cdot\nabla V.$$

Since $|u\cdot\nabla V| \leq \|u\|\cdot\|\nabla V\|$, we have,

$$\|u\| \geq \frac{|u\cdot\nabla V|}{\|\nabla V\|} = \frac{|\hat{u}\cdot\nabla V|}{\|\nabla V\|} = \|\hat{u}\|, \tag{8}$$

with equality if and only if $u=\hat{u}$. Since $u\neq\hat{u}$ on $\Omega_0\cap\Omega_2$, we have $\|\hat{u}\| < \|u\|$. To sum up, we conclude that $\|\hat{u}\| < \|u\|$ on Ω_0; and that $\hat{u}\cdot\nabla V = u\cdot\nabla V$ on Ω_0, hence on all Ω. The first conclusion implies that $J(\hat{u}) < J(u)$. The second conclusion implies that $\frac{1}{2}\Delta V + \hat{u}\cdot\nabla V + 1 = 0$ on Ω. Thus V is also the expected sojourn time for the control $\hat{u}$. Therefore, $W(\hat{u}) = W(u)$, implying that $\hat{u}$ is a better solution than u to Problem 2. This completes the proof of equation (7).

The condition that $1+\frac{1}{2}\Delta V\leq 0$ a.e. on Ω can be proved similarly. We refer the interested readers to [1] for detail. ■

Intuitively, Lemma 2 implies that for optimal solutions, whenever $\nabla V\neq 0$, the control u will always point to the same direction as ∇V. In addition, by (7),

$$J(u) = \int_\Omega \|u\|^2\,dx = \int_\Omega \frac{|1+\frac{1}{2}\Delta V|^2}{\|\nabla V\|^2}\,dx,$$

where the integrand $\frac{|1+\frac{1}{2}\Delta V|^2}{\|\nabla V\|^2}$ is understood to be zero whenever $\|\nabla V\| = 0$ (hence $1+\frac{1}{2}\Delta V = 0$ as well).

As a result of the discussion at the beginning of this section and Lemma 2, we have

Lemma 3: Suppose that $V:\Omega\to\mathbb{R}$ is the the expected sojourn time corresponding to some optimal control u that solves Problem 1 (or 2). Then V satisfies

1) V is positive, and vanishes on $\partial\Omega$;
2) V is second order differentiable with piecewise continuous second order derivatives;
3) $1+\frac{1}{2}\Delta V\leq 0$ a.e., and $1+\frac{1}{2}\Delta V = 0$ whenever $\nabla V = 0$;
4) $\int_\Omega \frac{|1+\frac{1}{2}\Delta V|^2}{\|\nabla V\|^2}\,dx < \infty$, where the integrand $\frac{|1+\frac{1}{2}\Delta V|^2}{\|\nabla V\|^2}$ is understood to be zero at those x where $\nabla V = 0$.

Define $\mathcal{V}(\Omega)$ as the set of all $V:\Omega\to\mathbb{R}$ satisfying the above four properties. Then Lemma 3 implies that we need only to focus on the functions V in $\mathcal{V}(\Omega)$ to find the optimal ones, i.e., the ones corresponding to optimal controls u. Hence an equivalent formulation of Problem 1 in terms of V is

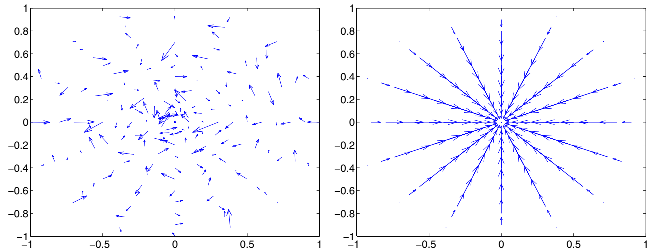

Fig. 2. Numerical solutions for radially symmetric safe sets in $\mathbb{R}^2$. Left: initial guess of u; Right: converged solution u (courtesy of Robbin Raffard).

