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$\mathcal{H}_\infty$ Control for uncertain systems under time domain constraints

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***Abstract*— This work deals with $\mathcal{H}_\infty$ design for continuous-time linear systems subject to parameter uncertainty and state, control and output constraints. The $\mathcal{H}_\infty$ control problem by dynamic output feedback is considered for systems under polytopic uncertainties. The designed controller can be of reduced or full order. The approach is based on the existence of an ellipsoidal positively $(\mathcal{D},\mathcal{R})$-invariant set, generated by quadratic Lyapunov function, contained in the system's domain of linearity. Using LMI, a hybrid genetic algorithm is proposed for solving this constrained $\mathcal{H}_\infty$ robust control problem.**

I. Introduction

In the last two decades a large amount of effort has been devoted to the design of robust controllers with guaranteed performance in the face of plant uncertainty. If there are uncertainties in the system model, the norm $\mathcal{H}_\infty$ can be a desirable measure of a system's robust performance [19]. The theoretic motivation for the $\mathcal{H}_\infty$ control problem and important results about output feedback control can be found in [3], [5], [13] and the references therein.

An important characteristic occurring in practical problems is the presence of state, control or output constraints [14], [16], [7] due to physical limitations and/or non-linearities in the plant. Most of realistic control problems involve both some type of time domain constraints and model uncertainties. Previous researches into this problem include the works of [17], [1], [2], [12]. In the current literature, many results are available for the robust constrained control problem mainly regarding control saturation.

In this work a procedure is proposed to solve the $\mathcal{H}_\infty$ control problem by dynamic output feedback, for continuous-time linear systems subject to polytopic parameter uncertainties and time domain constraints.

Due to the time domain constraints the concept of positive $(\mathcal{D},\mathcal{R})$-invariance, extension of the concept of positive invariance for systems subject to the additive disturbances, is used. A sufficient condition assuring positive $(\mathcal{D},\mathcal{R})$-invariance of an ellipsoidal set, defined by a Lyapunov matrix, is obtained.

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The bilinear character of some constraints in the resulting optimization problem leads to the proposed hybrid procedure, based on Genetic Algorithms (GAs) [8], [9] and linear matrix inequalities (LMIs). This procedure can be used for the synthesis of reduced or full order controllers.

This paper is organized as follows. Section 2 presents the preliminary assumptions related to the system description and the formulation of the constrained $\mathcal{H}_\infty$ robust control problem. In section 3, considering a sufficient condition for the positive $(\mathcal{D},\mathcal{R})$-invariance, the problem is formulated using matrix inequalities. A programming procedure for determining a solution to the stated problem is proposed in section 4. A numerical example is presented in section 5.

II. Problem Statement

Consider an uncertain continuous-time linear system described by the following state-space equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ y(t) = C_y x(t) \\ z(t) = Cx(t) + D_1 w(t) + D_2 u(t), \end{cases} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control vector, $y(t) \in \Re^p$ is the measured output vector, $w(t) \in \Re^l$ is the disturbance vector and $z(t) \in \Re^q$ is the controlled output vector. All matrices are real of appropriate dimensions. Assume that the matrices A and B_2 belong to the convex-bounded domains defined as

$$\mathcal{D}_A = \left\{ A;\, A = \sum_{i=1}^{N} \lambda_i A_i,\, \sum_{i=1}^{N} \lambda_i = 1,\ \lambda_i \geq 0 \right\}, \tag{2}$$

$$\mathcal{D}_B = \left\{ B_2;\, B_2 = \sum_{j=1}^{M} \theta_j B_{2j},\, \sum_{j=1}^{M} \theta_j = 1,\ \theta_j \geq 0 \right\}. \tag{3}$$

All pairs (A, B_2) are assumed to be stabilizable and C_y has full row rank.

