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A Stabilizing Time-switching Control Strategy for the Rolling Sphere

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Abstract— **The problem of the asymptotic stabilization of a five dimensional nonholonomic systems, namely the "ball and plate" or "rolling sphere" system, is discussed and solved by means of a hybrid control law relying on a suitable finite state machine. A control law is associated to each state of the machine and, by using a simple switching strategy, the origin is proven to be globally asymptotically stable in the sense of Lyapunov. Moreover, a particular function is proven to be a Lyapunov function for the considered hybrid system. The chosen control law takes naturally into account the presence of possible control saturations. Simulations are presented showing the effectiveness of the proposed control scheme.**

I. Introduction

In recent years, non-holonomic systems have been widely analized since they represent a paradigm for a number of mechanical systems such as multi-fingered robot hands and wheeled mobile robots (see, for example [1] and [2]). These systems are controllable, hence the controllability or motion planning problem have been widely investigated (see, for instance, [3], [4], [5] and the references cited therein).

However, as emphasized by Brockett's theorem ([6], see also [7]), these systems are not stabilizable by means of smooth (or a class of discontinuous) control laws. This important fact has motivated the use either of time-varying or of discontinuous feedback laws. While several results are available for systems in "chained" (or "power") form, see e.g. [8], [9], [10], [11], [12], [13], for systems not in these form (or not feedback equivalent to these form) either very general results exist [14], [15], [16] – which however cannot be easily used to explicitly find a control – or dedicated solutions for particular systems have been proposed [17], [18], [19], [20], [21], [22], [23].

The system taken into account in this paper, namely the "ball and plate" system, cannot be transformed into a chained form since it doesn't fulfill the conditions pointed out in [24]; as a consequence, results valid for such a class of systems cannot be applied in order to asymptotically stabilize the zero equilibrium. We propose a solution to the asymptotic stabilization problem by means of a switching controller. Our approach is based on the definition of a candidate Lyapunov function (LF in the following) for the resulting hybrid system and the control law is consequently designed. Finally, control saturations can be naturally incorporated in the design.

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II. Description of the System

The problem of controlling a sphere rolling on a plane arises in many applications (see, for example, [25]). The system consists of a ball moving without friction and without slipping between two planes; we suppose that one of the planes, say the bottom one, is fixed while the other is used to apply the control actions to the sphere at its contact point. The system represents the paradigm for a local approximation of all rolling manipulation situations, the two planes corresponding to two fingers and the ball to the object to be manipulated. No further details are given here on the physics of the motion (for details see [2], [26] and [20]). Our starting point is the set of differential equations locally describing the system [26], namely: $\dot{x} = u$, $\dot{y} = v$, $\dot{z} = xv - yu$, $\dot{\mu} = x^2 v$, $\dot{\nu} = y^2 u$, where u and v are the input variables. With a simple coordinates transformation, namely $z_1 \triangleq (xy+z)/2$, $z_2 \triangleq \mu$ and $z_3 \triangleq (y^2 x - \nu)/2$, the above system becomes:

$$\left\{ \begin{array}{lcl} \dot{x} & = & u \\ \dot{y} & = & v \\ \dot{z}_1 & = & xv \\ \dot{z}_2 & = & x^2 v \\ \dot{z}_3 & = & xyv \,, \end{array} \right. \tag{1}$$

for which the zero–equilibrium is preserved. We denote by $\mathbf{s} \triangleq (x, y, z_1, z_2, z_3)^\top$ the state vector of the system and we introduce the following sets:

$$\begin{array}{lcl} \mathcal{S}_0 & \triangleq & \{\mathbf{s} \in \mathbb{R}^5 \mid x = 0; y = 0\} \,, \\ \mathcal{S}_1 & \triangleq & \{\mathbf{s} \in \mathbb{R}^5 \mid x = 0; y = 0; z_1 = 0\} \,, \\ \mathcal{S}_2 & \triangleq & \{\mathbf{s} \in \mathbb{R}^5 \mid x = 0; y = 0; z_1 = 0; z_2 = 0; z_3 = 0\} \,. \end{array}$$

Notice that $\{0\} = \mathcal{S}_2 \subset \mathcal{S}_1 \subset \mathcal{S}_0$. Finally we introduce the *sign* and *saturation* functions:

$$\mathrm{sg}(x) \triangleq \left\{ \begin{array}{rl} 1 & \text{if } x \geqslant 0 \\ -1 & \text{otherwise} \end{array} \right. , \ \mathrm{sat}(x) \triangleq \left\{ \begin{array}{rl} 1 & \text{if } x > 1 \\ x & \text{if } |x| \leqslant 1 \\ -1 & \text{if } x < -1 \,. \end{array} \right.$$

The problem to be addressed is the following.

Problem 2.1: Given the system (1), design a control scheme that globally asymptotically stabilizes the zero equilibrium of the closed loop system.

In the paper, a solution to the above problem will be proposed based on a switching control strategy; the choice of a time varying strategy is due to the advantage that we are able to construct a LF thus proving stability of the system.

We underline the fact that the strategy leads to a closed loop control scheme, differently from the open-loop solutions proposed, for example, in [22]. A former example of a closed loop solution can be found in [20] where, nevertheless, one of the state coordinates is not taken into account in the design of the stabilizing strategy.

