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A Generalized Framework for Global Output Feedback Stabilization of Genuinely Nonlinear Systems

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***Abstract*— In this paper, we introduce a generalized framework for global output feedback stabilization of a class of uncertain, *genuinely* nonlinear systems of a particularly complex nature since their linearization is not guaranteed to be either controllable or observable. Based on a subtle homogeneous observer/controller construction and *homogeneous domination* design, this new framework not only unifies the existing output feedback stabilization results [12], [14], but also leads to more general results which have never before been achieved.**

I. INTRODUCTION

A formidable problem in the nonlinear control literature is the global stabilization of a nonlinear dynamic system by output feedback. Such a problem formulation is inherently practical in that only partial sensing is utilized to feedback state information, an efficient and cost-effective solution in many applications while a necessity in others. Unfortunately, the existing output feedback stabilization schemes have been very limited in what types of nonlinearities could be handled, an issue made more complex due to the lack of a true "separation principle" for nonlinear systems. In this work, we investigate more generic systems, for $j = 1, \ldots, n-1$

$$\dot{x}_j = x_{j+1}^{p_j} + \phi_j(t,x,u),\ \dot{x}_n = u + \phi_n(t,x,u),\ y = x_1, \quad (1)$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input and output, respectively. For $i = 1, \cdots, n$, $\phi_i(t,x,u)$ is an unknown nonlinear function of all the states and the control input, and $p_i \in \mathbb{R}_{odd}^{\geq 1} = \{q \in \mathbb{R} : q \geq 1, q$ is a ratio of odd integers$\}$ with $p_n = 1$.

When $p_i = 1$, the results on global output feedback stabilization of system (1) were based on quite restrictive conditions imposed on the nonlinear terms $\phi_i(\cdot)$, mainly attributed to finite escape time phenomena [8]. These stabilization results include systems where the nonlinear function is only dependent on the output [7], [6]; or is Lipschitz or linear in the unmeasurable states [2], [5], [13]. Until recently, there was no systematic way of dealing with systems whose dynamics are highly nonlinear in the unmeasured states. In the work of [9], the nonlinearity restriction was relaxed to globally stabilize more general systems by output feedback under a less restrictive polynomial growth condition.

However, when p_i are of higher order, i.e. the system (1) has uncontrollable/unobservable linearization, the global output feedback stabilization solutions are few, where the *state feedback* problem was resolved in [10] for the stabilization of systems such as (1) with $p_i \geq 1$ being odd integers and $\phi_i(\cdot)$ bounded by a lower-triangular function. Current output feedback stabilization results for these higher order systems, on the other hand, are contained in the works [12] for different p_i with a lower order growth rate for $\phi_i(\cdot)$ and [14] which necessitates that the p_i are all the same odd integer and $\phi_i(\cdot)$'s are Lipschitz-like. Note that the methods in [12] and [14] are quite different from each other. An interestingly unresolved problem is if we can find a generalized framework to unify these two solutions, while also containing the more general work [9] when $p_i = 1$. To handle this issue, we employ the concept of *homogeneous domination* to cover a larger class of nonlinear systems. In doing so, we allow $p_i \in \mathbb{R}_{odd}^{\geq 1}$ and bound $\phi_i(\cdot)$ by a high order growth rate. This formalism will allow for more complex nonlinearities than those seen in [9], [12], [14], and generalizes *homogeneous domination* introduced in [9]. This new design scheme will allow for the stabilization of systems such as

$$\dot{x}_1 = x_2^3, \quad \dot{x}_2 = u + d(t)x_2^q, \quad y = x_1, \quad (2)$$

with a bounded disturbance $d(t)$. When $q = 1$, the system (2) was stabilized in [12], and when $q = 3$, the stabilization of (2) was dealt with in [11], however, when $q = 2$, the global stabilization of (2) is unresolved. Nevertheless, we show that (2) can now be controlled via output feedback for, though not limited to, $q = 2$ by methods described herein.

II. STABILIZATION BY HOMOGENEOUS STATE FEEDBACK

Homogeneity has previously played an important role in the analysis of nonlinear dynamic systems. Utilizing this notion has allowed the undertaking of the concepts of controllability and controller design for nonlinear systems to be realizable [1], [3], [4]. Of particular importance here are homogeneous systems with weighted dilation and the homogeneous norm [1], [4], [9], where the particular work of [9] is inspiration for the methodology discussed in this paper. Using homogeneity, we propose in this section a new design method for a state feedback stabilizer for (1) under the following hypothetical assumption:

A2.1: There is a constant $\tau \geq 0$ such that for $i = 1, \cdots, n$,

$$|\phi_i(t,x,u)| \leq c\left(|x_1|^{\frac{r_i+\tau}{r_1}} + \cdots + |x_i|^{\frac{r_i+\tau}{r_i}}\right), \quad (3)$$

for a constant $c > 0$ with r_i defined as (for $p_n = 1$)

$$r_1 = 1, \qquad r_i + \tau = r_{i+1}p_i. \quad (4)$$

For simplicity, we assume the degree of homogeneity, $\tau = \frac{q}{d}$, with q an even integer and d an odd integer. Under this

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assumption, and taking into account the odd, not necessarily equivalent, powers of (1), the homogeneous weights, r_i, will always be odd numbers. Note an equivalent result will be achieved for the case when the r_i are not odd.