Furthermore, consider that the state, control and output vectors are subject to physical constraints. The sets of admissible state, control and output are given respectively by the convex polytopes:

$$\mathcal{D}(g,\rho)=\left\{x\in\Re^n : g_i^T x \le \rho_i,\, i=1,2,\cdots,n_g\right\} \tag{4}$$

$$\mathcal{D}(h,\mu)=\left\{u\in\Re^m : h_i^T u \le \mu_i,\, i=1,2,\cdots,n_h\right\} \tag{5}$$

$$\mathcal{D}(\eta,\xi)=\left\{y\in\Re^p : \eta_i^T y \le \xi_i,\, i=1,2,\cdots,n_n\right\} \tag{6}$$

where $g_i \in \Re^n$, $g_i \ne 0$, $\rho_i > 0$, $h_i \in \Re^m$, $h_i \ne 0$, $\mu_i > 0$, $\eta_i \in \Re^p$, $\eta_i \ne 0$, $\xi_i > 0, \forall i$.

By definition, $\mathcal{D}(g,\rho)$ contains the origin in its interior. Let us also consider a bounded polyhedral set of admissible initial states $x_0 = x(t_0)$:

$$\mathcal{D}(g_0,\rho_0)=\left\{x_0 = x(t_0)\in\Re^n : g_{0i}^T x_0 \le \rho_{0i},\, i=1,2,\cdots,n_o\right\} \tag{7}$$

where $g_{0i} \in \Re^n, g_{0i} \ne 0, \rho_{0i} > 0, i=1,2,\cdots,n_o$. Let $v_i \in \Re^n$, $i=1,\ldots,s$, denote the vertices of the polytope $\mathcal{D}(g_0,\rho_0)$.

The disturbance vector belongs to the following set:

$$\mathcal{D}(w_0)=\left\{w\in\Re^l : \|w\| \le w_0,\, w_0 > 0\right\}, \tag{8}$$

where $\|\cdot\|$ denotes the Euclidean norm. Thus, the disturbance $w(t)$ is constrained to a hypersphere of radius w_0.

The dynamic compensator is given by

$$\begin{cases}\dot{\zeta}(t) = A_k\zeta(t) + B_k e(t)\\ u(t) = C_k\zeta(t) + D_k e(t),\end{cases} \tag{9}$$

where $\zeta(t)\in\Re^{nc}$, $\zeta(0)=0$, $e(t)=r(t)-y(t)$ is the error signal, $r(t)$ is the reference input, and A_k, B_k, C_k e D_k are unknown matrices of appropriate dimensions.

Assume also the reference input $r(t)$ belongs to the following bounded set:

$$\mathcal{R}=\left\{r\in\Re^p : r^T R^{-1} r \le 1,\, R=R^T\in\Re^{p\times p},\, R>0\right\} \tag{10}$$

The resulting closed-loop system can be written as follows:

$$\begin{cases}\dot{x}_f(t) = (\overline{A}+\overline{B}_2 L_k \overline{C}_y)x_f(t) + \overline{B}_{1f}w(t) + \overline{B}_2 L_k \Pi_1 r(t)\\ y(t) = -\Pi_1^T \overline{C}_y x_f(t)\\ z = (\overline{C}+\overline{D}_2 L_k \overline{C}_y)x_f(t) + D_1 w(t) + \overline{D}_2 L_k \Pi_1 r(t)\end{cases} \tag{11}$$

with $x_f(t)=\begin{pmatrix} x(t)\\ \zeta(t)\end{pmatrix}$ and the control output $u(t)$ described by

$$u(t) = +\Pi_2^T L_k \overline{C}_y x_f(t) + \Pi_2^T L_k \Pi_1 r(t), \tag{12}$$

where

$$L_k = \begin{bmatrix} D_k & C_k\\ B_k & A_k\end{bmatrix}. \tag{13}$$

and

$$\overline{A}=\begin{bmatrix} A & 0\\ 0 & 0_{nc}\end{bmatrix};\quad \overline{B}_2=\begin{bmatrix} B_2 & 0\\ 0 & I_{nc}\end{bmatrix};\quad \overline{C}_y=\begin{bmatrix} -Cy & 0\\ 0 & I_{nc}\end{bmatrix};$$

$$\overline{B}_{1f}=\begin{bmatrix} B_1\\ 0_{nc\times l}\end{bmatrix};\quad C=\begin{bmatrix} C & 0_{q1\times nc}\end{bmatrix};\quad \overline{D}_2=\begin{bmatrix} D_2 & 0_{q1\times nc}\end{bmatrix};$$

$$\Pi_1=\begin{bmatrix} I_p\\ 0_{nc\times p}\end{bmatrix};\quad \Pi_2=\begin{bmatrix} I_m\\ 0_{nc\times m}\end{bmatrix}.$$