III. The switching control strategy

The general structure of a switching control scheme [27] can easily be represented by means of a finite state machine (FSM), each state of which corresponds to one of the controllers. Therefore, we introduce the FSM and denote with q the discrete variable associated to the current state of the machine. Moreover, let $\mathcal{Q} \triangleq \{0,1,2,3\}$ be the set of all possible values of q and let τ denote the generic switching time–instant. In the following, we describe the rationale of the control law $w_q \triangleq [u_q(\mathbf{s}), v_q(\mathbf{s})]^\top$ for $q \in \mathcal{Q}$, associated to the different states of the machine. The idea is to use the k-th order time derivative of the LF in order to predict its behaviour when the first $k-1$ derivatives vanish. The approach is similar to the one followed in [28].

• Let $q = 0$. Given a suitable candidate LF V (which will be defined in the next Section), w_0 is chosen in such a way that, regardless of the particular value $\mathbf{s}(\tau)$ of the continuous state of the system at the switching instant τ, $V(t)$ is non-increasing for all $t \geqslant \tau$ i.e.[1]

$$\dot{V}[\mathbf{s}(t)]\Big|_{q=0} \leqslant 0\,, \quad \forall\, t \geqslant \tau\,,\ \forall\, \mathbf{s}(\tau) \in \mathbb{R}^5$$

and both x and y tend to zero as time tends to infinity. i.e. $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$.

• Let $q = 1$. w_1 is chosen in such a way that, at the switching instant τ, the first order time derivative of the LF is non-positive regardless of $\mathbf{s}(\tau)$, while the second derivative is strictly negative if $\mathbf{s}(\tau) \in \mathcal{S}_0 \backslash \mathcal{S}_1$; namely

$$\dot{V}[\mathbf{s}(t)]\Big|_{q=1,\, t=\tau} \leqslant 0\,, \qquad \forall\, \mathbf{s}(\tau) \in \mathbb{R}^5$$

$$\ddot{V}[\mathbf{s}(t)]\Big|_{q=1,\, t=\tau} < 0\,, \qquad \forall\, \mathbf{s}(\tau) \in \mathcal{S}_0 \backslash \mathcal{S}_1\,.$$

• Let $q \in \{2,3\}$. w_2 and w_3 are designed together and chosen in such a way that, at the switching instant τ, the first order time derivative of the LF is non-positive regardless of $\mathbf{s}(\tau)$, while if $\mathbf{s}(\tau) \in \mathcal{S}_1 \backslash \mathcal{S}_2$ the second order time derivative is zero and, at least for one of the two possible values of q, the third order time derivative is strictly negative; namely

$$\dot{V}[\mathbf{s}(t)]\Big|_{q\in\{2,3\},\, t=\tau} \leqslant 0\,,\ \forall\, \mathbf{s}(\tau) \in \mathbb{R}^5$$

$$\ddot{V}[\mathbf{s}(t)]\Big|_{q\in\{2,3\},\, t=\tau} = 0\,,\ \forall\, \mathbf{s}(\tau) \in \mathcal{S}_1 \backslash \mathcal{S}_2\,,$$

$$\exists\, \bar{q} \in \{2,3\} \mid \dddot{V}[\mathbf{s}(t)]\Big|_{q=\bar{q},\, t=\tau} < 0\,,\ \forall\, \mathbf{s}(\tau) \in \mathcal{S}_1 \backslash \mathcal{S}_2\,.$$

[1]With a little abuse of notation, here and in the following, for a generic function $W(\mathbf{s})$ not depending on the discrete variable q, we denote by $W(\mathbf{s})|_{q=i}$ the value of $W(\mathbf{s})$ when $q = i$.

Based on this rationale, the proposed control law is:

$$u_q(\mathbf{s}) = \begin{cases} -x\,, & \text{if } q = 0 \\ c_1(q,\mathbf{s})\,, & \text{if } q \in \{1,2,3\} \end{cases} \tag{2}$$

$$v_q(\mathbf{s}) = \begin{cases} -y^3 - z_1 x - z_2 x^2 - z_3 xy\,, & \text{if } q = 0 \\ c_2(q,\mathbf{s})\,, & \text{if } q \in \{1,2,3\} \end{cases} \tag{3}$$

where c_1 and c_2 are suitable constant[2] to be updated every time that the FSM switches to one of the states in the set $\mathcal{Q} = \{1,2,3\}$ according to the value $\mathbf{s}(\tau)$ that the continuous state of the system takes on at the switching time instant τ.

To analyze the properties of the above control law, we introduce some further notations. Let q_τ and $\mathbf{s}(\tau)$ be the discrete state of the machine and the continuous state of the system (1) at the instant $t = \tau$, respectively, and let us introduce the following time instant[3]:

$$T_{\min}[q_\tau, \mathbf{s}(\tau)] \triangleq \sup_{t \geqslant \tau} \left\{ t \,\Big|\ \dot{V}[\mathbf{s}(\sigma)]\Big|_{q=q_\tau} \leqslant 0, \forall\, \sigma \in [\tau, t) \right\} - \tau\,. \tag{4}$$

Moreover, for a given $T_D > 0$ and for $i \in \{0,1,2,3\}$, let

$$\mathcal{I}(i,\tau) \triangleq \{ j \in \{1,2,3\},\ j > i \,|\, T_{\min}[j, \mathbf{s}(\tau)] > T_D \}\,,$$

and, if $\mathcal{I}(i,\tau) \neq \emptyset$, let $l(i) \triangleq \min \mathcal{I}(i,\tau)$. The discrete state q of the machine is switched to a new state q^+ at $t = kT_D$, with $k \in \mathbb{N}\backslash\{0\}$, according to the following switching law.