Theorem 2.1: By A2.1 there exists a homogeneous state feedback controller such that the nonlinear system (1) is guaranteed globally asymptotically stable[1].

Proof. The inductive proof relies on the simultaneous construction of a C^1 Lyapunov function which is positive definite and proper, as well as a homogeneous stabilizer at each iteration. **Initial Step.** Let $\mu \in \mathbb{R}_{\text{odd}}$[2] and $\mu \geq \max\{r_i p_{i-1}\}_{2\leq i\leq n+1}$, where r_i is defined as in (4). Choose

$$V_1 = \int_0^{x_1} \left(s^{\frac{\mu}{r_1}} - 0\right)^{(2\mu-\tau-r_1)/\mu} ds.$$

The time derivative of V_1 along the trajectory of (1) is

$$\dot{V}_1 = x_1^{(2\mu-\tau-r_1)/r_1}\left[x_2^{p_1} + \phi_1(t,x)\right]. \quad (5)$$

By A2.1, $\dot{V}_1 \leq x_1^{\frac{2\mu-\tau-r_1}{r_1}}\left[x_2^{p_1} - x_2^{*p_1} + x_2^{*p_1} + x_1^{r_2p_1/r_1}c\right]$. Then, the virtual controller $x_2^{*p_1}$ defined by $x_2^{*p_1} = -x_1^{(\tau+r_1)/r_1}(n+c) := -x_1^{r_2p_1/r_1}\beta_1$, yields

$$\dot{V}_1(x_1) \leq -nx_1^{2\mu/r_1} + x_1^{(2\mu-\tau-r_1)/r_1}\left[x_2^{p_1} - x_2^{*p_1}\right]. \quad (6)$$

Inductive Step. Suppose at step $k-1$, there is a C^1 Lyapunov function $V_{k-1} : \mathbb{R}^{k-1} \to \mathbb{R}$, which is homogeneous with respect to (4), and a set of C^0 virtual controllers $x_2^{*p_1}, \cdots, x_k^{*p_{k-1}}$, defined for $i = 2, \cdots, k$

$$x_i^{*p_{i-1}} = -\xi_{i-1}^{r_ip_{i-1}/\mu}\beta_{i-1}, \qquad \xi_i = x_i^{\mu/r_i} - x_i^{*\mu/r_i}, \quad (7)$$

with $\xi_1 = x_1^{\mu/r_1}$ and constants $\beta_1, \cdots, \beta_{k-1} > 0$, giving

$$\begin{aligned}\dot{V}_{k-1} \quad &\leq \quad -(n-k+2)\left(\xi_1^2 + \cdots + \xi_{k-1}^2\right) \\ &\qquad + \xi_{k-1}^{(2\mu-\tau-r_{k-1})/\mu}(x_k^{p_{k-1}} - x_k^{*p_{k-1}}). \quad (8)\end{aligned}$$

We claim (8) also holds at step k. To prove this, we set

$$W_k = \int_{x_k^*}^{x_k} \left(s^{\frac{\mu}{r_k}} - x_k^{*\frac{\mu}{r_k}}\right)^{(2\mu-\tau-r_k)/\mu} ds$$

and consider the Lyapunov function $V_k : \mathbb{R}^k \to \mathbb{R}$, as

$$V_k(x_1, \cdots, x_k) = V_{k-1}(x_1, \cdots, x_{k-1}) + W_k(x_1, \cdots, x_k) \quad (9)$$

which can be proven to be C^1 using a similar method as in [12]. The derivative of V_k along (1) is

$$\begin{aligned}\dot{V}_k \quad &= \quad \dot{V}_{k-1} + \sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l + \xi_k^{\frac{2\mu-\tau-r_k}{\mu}}\dot{x}_k \\ &\leq \quad -(n-k+2)\left(\xi_1^2 + \cdots + \xi_{k-1}^2\right) \\ &\qquad + \xi_{k-1}^{\frac{2\mu-\tau-r_{k-1}}{\mu}}(x_k^{p_{k-1}} - x_k^{*p_{k-1}}) + \sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l \\ &+ \xi_k^{\frac{2\mu-\tau-r_k}{\mu}}\left(x_{k+1}^{*p_k} + \phi_k(\cdot)\right) + \xi_k^{\frac{2\mu-\tau-r_k}{\mu}}(x_{k+1}^{p_k} - x_{k+1}^{*p_k}). \quad (10)\end{aligned}$$

In order to proceed further, an estimate for each term in the right hand side of (10) is needed. First, it follows from $r_kp_{k-1}/\mu \leq 1$ and Lemma A.1 that

$$\begin{aligned}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) &\leq \left|\left(x_k^{\mu/r_k}\right)^{r_kp_{k-1}/\mu} - \left(x_k^{*\mu/r_k}\right)^{r_kp_{k-1}/\mu}\right| \\ &\leq \quad 2^{1-r_kp_{k-1}/\mu}\left|\xi_k\right|^{r_kp_{k-1}/\mu}, \quad (11)\end{aligned}$$

and by Lemma A.2 it can be seen that, for a constant $c_k > 0$,

$$\xi_{k-1}^{2\mu-\tau-r_{k-1}/\mu}(x_k^{p_{k-1}} - x_k^{*p_{k-1}}) \leq \xi_{k-1}^2/3 + c_k\xi_k^2. \quad (12)$$