From (5) and (6), the sets $\mathcal{D}(L_k,h,\mu,r)$ and $\mathcal{D}(L_k,\eta,\xi)$ defined by

$$\begin{aligned}\mathcal{D}(L_k,h,\mu,r) = \{x_f \in\Re^{n+nc} : h_i^T \Pi_2^T L_k \overline{C}_y x_f(t) +\\ + h_i^T \Pi_2^T L_k \Pi_1 r(t) \le \mu_i,\, i=1,2,\cdots,n_h\}\end{aligned} \tag{14}$$

$$\mathcal{D}(L_k,\eta,\xi)=\left\{x_f\in\Re^{n+nc} : \eta_i^T [C_y \quad 0] x_f \le \xi_i,\, i=1,2,\cdots,n_n\right\} \tag{15}$$

are the regions in the state space in which control and output saturation respectively do not occur. The constraints on x and x_0 can be rewritten as function of the closed loop states x_f.

$$\mathcal{D}_f(g_f,\rho)=\left\{x_f\in\Re^{n+nc} : g_{fi}^T x_f \le \rho_i,\, i=1,2,\cdots,n_g\right\}, \tag{16}$$

$$\mathcal{D}_f(g_{f0},\rho_0)=\left\{x_{f0}\in\Re^{n+nc} : g_{f0i}^T x_{f0} \le \rho_{0i},\, i=1,2,\cdots,n_0\right\}, \tag{17}$$

where $g_{fi}^T = \begin{bmatrix} g_i^T & 0_{1\times nc}\end{bmatrix}, i=1,2,\cdots,n_g$; $g_{f0i}^T = \begin{bmatrix} g_{0i}^T & 0_{1\times nc}\end{bmatrix}$, $i=1,2,\cdots,n_0$. Hence, from (16), (14) and (15), it is worth noticing that the resulting closed-loop system is valid only for the states x_f belonging to

$$\mathcal{D}_f(g_f,\rho)\cap\mathcal{D}(L_k,h,\mu,r)\cap\mathcal{D}(L_k,\eta,\xi). \tag{18}$$

Defining the matrices

$$\begin{aligned}\overline{A}_f &= \overline{A}+\overline{B}_2 L_k \overline{C}_y\\ \overline{B}_{2f} &= \overline{B}_2 L_k \Pi_1\\ \overline{C}_f &= \overline{C}+\overline{D}_2 L_k \overline{C}_y\\ \overline{D}_{2f} &= \overline{D}_2 L_k \Pi_1,\end{aligned} \tag{19}$$

the closed-loop transfer function from w to z is given by

$$H_f(s) = \overline{C}_f (sI-\overline{A}_f)^{-1}\overline{B}_{1f} + D_1, \tag{20}$$

with $A \in \mathcal{D}_A$ and $B_2 \in \mathcal{D}_B$.

The constrained $\mathcal{H}_\infty$ robust control problem can be formulated as follows.

Problem 1. Find a stabilizing linear dynamic output feedback controller $L_k \in \Re^{(m+nc)\times(p+nc)}$ *of fixed order* nc*, and a positive scalar* γ *such that the following specifications are satisfied:*

i) $L_k = \arg \min\{\gamma : \|H_f(s)\|_\infty \le \gamma, \forall A \in \mathcal{D}_A, \forall B_2 \in \mathcal{D}_B\}$

ii) the constraints (4), (5) and (6) are respected for all $x_0 \in \mathcal{D}(g_0, \rho_0)$ *and any admissible disturbance* $w(t) \in \mathcal{D}(w_0)$ *and reference input* $r(t) \in \mathcal{R}$.