Switching law. If $q_{T_D} = i$ with $i \in \{0,1,2,3\}$, then

$$q^+[i, \mathbf{s}(kT_D)] = \begin{cases} l(i) & \text{if } \mathcal{I}(i, kT_D) \neq \emptyset \\ 0 & \text{otherwise}\,. \end{cases} \tag{5}$$

In Figure 1 a scheme is depicted showing the modes of behavior of the FSM.

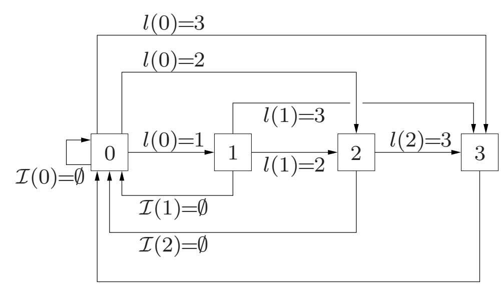

Fig. 1. Modes of behavior of the FSM (the variable q is emphasized and the time instant τ is not shown for the sake of simplicity).

IV. Stability analysis

In this section, we will show that for system (1) there exists a LF and a suitable choice of the scalars T_D, c_1 and c_2 such that the swtiching control law described in the previous section asympotically stabilizes the zero equilibrium of (1). For, let

$$V(\mathbf{s}) \triangleq \frac{x^4}{4} + \frac{y^4}{4} + \frac{z_1^2}{2} + \frac{z_2^2}{2} + \frac{z_3^2}{2} \tag{6}$$

[2]The term "constant" means that the scalars c_1 and c_2 are determined at each switching instant and are kept constant until the next switching event.

[3]Here we suppose that the set defined in the bracketed expression is not empty and admits a supremum. It will be clear in the following that this is always the case; however, we let $T_{\min}[q_\tau, \mathbf{s}(\tau)] \triangleq 0$ when the given definition is meaningless.

be the candidate LF. The following simple result can be immediately proven.

Lemma 4.1: Consider system (1) and the candidate LF (6). If $q = 0$ and the control law (2) and (3) is applied, then

$$T_{\min}(0, \mathbf{s}) = +\infty\,, \quad \forall\, \mathbf{s} \in \mathbb{R}^5\,. \tag{7}$$

□

Proof: Note that

$$\begin{aligned} \dot V(t)\Big|_{q=1} &= x^3 u + (y^3 + z_1 x + z_2 x^2 + z_3 xy) v \\ &= -x^4 - (y^3 + z_1 x + z_2 x^2 + z_3 xy)^2 \leqslant 0\,, \end{aligned} \tag{8}$$

from which (7) immediately follows. ■

When $q \in \{1, 2, 3\}$ a result as Lemma 4.1 does not hold[4]; nevertheless, by a different reasoning, it can be proven that there exists a suitable choice of $T_D > 0$, *independent from the discrete state* q, such that $T_{\min}(q, \mathbf{s}) > T_D$ for every $\mathbf{s} \in \mathcal{S}_0$. For, let us first determine the solutions of the differential system (1) for *constant* inputs $u = c_1$ and $v = c_2$ and when the initial condition is the value $\mathbf{s}(\tau)$ that the continuous state of the system takes on at the switching instant τ:

$$\begin{aligned} x(t) &= x(\tau) + c_1 r \\ y(t) &= y(\tau) + c_2 r \\ z_1(t) &= z_1(\tau) + c_2 x(\tau) r + \tfrac{1}{2} c_1 c_2 r^2 \\ z_2(t) &= z_2(\tau) + c_2 x(\tau)^2 r + c_1 c_2 x(\tau) r^2 + \tfrac{1}{3} c_1^2 c_2 r^3 \\ z_3(t) &= z_3(\tau) + c_2 x(\tau) y(\tau) r + \\ &\quad + \tfrac{1}{2}\left[c_1 c_2 y(\tau) + c_2^2 x(\tau)\right] r^2 + \tfrac{1}{3} c_1 c_2^2 r^3\,, \end{aligned} \tag{9}$$

where $r \triangleq t - \tau$. Now, let $A \triangleq x(\tau)^3$ and $B \triangleq y(\tau)^3 + z_1(\tau)x(\tau) + z_2(\tau)x(\tau)^2 + z_3(\tau)x(\tau)y(\tau)$ and set

$$c_1 = -\varphi_1\, \psi \quad \text{and} \quad c_2 = -\varphi_2\, \psi\,, \tag{10}$$

with φ_1 and φ_2 suitable scalar functions to be determined and $\psi = \mathrm{sg}(A\varphi_1 + B\varphi_2)$. It is easy to see that, for these selections of c_1 and c_2, the value of the first order time derivative of the LF is not positive if computed at the switching time instant. Specifically:

$$\begin{aligned} \dot V(t)\Big|_{q\in\{1,2,3\}, t=\tau} &= Ac_1 + Bc_2 = -(A\varphi_1 + B\varphi_2)\psi = \\ &= -\left|A\varphi_1 + B\varphi_2\right| \leqslant 0\,. \end{aligned} \tag{11}$$

We now prove some useful results.