Problem 3: Among all $V\in\mathcal{V}(\Omega)$, find the ones that

$$\begin{aligned} &\text{maximize } W(u) = \int_\Omega w(V)\,dx\\ &\text{subject to } J(u) = \int_\Omega \frac{(1+\frac{1}{2}\Delta V)^2}{\|\nabla V\|^2}\,dx \leq J_0. \end{aligned}$$

IV. Symmetry of Solutions

In [7], we propose a numerical solution of the problems formulated in the previous sections using the adjoint method, which is essentially a gradient descent method in functional spaces. From numerical experiments, we observe that, if the safe set Ω is radially symmetric, i.e., a ball, then regardless of the initial guesses of the feedback control law u, the numerical algorithm will always converge to a radially symmetric solution, suggesting that this is the optimal solution. See Fig. 2 for an instance of the two dimensional case. In this section, we shall establish this analytically.

The key to our results is the following lemma.

Lemma 4: Given a domain Ω and a function $V\in\mathcal{V}(\Omega)$, there exist a radially symmetric domain $\bar{\Omega}$ (i.e., a ball) in $\mathbb{R}^n$ with the same volume as Ω, and a radially symmetric function $\bar{V}\in\mathcal{V}(\bar{\Omega})$, such that

$$\int_\Omega \frac{(1+\frac{1}{2}\Delta V)^2}{\|\nabla V\|^2}\,dx = \int_{\bar{\Omega}} \frac{(1+\frac{1}{2}\Delta\bar{V})^2}{\|\nabla\bar{V}\|^2}\,dx \tag{9}$$

$$\text{and} \quad \int_\Omega w(V)\,dx \leq \int_{\bar{\Omega}} w(\bar{V})\,dx. \tag{10}$$

The following two theorems follow directly from Lemma 4; thus their proofs are omitted.

Theorem 1 (Symmetry of solutions on symmetric domain): Suppose that Ω is radially symmetric. Then there exists at least one radially symmetric solution to Problem 1 (Problem 2, or Problem 3).

Theorem 2 (Optimality of symmetric domain): Suppose that in Problem 3 the constant J_0 is fixed. Denote by $W_{max}(\Omega)$ the maximal possible value of $W(u)$ subject to the constraint $J(u)\leq J_0$. Then among all Ω with the same volume, there is a radially symmetric one $\bar{\Omega}$ for which $W_{max}(\bar{\Omega})\geq W_{max}(\Omega)$ for all other Ω.

Since Problem 3 is equivalent to Problem 1, and dual to Problem 2, we have

Corollary 1: Suppose that in Problem 1 the constant J_0 is fixed. Denote by $W_{max}(\Omega)$ the maximal possible value of $W(u)$ subject to the constraint $J(u)\leq J_0$. Then among all

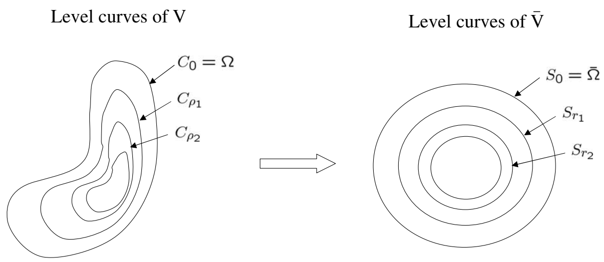

Fig. 3. Level curves of the function V and its symmetrization $\bar{V}$.

Ω with the same volume, there is a radially symmetric one $\bar{\Omega}$ for which $W_{max}(\bar{\Omega}) \geq W_{max}(\Omega)$ for all other Ω.

Corollary 2: Suppose that in Problem 2 the constant W_0 is fixed. Denote by $J_{min}(\Omega)$ the minimal possible value of $J(u)$ subject to the constraint $W(u) \geq W_0$. Then among all Ω with the same volume, there is a radially symmetric one $\bar{\Omega}$ for which $J_{min}(\bar{\Omega}) \leq J_{min}(\Omega)$ for all other Ω.

Roughly speaking, the corollaries state that, in Problem 1 and 2, if the domain Ω has a fixed volume, then its "best" shape in terms of designing u with a large $W(u)$ and a small $J(u)$ is a radially symmetric one. And on this domain, the optimal u and V are both radially symmetric by Theorem 1.

The rest of this section is devoted to proving Lemma 4. In particular, given a domain Ω and a function $V \in \mathcal{V}(\Omega)$, we will define a symmetrization operator that can transform Ω into a radially symmetric $\bar{\Omega}$ with the same volume, and V into a radially symmetric $\bar{V}$ satisfying both (9) and (10).