Note that the controller L_k is a solution to Problem 1 if and only if the closed loop system (11) is asymptotically stable and no trajectory $x_f(t; x_{f0})$ emanating from the region $\mathcal{D}_f(g_{f0}, \rho_0)$ leaves the linearity domain (18) for any admissible disturbance and reference input.

III. MAIN RESULTS

The following definition will be useful for establishing some of the results in this paper.

Definition 1. Let $\mathcal{D}$ *and* $\mathcal{R}$ *be compact and convex sets containing the origin and let* Ω *be a non-empty set.* Ω *is said to be a positively* $(\mathcal{D},\mathcal{R})$*-invariant set with respect to the system (11) if for every initial state* $x_f(t_0) \in \Omega$ *and every disturbance sequence* $w(t) \in \mathcal{D}, x_f(t) \in \Omega$, $\forall t \ge t_0$ *and* $\forall r(t) \in \mathcal{R}$.

Let $\mathcal{L}$ denote the set of $L_k \in \Re^{(m+nc)\times(p+nc)}$ such that $\overline{A}_f$ is asymptotically stable $\forall A \in \mathcal{D}_A$ and $\forall B_2 \in \mathcal{D}_B$. The next proposition follows directly from Definition 1.

Proposition 1. The dynamic output feedback L_k *is a solution to Problem 1 if and only if*

$$L_k = \arg \min\{\gamma : \|H_f(s)\|_\infty \le \gamma, L_k \in \mathcal{L}\}$$

and there exists a positively $(\mathcal{D},\mathcal{R})$*-invariant set* $\Omega \subset \Re^{nxnc}$ *with respect to the closed-loop system (11) such that*

$$\mathcal{D}_f(g_{f0}, \rho_0) \subseteq \Omega \subseteq \mathcal{D}_f(g_f, \rho)$$

$$\Omega \subseteq \mathcal{D}(L_k, h, \mu, r) \cap \mathcal{D}(L_k, \eta, \xi).$$

It is well-known that the Lyapunov functions generate positively invariant sets for asymptotically stable systems. In this paper, we are interested in ellipsoidal positively invariant sets generated by quadratic Lyapunov functions of the type $v(x) = x^T P x$, where $P = P^T > 0$. Thus, consider the set Ω defined as follows:

$$\Omega = \left\{x_f \in \Re^{n+nc}; x_f^T W_1^{-1} x_f \le 1, W_1 = W_1^T > 0\right\} \qquad (21)$$

where $W_1 \in \Re^{(n+nc)\times(n+nc)}$.

The next result presents a sufficient condition concerning the positive $(\mathcal{D},\mathcal{R})$-invariance.

Lemma 1. Consider the sets $\mathcal{R}$, $\mathcal{D}(w_0)$ *and* Ω *defined respectively in (10), (8) and (21). Let* $w_0 > 0$, $W_1 = W_1^T \in \Re^{(n+nc)\times(n+nc)}$ *and* L_k *be given. Assume* $\Omega \subseteq \mathcal{D}_f(g_f, \rho)$ *and* $\Omega \subseteq \mathcal{D}(L_k, h, \mu, r) \cap \mathcal{D}(L_k, \eta, \xi)$. *If there exist* $\alpha_l \ge 0$ *and* $\sigma_l \ge 0$, $l = 1, 2, \cdots, N*M$, *satisfying*

$$\begin{bmatrix} \overline{A}_f W_1 + W_1 \overline{A}_f^T + \alpha_l W_1 & \overline{B}_{1f} & \overline{B}_{2f} \\ \overline{B}_{1f}^T & -\sigma_l I & 0 \\ \overline{B}_{2f}^T & 0 & -(\alpha_l - \sigma_l w_0^2) R^{-1} \end{bmatrix} \le 0,$$

$\forall A \in \mathcal{D}_A$ and $\forall B_2 \in \mathcal{D}_B$, *then* Ω *is a positively* $(\mathcal{D},\mathcal{R})$*-invariant set with respect to the uncertain system (11).*

Proof: It is based on S-procedure [18]. For the sake of brevity this proof is omitted here.

The next theorem provides a solution for the constrained $\mathcal{H}_\infty$ robust control problem for dynamic output feedback.