Lemma 4.2: Suppose that $q = 1$ and that the control law applied to system (1) takes on the form (10) with:

$$\varphi_1\big|_{q=1} \triangleq \frac{\mathrm{sat}(z_1(\tau))}{D_1} \quad \text{and} \quad \varphi_2\big|_{q=1} \triangleq -\left|\varphi_1\big|_{q=1}\right|\,, \tag{12}$$

where $D_1 \triangleq \sqrt{1 + z_1(\tau)^2 + z_2(\tau)^2 + z_3(\tau)^2}$. Then, for any continuous state $\mathbf{s}_0 \in \mathcal{S}_0 \backslash \mathcal{S}_1$ and for the candidate LF (6),

$$\exists\, T_F > 0 \mid T_{\min}(1, \mathbf{s}_0) > T_F\,.$$

□

Proof: Note that, if the continuous state at the switching instant is $\mathbf{s}(\tau) = \mathbf{s}_0 \in \mathcal{S}_0 \backslash \mathcal{S}_1$, then $A = B = 0$, which, in turn, implies the following two facts.

[4]For constant inputs, in fact, x and y diverge for $t \to \infty$ and so will $V[\mathbf{s}(\tau)]$; this means that $T_{\min}(q, \mathbf{s}) < +\infty$ if $q \in \{1, 2, 3\}$.

- By the first equality in (11),

$$\dot V(t)\Big|_{q=1, t=\tau} = 0\,. \tag{13}$$

- $c_1\big|_{q=1} = -\,\varphi_1\big|_{q=1}$ and $c_2\big|_{q=1} = -\,\varphi_2\big|_{q=1}$.

Moreover, the expression of the first order time derivative for $\mathbf{s} \in \mathcal{S}_0 \backslash \mathcal{S}_1$ is

$$\begin{aligned} \dot V(t)\Big|_{q=1} &= c_1 c_2 z_1(\tau) r + \left[c_1^2 c_2 z_2(\tau) + c_1 c_2^2 z_3(\tau)\right] r^2 + \\ &+ \left(c_1^4 + c_2^4 + \frac{1}{2} c_1^2 c_2^2\right) r^3 + \frac{1}{3}\left(c_1^4 c_2^2 + c_1^2 c_2^4\right) r^5\,, \end{aligned} \tag{14}$$

where $r = t - \tau$. Hence

$$\ddot V(t)\Big|_{q=1, t=\tau} = \varphi_1\big|_{q=1}\, \varphi_2\big|_{q=1}\, z_1(\tau) < 0\,, \tag{15}$$

where the last inequality immediately follows by considering (12) and recalling that $z_1(\tau) \neq 0$ for $\mathbf{s}_0 \in \mathcal{S}_0 \backslash \mathcal{S}_1$. Equations (13) and (15) imply that the candidate LF decreases at least for $t \in (\tau, \tau + T)$, where T is a finite (though possibly small) positive number. What we want to show is that there exists a strictly positive lower bound T_F for T. For, note that $\varphi_2\big|_{q=1} = -\mathrm{sg}[z_1(\tau)]\, \varphi_1\big|_{q=1}$ and rewrite (14) as

$$\begin{aligned} \dot V(t) &= -\varphi_1^2\, |z_1(\tau)|\, r + \varphi_1^3 \rho_1 r^2 + \frac{5}{2}\varphi_1^4 r^3 + \frac{2}{3}\varphi_1^6 r^5 = \\ &= \varphi_1^2 |z_1(\tau)| r \left(-1 + \frac{\varphi_1 \rho_1 r}{|z_1(\tau)|} + \frac{5}{2}\frac{\varphi_1^2 r^2}{|z_1(\tau)|} + \frac{2}{3}\frac{\varphi_1^4 r^4}{|z_1(\tau)|}\right) \end{aligned} \tag{16}$$

where $\rho_1 = \mathrm{sg}[z_1(\tau)] z_2(\tau) - z_3(\tau)$ and the dependency of φ_1 from q has been dropped to simplify the notation.

We now establish a few inequalities, which will be used to conclude the proof.

- $\frac{\varphi_1 \rho_1}{|z_1(\tau)|} \leqslant \sqrt{2}$. In fact if $|z_1(\tau)| \geqslant 1$ it follows that[5]

$$\frac{\varphi_1 \rho_1}{|z_1(\tau)|} \leqslant \frac{|\rho_1|}{D_1 |z_1(\tau)|} \leqslant \frac{|\rho_1|}{\sqrt{1 + z_2(\tau)^2 + z_3(\tau)^2}} \leqslant \sqrt{2}$$

Otherwise, if $|z_1(\tau)| < 1$, we have

$$\frac{\varphi_1 \rho_1}{|z_1(\tau)|} = \frac{\rho_1}{D_1 \mathrm{sg}[z_1(\tau)]} \leqslant \frac{|\rho_1|}{D_1} \leqslant \sqrt{2}$$

- Analogously, it can be proven that $\frac{\varphi_1^4}{|z_1(\tau)|} \leqslant 1$ and $\frac{\varphi_1^2}{|z_1(\tau)|} \leqslant 1$.