We first introduce some notations. Some of them are adapted from those in [6, Note F, pp 232]. Denote by $V_m = \max\{V(x)|\ x \in \Omega\}$ the maximal value of V, which is achieved inside Ω as Ω has compact support and $V|_{\partial\Omega} \equiv 0$. The minimal value of V on Ω is 0, achieved on the boundary of Ω. For each $\rho \in [0, V_m]$, denote by C_ρ the level surface $\{x \in \Omega | V(x) = \rho\}$, and by $A(\rho)$ the volume enclosed by C_ρ. Note that C_ρ may have many connected components, and $A(\rho)$ is the total volume enclosed by all of them. As ρ increases from 0 to V_m, C_ρ shrinks from $C_0 = \partial\Omega$ to C_{V_m}, the set of all $x \in \Omega$ where V achieves its maximal value V_m; and $A(\rho)$ decreases from volume(Ω) to 0.

Next, we define the radially symmetric domain $\bar{\Omega}$ and function $\bar{V}$ on $\bar{\Omega}$. For each $\rho \in [0, V_m]$, a positive number r is chosen so that the $(n-1)$-sphere S_r in $\mathbb{R}^n$ with radius r centered at the origin encloses the same volume as C_ρ, namely, $A(\rho)$. Using the formula of the volume of n-dimensional ball, we have

$$\frac{\pi^{n/2} r^n}{\Gamma(1+\frac{n}{2})} = A(\rho) \Rightarrow r = \frac{1}{\sqrt{\pi}} \Gamma^{1/n}(1+\frac{n}{2}) A^{1/n}(\rho). \quad (11)$$

Thus r is a decreasing function of ρ that takes the value of $r_m = \frac{1}{\sqrt{\pi}}\Gamma^{1/n}(1+\frac{n}{2}) \cdot \text{volume}(\Omega)^{1/n}$ when $\rho = 0$, and decreases to 0 as $\rho \to V_m$. Let $\bar{\Omega}$ be the open n-ball enclosed by S_{r_m}. Then $\bar{\Omega}$ and Ω have the same volume. Define $\bar{V}$ so that it has constant value $f(\rho)$ on the sphere S_r, where r and ρ satisfy the relation (11), and f is a monotonically increasing function from $[0, V_m]$ to $\mathbb{R}$ to be determined later. We require that $f(0) = 0$ so that $\bar{V}|_{\partial\bar{\Omega}} \equiv 0$, and that f is sufficiently smooth so that $\bar{V}$ has piecewise continuous second order derivatives. We will show that for properly chosen f, the new $\bar{\Omega}$ and $\bar{V}$ satisfy the conditions (9) and (10), which will then prove Lemma 4.

Choose an arbitrary $\rho \in [0, V_m]$. At each point $x \in C_\rho$, denote by $\vec{n}$ the unit normal of C_ρ pointing inside of C_ρ, and by ds the area element of C_ρ. Then the volume element at x is $dx = ds \cdot dn$, where dn is the infinitesimal element along $\vec{n}$. The gradient of V has the same direction as $\vec{n}$, and an amplitude $\|\nabla V\| = \frac{dV}{dn} = \frac{d\rho}{dn}$. Therefore, $dn = \|\nabla V\|^{-1} d\rho$; and the volume element becomes

$$dx = ds\, dn = \|\nabla V\|^{-1} ds\, d\rho. \quad (12)$$

We then integrate (12) over the set enclosed by C_ρ to obtain

$$A(\rho) = \int_{\{x\in\Omega | V(x) \geq \rho\}} dx = \int_\rho^{V_m} \int_{C_\rho} \|\nabla V\|^{-1}\, ds\, d\rho.$$

In particular, let $\rho = 0$, we have $\int_0^{V_m}\int_{C_\rho} \|\nabla V\|^{-1}\, ds\, d\rho = A(0) = \text{volume}(\Omega) \leq \infty$, which implies that $\nabla V \neq 0$ a.e. (with respect to the measure $ds\, d\rho$). Differentiating with respect to ρ, we obtain

$$A'(\rho) = -\int_{C_\rho} \|\nabla V\|^{-1}\, ds. \quad (13)$$

Note that $A'(\rho) \leq 0$ as $A(\rho)$ is decreasing in ρ.