Theorem 1. Consider the uncertain system with time domain constraints (1) and the set Ω *defined in (21). Let* α_L *and* σ_L *denote the vectors* $[\alpha_1 \alpha_2 \cdots \alpha_{N*M}]$, $\alpha_i \ge 0, \forall i$, *and* $[\sigma_1 \sigma_2 \cdots \sigma_{N*M}]$, $\sigma_i \ge 0, \forall i$, *respectively. Let* $w_0 > 0$ *be given. Let* $(\hat{W}_1, \hat{L}_k, \hat{\alpha}_L, \hat{\sigma}_L)$ *be the solution of the following optimization problem*

$$\min_{W_1, L_k, \alpha_L, \sigma_L} \gamma \qquad (22)$$

s. t.

$$g_{fi}^T W_1 g_{fi} \le \rho_i^2, \quad i = 1, 2, \cdots, n_g, \qquad (23)$$

$$\begin{bmatrix} 1 & \begin{bmatrix} v_i & 0 \end{bmatrix} \\ \begin{bmatrix} v_i^T \\ 0 \end{bmatrix} & W_1 \end{bmatrix} \geq 0, \quad i = 1, 2, \cdots, n_v, \qquad (24)$$

$$h_i^T \Pi_2^T L_k \overline{C}_y W_1 \overline{C}_y^T L_k^T \Pi_2 h_i \leq \overline{c}_i^{\,2}, \quad i = 1, 2, \cdots, n_h, \qquad (25)$$

with $\overline{c}_i = \mu_i - \sqrt{h_i^T \Pi_2^T L_k \Pi_1 R \Pi_1^T L_k^T \Pi_2 h_i}$,

$$\eta_i^T \Pi_1^T \overline{C}_y W_1 \overline{C}_y^T \Pi_1 \eta_i \leq \xi_i^{\,2}, \quad i = 1, 2, \cdots, n_n, \qquad (26)$$

$$\begin{bmatrix} \overline{A}_f W_1 + W_1 \overline{A}_f^T & B_{1f} & W_1 \overline{C}_f^T \\ B_{1f}^T & -I & D_1^T \\ \overline{C}_f W_1 & D_1 & -\gamma^2 I \end{bmatrix} < 0, \qquad (27)$$

$$\begin{bmatrix} \overline{A}_f W_1 + W_1 \overline{A}_f^T + \alpha_l W_1 & \overline{B}_{1f} & \overline{B}_{2f} \\ \overline{B}_{1f}^T & -\sigma_l I & 0 \\ \overline{B}_{2f}^T & 0 & -(\alpha_l - \sigma_l w_0^2) R^{-1} \end{bmatrix} \leq 0, \qquad (28)$$

for $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, M$, *and* $l = 1, 2, \cdots, N * M$, *with* $W_1 = W_1^T \in \Re^{(n+nc)\times(n+nc)}$, $W_1 > 0$, $\forall A \in \mathcal{D}_A$ and $\forall B_2 \in \mathcal{D}_B$.

Then the controller $\hat{L}_k$ *is a solution of Problem 1. The upper bound of all feasible* $\left\| H_f(s) \right\|_\infty$ *is given by* γ *and the suitable ellipsoidal set* Ω *is generated by* $\hat{W}_1$ *for the closed-loop system (11).*

Proof: From the bounded real lemma [19] and the concept of quadratic stability [4], the LMIs (27) assure that this problem of minimization gives an upper bound γ of the $\mathcal{H}_\infty$ cost for the uncertain system (11), with the stabilizing controller $\hat{L}_k$. From LMIs (23) and (24) obtained by geometric considerations (Boyd *et al.*, 1994), one can conclude that $\mathcal{D}_f(g_{f0}, \rho_0) \subseteq \Omega \subseteq \mathcal{D}_f(g_f, \rho)$, *i.e.*, the sate constraint is satisfied for the controller $\hat{L}_k$. From LMIs (25) and (26), the ellipsoid Ω is contained in the region $\mathcal{D}(L_k, h, \mu, r) \cap \mathcal{D}(L_k, \eta, \xi)$. The procedure to obtain these LMIs is similar to the one discussed in [2]. Thus, the dynamic output feedback controller $\hat{L}_k$ guarantees that control and output constraints are respected. Finally, by Lemma 1, if the inequalities (28) hold, then Ω is a positively $(\mathcal{D},\mathcal{R})$-invariant set with respect to system (11). Consequently, the controller $\hat{L}_k$ and γ solve the constrained $\mathcal{H}_\infty$ robust control problem, and Ω is a suitable ellipsoidal positively $(\mathcal{D},\mathcal{R})$-invariant set.