Now, consider the function

$$f(r) \triangleq -1 + \sqrt{2}\, r + \frac{5}{2} r^2 + \frac{2}{3} r^4\,, \tag{17}$$

continuous and such that $f(0) = -1$ and $\lim_{r \to +\infty} f(r) = +\infty$. Then, $\exists\, r^\star \in (0, +\infty)$ such that $f(r^\star) = 0$; let $T_F \triangleq \min\{r \in (0, +\infty) \mid f(r) = 0\} \simeq 0.4048$. Clearly, $\forall\, r \in [0, T_F)$, $f(r) < 0$ that is:

$$1 > \sqrt{2} r + \frac{5}{2} r^2 + \frac{2}{3} r^4\,, \quad \forall\, r \in [0, T_F)\,.$$

This, using the above inequalities, yields, $\forall\, t \in [\tau, \tau + T_F)$

$$1 > \frac{\varphi_1 \rho_1}{|z_1(\tau)|} t + \frac{5}{2}\frac{\varphi_1^2}{|z_1(\tau)|} t^2 + \frac{2}{3}\frac{\varphi_1^4}{|z_1(\tau)|} t^4\,.$$

[5]Recall that for any a and b, we have $|a - b| < \sqrt{2(1 + a^2 + b^2)}$.

Multiplying by the positive quantity $\varphi_1^2\,|z_1(\tau)|\,r$ and recalling Equation (16), it is easy to conclude that $\dot{V}(t) < 0$ for $t \in (\tau, \tau+T_F)$. Finally, note that T_F is constant as it is one of the roots of a polynomial with constant coefficients. ■

Lemma 4.3: Suppose that $q{=}i$, with $i{\in}\{2,3\}$, and that the control law applied to system (1) takes the form (10) with:

$$\varphi_1|_{q=i} \triangleq \frac{\text{sat}(\rho_i)}{D_2} \quad \text{and} \quad \varphi_2|_{q=i} \triangleq \delta_i \varphi_1|_{q=i}\,, \tag{18}$$

where $D_2 \triangleq \sqrt{1+z_2(\tau)^2+z_3(\tau)^2}$, $\rho_i = z_2(\tau)+\delta_i z_3(\tau)$ and $\delta_2 > 0$, $\delta_3 > 0$, $\delta_2 \neq \delta_3$ are suitable scalars to be specified. Then, if the continuous state at the switching time instant is $\mathbf{s}_1 \in \mathcal{S}_1 \backslash \mathcal{S}_2$, there exists $i^\star \in \{2,3\}$ such that

$$\exists\, T_G > 0 \,|\, T_{\min}(i^\star, \mathbf{s}_1) > T_G\,. \quad \square$$

Proof: The proof is similar to the one of Lemma 4.2. First, note that when the continuous state is such that $\mathbf{s}_1 \in \mathcal{S}_1 \backslash \mathcal{S}_2$, again $A{=}B{=}0$. Moreover the following facts hold.

• The first and the second order time derivatives of the LF are zero if computed in $t=\tau$; i.e.

$$\dot{V}(t)\Big|_{q\in\{2,3\},t=\tau} = \ddot{V}(t)\Big|_{q\in\{2,3\},t=\tau} = 0\,, \tag{19}$$

as shown by the first equality in (11).

• $c_1|_{q\in\{2,3\}}{=}{-}\varphi_1|_{q\in\{2,3\}}$ and $c_2|_{q\in\{2,3\}}{=}{-}\varphi_2|_{q\in\{2,3\}}$.

¿From the expression (14) of the first order time derivative of V in $\mathcal{S}_0$, we are able to compute the third order time derivative of V in $\mathcal{S}_1 \backslash \{0\}$, that is

$$\dddot{V}(t)\big|_{q=i,t=\tau} = -2\varphi_1|_{q=i}^3 \delta_i \rho_i \leqslant 0\,, \tag{20}$$

where the last inequality follows by considering (18) and recalling that $\delta_i > 0$. Note that the above inequality holds in the strict sense at least for one value of δ_i, hence:

$$\exists\, i^\star \in \{2,3\} \,|\; \dddot{V}(t)\big|_{q=i^\star,t=\tau} < 0\,. \tag{21}$$

Equations (19) and (21) imply that the candidate LF decreases at least for $t \in (\tau, \tau+T)$ where $T>0$ is a finite (though possibly small) number. Again, we will prove that there exists a lower bound $T_G{>}0$ for T. Rewrite (14) as

$$\begin{aligned}\dot{V}(r) &= -\varphi_1^3\delta_i\rho_i r^2 + \varphi_1^4\left(1+\delta_i^4+\tfrac{1}{2}\delta_i^2\right) r^3 + \tfrac{1}{3}\varphi_1^6\left(\delta_i^2+\delta_i^4\right) r^5 \\ &= \varphi_1^3\delta_i\rho_i r^2\left(-1+\tfrac{\varphi_1\left(1+\delta_i^4+\frac{1}{2}\delta_i^2\right)}{\delta_i\rho_i} r + \tfrac{1}{3}\tfrac{\varphi_1^3\left(\delta_i^2+\delta_i^4\right)}{\delta_i\rho_i} r^3\right)\end{aligned} \tag{22}$$

where the dependency of φ_1 from the value of q has been dropped again for the sake of a simpler notation.