Define two nonnegative functions $P(\rho)$ and $Q(\rho)$ as

$$P(\rho) \triangleq \int_{C_\rho} \|\nabla V\|\, ds,\ Q(\rho) \triangleq \int_{C_\rho} \frac{(1+\frac{1}{2}\Delta V)^2}{\|\nabla V\|^3}\, ds. \quad (14)$$

By (12), the two integrals in (9) and (10) can be written as

$$\int_\Omega \frac{(1+\frac{1}{2}\Delta V)^2}{\|\nabla V\|^2}\, dx = \int_0^{V_m} Q(\rho)\, d\rho, \quad (15)$$

$$\int_\Omega w(V)\, dx = \int_0^{V_m} w(\rho) |A'(\rho)|\, d\rho. \quad (16)$$

Before we proceed in the proof, two useful inequalities involving $P(\rho)$ and $Q(\rho)$ will be introduced.

Lemma 5: The function $P(\rho)$ satisfies, $\forall \rho \in [0, V_m]$,

$$P(\rho)|A'(\rho)| \geq \frac{n^2 \pi A^{2-\frac{2}{n}}(\rho)}{\Gamma^{2/n}(1+\frac{n}{2})}. \quad (17)$$

Proof: Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} P(\rho)\cdot|A'(\rho)| &= \int_{C_\rho} \|\nabla V\|\, ds \cdot \int_{C_\rho} \|\nabla V\|^{-1}\, ds \\ &\geq \left(\int_{C_\rho} ds\right)^2 = B^2(\rho), \end{aligned}$$

where $B(\rho)$ is the area of the hyper-surface C_ρ. The desired conclusion is obtained by combining the above inequality with the n-dimensional isoperimetric inequality

$$B(\rho) \geq \frac{n\sqrt{\pi}}{\Gamma^{1/n}(1+\frac{n}{2})} A^{1-\frac{1}{n}}(\rho),$$

where $\Gamma(\cdot)$ is the Gamma function. ■

Lemma 6: The functions $P(\rho)$ and $Q(\rho)$ satisfy

$$Q^{1/2}(\rho)P^{1/2}(\rho) \geq A'(\rho) - \frac{1}{2}P'(\rho).$$

Proof: Apply the Cauchy-Schwarz inequality to obtain

$$Q^{1/2}(\rho)P^{1/2}(\rho) \geq \int_{C_\rho} \frac{|1+\frac{1}{2}\Delta V|}{\|\nabla V\|}\, ds.$$

Integrating on the sub-interval $[\rho_1, \rho_2]$ of $[0, V_m]$, we have

$$\int_{\rho_1}^{\rho_2} Q^{1/2}(\rho)P^{1/2}(\rho)\, d\rho \geq \int_{\rho_1}^{\rho_2}\int_{C_\rho} \frac{|1+\frac{1}{2}\Delta V|}{\|\nabla V\|}\, ds\, d\rho$$
$$= -\iint_{\rho_1 \leq V \leq \rho_2} (1+\frac{1}{2}\Delta V)\, dx$$
$$= -\iint_{\Omega_{\rho_1,\rho_2}} dx - \frac{1}{2}\iint_{\Omega_{\rho_1,\rho_2}} \Delta V\, dx, \tag{18}$$

where $\Omega_{\rho_1,\rho_2} \triangleq \{x \in \Omega | \rho_1 \leq V(x) \leq \rho_2\}$ is the region sandwiched between C_{ρ_1} and C_{ρ_2}, and in deriving the first equality we use (12) and the fact that $1 + \frac{1}{2}\Delta V \leq 0$ as $V \in \mathcal{V}(\Omega)$. The first term in (18) reduces to $-\iint_{\Omega_{\rho_1,\rho_2}} dx = -\text{volume}(\Omega_{\rho_1,\rho_2}) = A(\rho_2) - A(\rho_1)$, while the second terms is, by the divergence theorem in $\mathbb{R}^n$, $-\frac{1}{2}\iint_{\Omega_{\rho_1,\rho_2}} \Delta V\, dx = \frac{1}{2}\int_{C_{\rho_1}} \|\nabla V\|\, ds - \frac{1}{2}\int_{C_{\rho_2}} \|\nabla V\|\, ds = -\frac{1}{2}[P(\rho_2) - P(\rho_1)]$. Thus (18) becomes $\int_{\rho_1}^{\rho_2} Q^{1/2}(\rho)P^{1/2}(\rho)\, d\rho \geq A(\rho_2) - A(\rho_1) - \frac{1}{2}[P(\rho_2) - P(\rho_1)]$. The desired conclusion follows by dividing both sides by $\rho_2 - \rho_1$ and letting $\rho_2 \to \rho_1$. ■