Note that the problem (22) is not jointly convex in L_k, W_1 and α_L. The matrix inequalities associated to the control constraint (25) don't allow solving the problem with convex techniques even for a fixed matrix W_1. Nevertheless, for a fixed L_k, the optimization problem (22) is bilinear in α_L and W_1. This fact is explored by the proposed algorithm.

IV. SYNTHESIS ALGORITHM

In this section a hybrid design procedure of robust output feedback controller for solving the constrained $\mathcal{H}_\infty$ robust control problem (22) is introduced. The proposed procedure combines the reliability properties of the Genetic Algorithms [8] and their typical search heuristics with the accuracy and efficiency of the LMI solving methods [6].

Based on Genetic Algorithms and LMIs, this algorithm searches an optimal robust controller L_k (13) and α_L associated and consequently determines W_1 that solve the optimization problem (22). Note that for fixed L_k and α_L the constrained problem (22) is convex. Thus the algorithm works with a population of candidate solutions (individuals) $[\alpha_L, L_k]$. At each generation the optimization problem (22) is solved using Matlab package LMI-Lab [6] for all candidate solutions of a population of size p_s. The algorithm stops when a number of generations n_{gen} is reached. The fitness function that provides the mechanism for evaluating each individual is defined as

$$f(\alpha_L, L_k) = \frac{1}{1 + \min\limits_{W_1, \sigma_L} \gamma}.$$

4.1 Algorithm

BEGIN

Generate initial feasible population $x = [\alpha_L, L_k]$.

Evaluate population

WHILE n_{gen} *is not achieved*

Store the best individual of the population

Perform selection (roulette)

WHILE infeasible AND attempt $< n_r$

Perform arithmetical recombination ($pc = 1$) [9]

Choose $q \in [0,1]$

$x'_1 = q x_1 + (1-q) x_2$

$x'_2 = q x_2 + (1-q) x_1$

END_WHILE

Perform uniform mutation [15]

$p_m = 0.5(f_{\max} - f)/(f_{\max} - \bar{f}), \quad f \geq \bar{f}$

$p_m = 0.5, \quad f < \bar{f}$

Choose scalar a

WHILE infeasibility AND attempt < n_u

$$a' = \frac{a}{2^{attempt}},$$

$$x'_k = x_k + a'$$

$$x' = (x_1, \cdots, x'_k, \cdots, x_q)$$

END_WHILE

Substitute the worst individual for the best individual stored previously

END_WHILE

Solve LMIs for the best individual

END

Notice that pc and pm are, respectively, the probability of recombination and mutation. $f_{\max}$ is the maximum fitness value of the population, $\bar{f}$ is the average fitness value of the population and f is the fitness value of the solution. a is a number uniformly chosen in $[-\beta, \beta]$. The parameter β gives the possibility of changing the mutation at each iteration. Since the infeasibility of new candidate solutions can occur due to recombination and mutation procedures (matrix inequalities are quite sensitive to parameter changes), the algorithm may execute the recombination up to n_r times, choosing new q, and up to n_u times for the mutation, adjusting parameter a', in an attempt to reduce this problem. n_r and n_u must be adjusted to each kind of problem.

Since the search space is not convex and its bounds are unknown, the classical random initial population generation requires a strong computational effort. In order to reduce this effort generating feasible elements, an approach described in [10] is implemented. This approach gives a sufficient condition to find feasible controllers using the Lyapunov inequality associated to the $\mathcal{H}_\infty$-norm for the state feedback control. For this aim, the output feedback control problem is rewritten as a state feedback one. In this procedure, the transformation matrices carry the diversity of the initial population, since for each generated transformation a distinct initial solution is found. The control constraint is relaxed during the initialization step because the transformation doesn't generate a convex constraint. The vector α_L is randomly generated.