Now, analogously to Lemma 4.2, it can be proven that $\frac{\varphi_1^3}{\rho_i} \leqslant 1$ and $\frac{\varphi_1}{\rho_i} < 1$. Consider, then, the function ($\delta_\star = \delta_{i^\star}$):

$$g(r) \triangleq -1 + \frac{1+\delta_\star^4+\frac{1}{2}\delta_\star^2}{\delta_\star} r + \frac{1}{3}(\delta_\star+\delta_\star^3) r^3\,, \tag{23}$$

which is continuous and such that $g(0) = -1$ and $\lim\limits_{r\to+\infty} g(r) = +\infty$. This mean that there exists at least one value $r^\star \in (0,+\infty)$ such that $g(r^\star) = 0$; let T_G denote the minimum of all the values r having this property, that is

$$T_G = \min\{r \in (0,+\infty) \mid g(r) = 0\}\,. \tag{24}$$

Clearly, $g(r) < 0,\ \forall\, r \in [0, T_G)$, that is:

$$1 > \frac{1+\delta_\star^4+\frac{1}{2}\delta_\star^2}{\delta_\star} r + \frac{1}{3}(\delta_\star+\delta_\star^3) r^3 \quad \forall r \in [0, T_G)\,.$$

By using the above inequalities, $\forall\, r{\in}[0, T_G)$ it follows that

$$1 > \frac{\varphi_1\left(1+\delta_\star^4+\frac{1}{2}\delta_\star^2\right)}{\left(z_2(\tau)+\delta_\star z_3(\tau)\right)\delta_\star} r + \frac{1}{3}\frac{\varphi_1^3\left(\delta_\star+\delta_\star^3\right)}{z_2(\tau)+\delta_\star z_3(\tau)} r^3\,.$$

Multiplying by the positive quantity $\varphi_1^3\delta_\star(z_2(\tau){+}\delta_\star z_3(\tau))r^2$ and recalling equation (22), it is easy to conclude that $\dot{V}(r)\Big|_{q=i^\star} < 0$ for $r \in (0, T_G)$, that is

$$\dot{V}(t)\Big|_{q=i^\star} < 0 \quad \forall\, t \in (\tau, \tau+T_G)\,.$$

The proof is concluded by noticing that, once δ_2 and δ_3 are fixed, $g(r)$ is a polynomial with constant coefficients and, consequently, T_G does not depend on the value of q or $\mathbf{s}$. ■

Note that Lemmas 4.2 and 4.3 have been proven when the continuous state $\mathbf{s}$ of the system, at the switching time instant, belongs to a particular set, namely $\mathcal{S}_0 \backslash \mathcal{S}_1$ and $\mathcal{S}_1 \backslash \mathcal{S}_2$, respectively. Nevertheless, when the discrete state of the machine is $q \in \{1,2,3\}$, all the coefficient of V are analytic functions of the initial condition, as can be seen by the explicit solutions (9). This means that if $\mathbf{s}$ follows a trajectory approaching $\mathcal{S}_0 \backslash \mathcal{S}_1$ or $\mathcal{S}_2 \backslash \mathcal{S}_1$, respectively, the first order time derivative of the LF tends to equation (14). Therefore, the following two properties hold.

Property 4.1: $\forall\, \tilde{T}$ such that $0 < \tilde{T} < T_F$, there exist $\varepsilon_x(\tilde{T})$, $\varepsilon_y(\tilde{T})$ such that $T_{\min}(1,\mathbf{s}) > \tilde{T}$ for all $\mathbf{s} = (x,y,z_1,z_2,z_3)^\top$ with $\|x\| < \varepsilon_x(\tilde{T})$ and $\|y\| < \varepsilon_y(\tilde{T})$.

Property 4.2: $\forall\, \tilde{T}$ such that $0 < \tilde{T} < T_G$, there exist $\varepsilon_x(\tilde{T})$, $\varepsilon_y(\tilde{T})$, $\varepsilon_{z_1}(\tilde{T})$ such that $\max_{i\in\{2,3\}} T_{\min}(i,\mathbf{s}) > \tilde{T}$ for all $\mathbf{s} = (x,y,z_1,z_2,z_3)^\top$ with $\|x\| < \varepsilon_x(\tilde{T})$, $\|y\| < \varepsilon_y(\tilde{T})$ and $\|z_1\| < \varepsilon_{z_1}(\tilde{T})$.

We are now ready to prove the main stability result of the paper. For, let $T_D = \alpha \min\{T_F, T_G\}$, with $0{<}\alpha{<}1$. Then the asymptotic stabilization of the zero equilibrium of system (1) by means of the control law (2)-(3) and of the switching strategy (5) is guaranteed by the following theorem.

Theorem 4.1: Consider system (1). Let the control law be determined according to (2) and (3) with c_1 and c_2 given by equations (10) when $q \in \{1,2,3\}$. Let the discrete state q of the machine be updated according to the switching strategy (5) and pick φ_1 and φ_2 according to equations (12) if $q=1$ and to equations (18) if $q \in \{2,3\}$. Then the zero equilibrium of the resulting hybrid closed loop system is globally asymptotically stable in the sense of Lyapunov. □

Proof: First, note that the control law defined by (2) and (3) yields a null control action when the state vector is the origin of the state space. Then, the zero equilibrium is preserved, regardless the switching. We show that the function (6) is a LF for the hybrid closed loop system. Clearly, $V(0){=}0$ and $V(\mathbf{s}) > 0,\ \forall\, \mathbf{s} \in \mathbb{R}^5 \backslash \{0\}$. Moreover,

V is radially unbounded. Now, Lemmas 4.1, 4.2 and 4.3 imply that $V[\mathbf{s}(t)]$ is always non-increasing, i.e.

$$\forall \mathbf{s} \in \mathbb{R}^5 \backslash \{0\}, \quad \forall i \in \{0,1,2,3\}, \; \dot{V}[\mathbf{s}(t)]\Big|_{q=i} \leqslant 0$$

for all the time that $q = i$. As a consequence, the zero equilibrium of the closed loop system is stable.