We now return to the proof of Lemma 4. For the domain $\bar{\Omega}$ and the function $\bar{V}$, the infinitesimal version of (11) is

$$A'(\rho)\, d\rho = \frac{n\pi^{n/2} r^{n-1}}{\Gamma(1+\frac{n}{2})}\, dr. \tag{19}$$

Since $\bar{V} = f(\rho)$, by writing $\Delta\bar{V}$ in the spherical coordinate and using (11) and (19), we have

$$\Delta\bar{V} = \frac{1}{r^{n-1}}\frac{d}{dr}(r^{n-1}\frac{d\bar{V}}{dr})$$
$$= \frac{n^2\pi}{\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}\frac{d}{d\rho}\left(\frac{A^{2-2/n}(\rho)}{|A'(\rho)|}f'(\rho)\right). \tag{20}$$

Moreover,

$$\|\nabla\bar{V}\|^2 = \left(\frac{d\bar{V}}{dr}\right)^2 = \left(\frac{d\bar{V}}{d\rho}\cdot\frac{d\rho}{dr}\right)^2 = |f'(\rho)|^2\frac{n^2\pi^n r^{2n-2}}{\Gamma^2(1+\frac{n}{2})}$$
$$= |f'(\rho)|^2\frac{n^2\pi A^{2-2/n}(\rho)}{\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|^2}.$$

Using the fact that the (infinitesimal) volume of the set sandwiched between the sphere S_r and S_{r+dr} is given by (19), as well as the above two equations, we compute

$$\int_{\bar{\Omega}} \frac{(1+\frac{1}{2}\Delta\bar{V})^2}{\|\nabla\bar{V}\|^2}\, dx = \int_0^{r_m} \frac{(1+\frac{1}{2}\Delta\bar{V})^2}{\|\nabla\bar{V}\|^2}\frac{n\pi^{n/2}r^{n-1}}{\Gamma(1+\frac{n}{2})}\, dr$$
$$= \int_0^{V_m} \frac{(1+\frac{1}{2}\Delta\bar{V})^2}{\|\nabla\bar{V}\|^2}|A'(\rho)|\, d\rho$$
$$= \int_0^{V_m}\left[1+\frac{n^2\pi}{2\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}\frac{d}{d\rho}\left(\frac{A^{2-2/n}(\rho)}{|A'(\rho)|}f'(\rho)\right)\right]^2$$
$$\cdot\left[|f'(\rho)|^2\frac{n^2\pi A^{2-2/n}(\rho)}{\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|^2}\right]^{-1}\cdot|A'(\rho)|\, d\rho. \tag{21}$$

Lemma 7: Define a function f on $[0, V_m]$ by

$$f(\rho) = \frac{2\Gamma^{2/n}(1+\frac{n}{2})}{n^2\pi}\int_0^\rho \frac{|A'(\chi)|e^{-\int_0^\chi G(\eta)\,d\eta}}{A^{2-2/n}(\chi)}$$
$$\cdot\left[\int_\chi^{V_m} |A'(\xi)|e^{\int_0^\xi G(\eta)\,d\eta}\, d\xi\right], \tag{22}$$

where G is the nonnegative function on $[0, V_m]$ defined by

$$G(\rho) = \frac{2\Gamma^{1/n}(1+\frac{n}{2})|A'(\rho)|^{1/2}(\rho)}{n\pi^{1/2}A^{1-1/n}(\rho)}Q^{1/2}(\rho). \tag{23}$$

Then $f(\rho)$ is a strictly increasing function in ρ with $f(0) = 0$ and $f(\rho) < \infty$ for $\rho \in [0, V_m)$. Moreover, $\forall\rho \in [0, V_m)$,

$$1 + \frac{n^2\pi}{2\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}\frac{d}{d\rho}\left(\frac{A^{2-2/n}(\rho)}{|A'(\rho)|}f'(\rho)\right) \leq 0 \tag{24}$$

$$\left[1 + \frac{n^2\pi}{2\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}\frac{d}{d\rho}\left(\frac{A^{2-2/n}(\rho)}{|A'(\rho)|}f'(\rho)\right)\right]^2$$
$$\cdot\left[|f'(\rho)|^2\frac{n^2\pi A^{2-2/n}(\rho)}{\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|^2}\right]^{-1}\cdot|A'(\rho)| = Q(\rho). \tag{25}$$