V. NUMERICAL EXAMPLE

Consider the uncertain continuous-time system [11]. The system matrices are

$$A = \begin{bmatrix} 0.6812 & 0.2944 & -0.9223 \\ 0.8284 & -1.6680 & -0.4420 \\ 0.2091 & 1.5766 & -a_{33} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.9386 & -2.1884 \\ b_{21} & 0.2947 \\ -1.0445 & -0.2946 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $-1 \leq a_{33}(t) \leq 1$, $-0.4723 \leq b_{21}(t) \leq 0.9445$ and

$$B_1 = I_3; \; C_y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_y = [0], \quad D_{yw} = [0],$$

$$D_1 = [0], \quad D_2^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The state constraints are given by

$$-50 \leq x_i(t) \leq 50, \; i = 1,2,3.$$

The control is subject to the constraints

$$-40 \leq u_i(t) \leq 40, i = 1,2.$$

The output constraints are given by

$$\begin{cases} -40 \leq y_1(t) \leq 40 \\ -30 \leq y_2(t) \leq 30, \end{cases}$$

and the disturbance $w(t)$ is contained in a sphere of radius $w_0 = 0{,}5$.

The reference input satisfies the constraints

$$-2 \leq r_i \leq 2, \; i = 1, 2.$$

The region of admissible initial states is centered in the origin and defined by the inequalities

$$-1.25 \leq x_{0i}(t) \leq 1.25, i = 1,2,3.$$

The dynamic output feedback obtained by the synthesis algorithm is

$$A_k = \begin{bmatrix} -1.5417 & -0.3378 \\ -0.3378 & -1.3303 \end{bmatrix}, \quad B_k = \begin{bmatrix} 0.2446 & -0.0979 \\ -0.4218 & -0.4871 \end{bmatrix},$$

$$C_k = \begin{bmatrix} 0.9132 & 1.9913 \\ -0.3101 & -0.0778 \end{bmatrix}, \quad D_k = \begin{bmatrix} 0.1844 & -5.2202 \\ -2.2781 & -0.4089 \end{bmatrix}.$$

and the associated $\mathcal{H}_\infty$-norm upper bound is $\gamma = 14.4035$. The algorithm was executed for 30 generations with a population of 20 elements.

The positive $(\mathcal{D},\mathcal{R})$-invariant set Ω is determined by

$$W_1^{-1} = \begin{bmatrix} 0.0074 & 0.0020 & 0.0058 & -0.0035 & 0.0127 \\ 0.0020 & 0.0236 & 0.0089 & 0.0063 & -0.0411 \\ 0.0058 & 0.0089 & 0.0355 & 0.0063 & 0.0055 \\ -0.0035 & 0.0063 & 0.0063 & 0.0251 & -0.0332 \\ 0.0127 & -0.0411 & 0.0055 & -0.0332 & 0.1407 \end{bmatrix}$$

The associated best vector α_L is:

$$\alpha_L = [1.3992 \quad 1.3517 \quad 1.1522 \quad 1.9013].$$

VI. CONCLUSION

In this work the constrained $\mathcal{H}_\infty$ robust control problem by dynamic output feedback has been addressed, for systems subject to parameter uncertainties and time domain constraints. The problem formulation was based on the concepts of positive $(\mathcal{D},\mathcal{R})$-invariance and quadratic stability. A sufficient condition has been obtained that guarantees positive $(\mathcal{D},\mathcal{R})$-invariance of an ellipsoidal set contained in the linearity region of the system with disturbance.

A hybrid algorithm mixing genetic algorithms and linear matrix inequalities has been proposed, exploring the bilinear relation between the controller L_k and Lyapunov matrix W_1 that exists in the problem formulation by matrix inequalities. This algorithm has been applied to many others examples and the simulations results indicate that this approach can offer an effective and simple method to solve the constrained $\mathcal{H}_\infty$ robust control problem. The proposed algorithm is suitable for full or reduced order dynamic output controller design.

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