Observe that $V[\mathbf{s}(t)]$ is continuous and bounded from below, hence it has a well–defined limit $V_\infty \geqslant 0$ for $t \to \infty$. Now, Lemmas 4.1, 4.2 and 4.3 guarantee that, $\forall t \in (\tau, \tau + T_D)$, the following holds.

$$\forall \mathbf{s} \in \mathbb{R}^5 \backslash \mathcal{S}_0 \quad , \qquad \dot{V}[\mathbf{s}(t)]\Big|_{q=0} < 0 \,.$$

$$\forall \mathbf{s} \in \mathcal{S}_0 \backslash \mathcal{S}_1 \quad , \qquad \dot{V}[\mathbf{s}(t)]\Big|_{q=1} < 0 \,.$$

$$\forall \mathbf{s} \in \mathcal{S}_1 \backslash \mathcal{S}_2 \quad , \qquad \exists i \in \{2,3\} \mid \dot{V}[\mathbf{s}(t)]\Big|_{q=i} < 0 \,.$$

We can conclude that $\forall \mathbf{s} \in \mathbb{R}^5 \backslash \{0\} \;\; \exists i^\star \in \{0,1,2,3\}$ s.t. $\dot{V}(\mathbf{s}(t))\Big|_{q=i^\star} < 0 \;\; \forall t \in (\tau, \tau + T_D)$. Moreover, Property 4.1 guarantees that the condition for a switching from $q = 0$ to $q = 1$ is fulfilled in finite time; analogously, Property 4.2 guarantees that a switching from $q \in \{0,1\}$ to $q \in \{2,3\}$ occurs in finite time. Hence the discrete state cannot take the same value for an infinite time: $\nexists (\hat{T}, \hat{i}) \;\;$ s. t. $\;\; q = \hat{i} \;\; \forall t > \hat{T}$. Therefore, in a finite time the discrete variable q will take the value $i^\star$. This implies $V_\infty = 0$, hence asymptotic stability of the zero equilibrium of the closed–loop system. ■

V. The case of saturated control actions

We now show that the results obtained in the previous sections can be extended to the case in which the modulus of the control variables should not exceed given upper bounds. For, suppose that the resctrictions $-\alpha_1 \leqslant u \leqslant \beta_1$ and $-\alpha_2 \leqslant v \leqslant \beta_2$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ hold and define for simplicity $M \triangleq \min\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. The following holds.

• If $q = 0$, by applying to system (1) the input

$$w_0(\mathbf{s}) = -M[\mathrm{sat}(x),\; \mathrm{sat}(y^3 + z_1 x + z_2 x^2 + z_3 xy)]^\top , \quad (25)$$

the conclusions of Lemma 4.1 are still valid.

• If $q = 1$, applying inputs (10) with

$$\varphi_1\big|_{q=1} \triangleq \frac{M\mathrm{sat}[z_1(\tau)]}{D_1} \quad \text{and} \quad \varphi_2\big|_{q=1} \triangleq -\left|\varphi_1\big|_{q=1}\right| , \quad (26)$$

a result analogous to Lemma 4.2 can be found. In this case $\frac{\varphi_1^4}{|z_1(\tau)|} \leqslant M^4$,$\frac{\varphi_1^2}{|z_1(\tau)|} \leqslant M^2$ and $\frac{\varphi_1 \rho_1}{|z_1(\tau)|} \leqslant M\sqrt{2}$ and the proof can be carried out by considering the function $f_B(r) \triangleq -1 + M\sqrt{2}r + \frac{5}{2}M^2 r^2 + \frac{2}{3}M^4 r^4$, instead of function (17) and $\tilde{T}_F = \frac{1}{M}T_F$ instead of T_F.

• If $q \in \{2,3\}$, applying inputs (10) with

$$\varphi_1|_{q=i} \triangleq M\frac{\mathrm{sat}(\rho_i)}{D_2} \quad \text{and} \quad \varphi_2|_{q=i} \triangleq M\mathrm{sat}\left(\delta_i \varphi_1|_{q=i}\right) , \quad (27)$$

a result analogous to Lemma 4.3 can be found. In this case $\frac{\varphi_1^3}{\rho_i} \leqslant M^3$ and $\frac{\varphi_1}{\rho_i} \leqslant M$ and the proof can be carried out by considering, instead of function (23), the function $g_B(r) \triangleq -1 + M\frac{1+\delta_\star^4 + \frac{1}{2}\delta_\star^2}{\delta_\star} r + \frac{1}{3}M^3(\delta_\star + \delta_\star^3) r^3$, and the time-constant $\tilde{T}_G = \frac{1}{M}T_G$ instead of T_G.

As a consequence, the following Corollary of Theorem 4.1 can be easily proven.