Proof: That $f(0) = 0$ is obvious from (22). Taking the derivative of (22), we have

$$f'(\rho) = \frac{2\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|e^{-\int_0^\rho G(\eta)\,d\eta}}{n^2\pi A^{2-2/n}(\rho)}$$
$$\int_\rho^{V_m} |A'(\xi)|e^{\int_0^\xi G(\eta)\,d\eta}\, d\xi. \tag{26}$$

Thus for $\rho \in [0, V_m)$,

$$1 + \frac{n^2\pi}{2\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}\frac{d}{d\rho}\left(\frac{A^{2-2/n}(\rho)}{|A'(\rho)|}f'(\rho)\right)$$
$$= -\frac{G(\rho)e^{-\int_0^\rho G(\eta)\,d\eta}}{|A'(\rho)|}\int_\rho^{V_m} |A'(\xi)|e^{\int_0^\xi G(\eta)\,d\eta}\, d\xi \leq 0,$$

which proves (24). To prove (25), we verify, for $\rho \in [0, V_m)$,

$$\left[1 + \frac{n^2\pi}{2\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}\frac{d}{d\rho}\left(\frac{A^{2-2/n}(\rho)}{|A'(\rho)|}f'(\rho)\right)\right]^2$$
$$\cdot\left[|f'(\rho)|^2\frac{n^2\pi A^{2-2/n}(\rho)}{\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|^2}\right]^{-1}\cdot|A'(\rho)|$$
$$= \frac{G^2(\rho)}{|A'(\rho)|^2}\cdot\left[\frac{4\Gamma^{2/n}(1+\frac{n}{2})}{n^2\pi A^{2-2/n}(\rho)}\right]^{-1}\cdot|A'(\rho)| = Q(\rho).$$

which proves the desired result. ■

As a result, we have

Corollary 3: Let $\bar{V}$ be the radially symmetric function on $\bar{\Omega}$ defined by $\bar{V}(r) = f(\rho)$ where ρ is determined from r by (11). Then equation (9) holds. Moreover, $\bar{V} \in \mathcal{V}(\bar{\Omega})$.

Proof: By condition (25) proved in Lemma 7, equation (21) implies

$$\int_{\bar{\Omega}} \frac{(1+\frac{1}{2}\Delta\bar{V})^2}{\|\nabla\bar{V}\|^2}\, dx = \int_0^{V_m} Q(\rho)\, d\rho = \int_\Omega \frac{(1+\frac{1}{2}\Delta V)^2}{\|\nabla V\|^2}\, dx,$$

which is exactly the condition (9). Since $V \in \mathcal{V}(\Omega)$, the last integral in the above equation is finite, hence $\int_{\bar\Omega} \frac{|1+\frac{1}{2}\Delta \bar V|^2}{\|\nabla \bar V\|^2}\,dx \leq \infty$. Also, by (20) and condition (25) in Lemma 7, we have $1 + \frac{1}{2}\Delta \bar V = 1 + \frac{n^2\pi}{2\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}\frac{d}{d\rho}\left(\frac{A^{2-2/n}(\rho)}{|A'(\rho)|}f'(\rho)\right) \leq 0$. Hence $\bar V \in \mathcal{V}(\bar\Omega)$. ■

To prove Lemma 4, the only thing remained is to establish condition (10). To this end, we first prove

Lemma 8: The function f defined in (22) satisfies $f'(\rho) \geq 1$ for all $\rho \in [0, V_m)$.

Proof: Let $\rho \in [0, V_m)$ be arbitrary. Then

$$\frac{1}{4}P(\rho)G^2(\rho) = P(\rho)\frac{\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}{n^2\pi A^{2-\frac{2}{n}}(\rho)}Q(\rho) \geq Q(\rho),$$

where the inequality follows from Lemma 5. The square root of the above equation yields

$$\frac{1}{2}P^{1/2}(\rho)G(\rho) \geq Q^{1/2}(\rho).$$

Multiplying both sides by $P^{1/2}(\rho)$ and applying Lemma 6, we have

$$\frac{1}{2}P(\rho)G(\rho) \geq P^{1/2}(\rho)Q^{1/2}(\rho) \geq A'(\rho) - \frac{1}{2}P'(\rho),$$

which in turn implies

$$\begin{aligned}
& 2|A'(\rho)| = -2A'(\rho) \geq -P'(\rho) - P(\rho)G(\rho) \\
\Rightarrow\quad & \frac{2\Gamma^{2/n}(1+\frac{n}{2})|A'(\rho)|}{n^2\pi}e^{\int_0^\rho G(\eta)\,d\eta} \\
& \geq \frac{\Gamma^{2/n}(1+\frac{n}{2})}{n^2\pi}\left[-P'(\rho) - P(\rho)G(\rho)\right]e^{\int_0^\rho G(\eta)\,d\eta} \\
\Rightarrow\quad & H(\rho)e^{\int_0^\rho G(\eta)\,d\eta} \\
& \geq \frac{\Gamma^{2/n}(1+\frac{n}{2})}{n^2\pi}\left[-P'(\rho) - P(\rho)G(\rho)\right]e^{\int_0^\rho G(\eta)\,d\eta}. \qquad (27)
\end{aligned}$$

Since the inequality (27) holds for all $0 \leq \rho < V_m$, we can integrate it from ρ to V_m to obtain

$$\begin{aligned}
& \int_\rho^{V_m} H(\xi)e^{\int_0^\xi G(\eta)\,d\eta}\,d\xi \\
& \geq \int_\rho^{V_m} \frac{\Gamma^{2/n}(1+\frac{n}{2})}{n^2\pi}\left[-P'(\xi) - P(\xi)G(\xi)\right]e^{\int_0^\xi G(\eta)\,d\eta}\,d\xi \\
& = \frac{\Gamma^{2/n}(1+\frac{n}{2})}{n^2\pi}\left[-P(\xi)e^{\int_0^\xi G(\eta)\,d\eta}\right]\Bigg|_{\xi=\rho}^{\xi=V_m} \\
& = \frac{\Gamma^{2/n}(1+\frac{n}{2})}{n^2\pi}P(\rho)e^{\int_0^\rho G(\eta)\,d\eta}, \quad \forall \rho \in [0, V_m), \qquad (28)
\end{aligned}$$

where we have used the fact that $P(V_m) = 0$ and $e^{\int_0^{V_m} G(\eta)\,d\eta} < \infty$. Therefore,

$$\begin{aligned}
f'(\rho) &= \frac{|A'(\rho)|}{A^{2-2/n}(\rho)}e^{-\int_0^\rho G(\eta)\,d\eta}\int_\rho^{V_m} H(\xi)e^{\int_0^\xi G(\eta)\,d\eta}\,d\xi \\
&= \frac{P(\rho)|A'(\rho)|\Gamma^{2/n}(1+\frac{n}{2})}{n^2\pi A^{2-2/n}(\rho)} \\
&\geq 1,
\end{aligned}$$

for all $\rho \in [0, V_m)$, which is the desired result. Note that Lemma 5 is applied for the last inequality. ■

We now prove Lemma 4 at the beginning of this section.

Proof: [Proof of Lemma 4] Let $\bar\Omega$ be chosen as described earlier in this section, and let $\bar V$ be defined as $\bar V(r) = f(\rho)$ where f is given in (22). Then, by Lemma 7, $\bar V \in \mathcal{V}(\bar\Omega)$ and equality (9) holds. To prove (10), we apply Lemma 8 to obtain $f'(\rho) \geq 1$ for $\rho \in [0, V_m]$. Since $f(0) = 0$, this implies that $f(\rho) \geq \rho$, and thus

$$\begin{aligned}
& \int_{\bar\Omega} w(\bar V)\,dx = \int_0^{V_m} w[f(\rho)]|A'(\rho)|\,d\rho \\
& \geq \int_0^{V_m} w(\rho)|A'(\rho)|\,d\rho = \int_\Omega w(V)\,dx.
\end{aligned}$$

■

V. Conclusions and Future Directions

In this paper, we show that the solutions to the optimal sojourn time control problem is radially symmetric when the safe set is radially symmetric. In addition, among all safe sets with the same volume, the radially symmetric one has the best performing optimal sojourn time control law.

One implication of our results is, for multi-dimensional radially symmetric safe sets, the optimal sojourn time control law is indeed a one-dimensional object. This reduces the complexity of finding the analytical or numerical solution considerably, which will be one of our future research directions. A curious fact is that the results in this paper hold only when the dimension of the state space is at least two. For the (seemingly simpler) one dimension state space case, the symmetrization technique cannot be directly applied. Thus alternative proofs need to be sought in the latter case.

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