Corollary 5.1: Consider system (1). Let α_1, α_2, β_1 and β_2 be positive constant such that $-\alpha_1 \leqslant u \leqslant \beta_1$ and $-\alpha_2 \leqslant v \leqslant \beta_2$. Moreover, let the control be as in (25) if $q = 0$, and be equal to $[c_1, c_2]^\top$, with c_1 and c_2 given by equation (10) when $q \in \{1,2,3\}$. Finally, let q be updated according to the strategy (5) and pick φ_1 and φ_2 according to equations (26) if $q = 1$ and to equations (27) if $q \in \{2,3\}$. Then the zero equilibrium of the system (1) is globally asymptotically stable in the sense of Lyapunov. □

VI. Simulation results

In this section, some simulation results are given in order to illustrate the performance of the proposed switching control law. The plots refer to an initial condition $\mathbf{s}_0 = (0,0,0,-1,1)^\top$. The δ's of the third and fourth controllers have been chosen as follows: $\delta_2 = 2.1$, $\delta_3 = 0.01$; with these values, the two values of T_G in Equation (24) are $T_{G1} \simeq 9.245 \times 10^{-2}$ and $T_{G2} \simeq 9.999 \times 10^{-3}$; as a consequence, we have chosen $T_D = 5 \times 10^{-3}$. To improve the performance of the scheme, when $q^+ \in \{1,2,3\}$ the associated controller is used for $T_{\min}(q, \mathbf{s}(\tau))$ time units.

The time-behaviour of the LF and of four of the state coordinates are reported in Figures 2 and 3[6], respectively. Note that the proposed strategy has two drawbacks from the practical point of view, namely a slow convergence and an oscillating behaviour in some of the coordinates. Finally, in Figure 4 an enlargement of Figure 2 is reported together with the value of the discrete variable q for a small time interval.

VII. Conclusions

We have taken into account the rolling sphere problem and proposed a swiching control strategy in order to stabilize the zero equilibrium of the system. By means of a LF for the hybrid system, the strategy has been proven able to globally asymptotically stabilize the zero equilibrium in the sense of Lyapunov. Some simulations have been carried out to show the effectiveness stabilizing property of the method. Future works will be devoted to the extension of the method to a general class of n-dimensional systems.

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[6] The second coordinate, y has been neglected, having a behaviour similar to x.

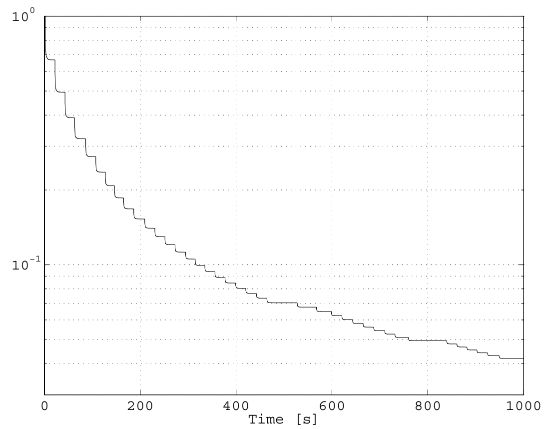

Fig. 2. Time history of the LF.

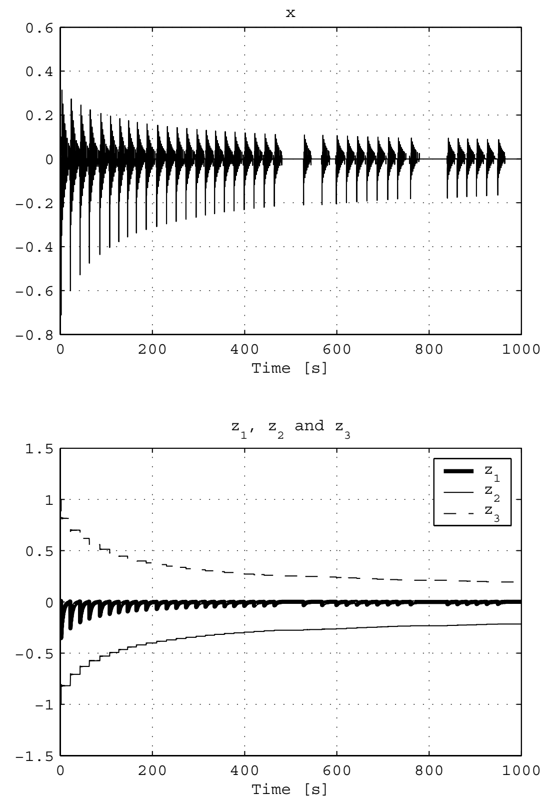

Fig. 3. Time histories of x, z_1, z_2, z_3.

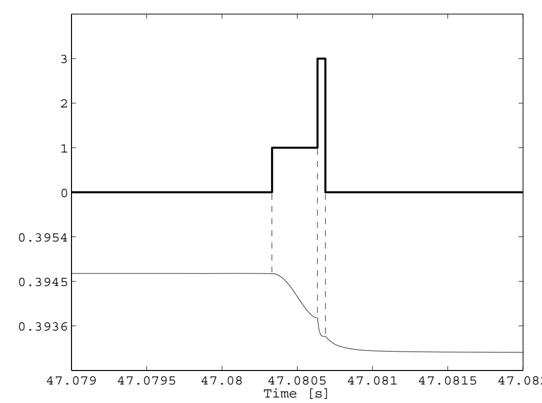

Fig. 4. Time histories of the discrete variable (bold line) and of the LF for $t \in [47.079, 47.082]$.

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