Using Lemma A.1, condition A2.1 can be rewritten as

$$|\phi_k(t,x,u)| \leq \bar{c}_k\left(|\xi_1|^{r_{k+1}p_k/\mu} + \cdots + |\xi_k|^{r_{k+1}p_k/\mu}\right) \quad (13)$$

for a constant $\bar{c}_k > 0$. By Lemma A.2 and (13)

$$\begin{aligned}\xi_k^{\frac{2\mu-\tau-r_k}{\mu}}\phi_k(\cdot) \quad &\leq \quad |\xi_k|^{\frac{2\mu-\tau-r_k}{\mu}}\bar{c}_k\sum_{i=1}^{k}|\xi_i|^{\frac{r_{k+1}p_k}{\mu}} \\ &\leq \quad \frac{1}{2}\sum_{i=1}^{k-2}\xi_i^2 + \frac{1}{3}\xi_{k-1}^2 + \tilde{c}_k\xi_k^2 \quad (14)\end{aligned}$$

for a constant $\tilde{c}_k > 0$. The third term in (10), namely $\sum_{l=1}^{k-1}\dot{x}_l\partial W_k/\partial x_l$ can be estimated as the following proposition whose proof is included in the Appendix.

Proposition 2.1: There is a constant $\hat{c}_k > 0$ such that

$$\left|\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot{x}_l\right| \leq \frac{1}{2}\sum_{i=1}^{k-2}\xi_i^2 + \frac{1}{3}\xi_{k-1}^2 + \hat{c}_k\xi_k^2.$$

Substituting the estimates (12), (14), and the result of Proposition 2.1 into (10), we arrive at

$$\begin{aligned}\dot{V}_k \quad &\leq \quad -(n-k+1)\left(\xi_1^2 + \cdots + \xi_{k-1}^2\right) \\ &\qquad + \xi_k^{2\mu-\tau-r_k/\mu}\left(x_{k+1}^{*p_k} + (c_k + \tilde{c}_k + \hat{c}_k)\xi_k^{r_{k+1}p_k/\mu}\right) \\ &\qquad + \xi_k^{2\mu-\tau-r_k/\mu}(x_{k+1}^{p_k} - x_{k+1}^{*p_k}).\end{aligned}$$

Observe that a virtual controller of the form $x_{k+1}^{*p_k} = -\xi_k^{r_{k+1}p_k/\mu}\beta_k = -\xi_k^{r_{k+1}p_k/\mu}[n-k+1+c_k+\tilde{c}_k+\hat{c}_k]$, yields

$$\dot{V}_k \leq -(n-k+1)\sum_{i=1}^{k}\xi_i^2 + \xi_k^{\frac{2\mu-\tau-r_k}{\mu}}(x_{k+1}^{p_k} - x_{k+1}^{*p_k}).$$

This completes the inductive proof. The inductive argument shows that (8) holds for $k = n+1$ with a set of virtual controllers (7). Hence, at the last step, choosing

$$u = x_{n+1} = x_{n+1}^* = -\xi_n^{(r_n+\tau)/\mu}\beta_n \quad (15)$$

yields $\dot{V}_n \leq -\left(\xi_1^2 + \cdots + \xi_n^2\right)$ where $\dot{V}_n < 0, \forall x \neq 0$ under (7), and $V_n(x_1, \cdots, x_n)$ is a Lyapunov function of the form (9). Thus, (1)–(15) is *globally asymptotically stable.* ∎

Remark 2.1: In the case when τ is any nonnegative real number, we are still able to design a homogenous controller globally stabilizing the system (1) with necessary modification to preserve the sign of function $[\cdot]^{r_ip_{i-1}/\mu}$.Specifically, for any real number $r_ip_{i-1}/\mu > 0$, we define $[\cdot]^{r_ip_{i-1}/\mu} = \text{sign}(\cdot)|\cdot|^{r_ip_{i-1}/\mu}$. For brevity, the details are excluded.

[1]Under certain conditions global strong stability may be the only achievable result. See [11], [12] and the references therein for details of these conditions and global strong stability.

[2]In this paper we define $\mathbb{R}_{\text{odd}} = \{\text{b} \in \mathbb{R} : \text{b is a ratio of odd integers}\}$.

III. Stabilization of (1) by Output Feedback

In this section, we show that under A2.1, the problem of global output feedback stabilization for system (1) is solvable. We will first construct a homogeneous output feedback controller for the nominal chain of power integrators:

$$\dot{z}_1 = z_2^{p_1},\ \dot{z}_2 = z_3^{p_2},\ \cdots,\ \dot{z}_n = v,\ y = z_1, \tag{16}$$

with $p_i \in \mathbb{R}_{\text{odd}}^{\geq 1},\ \mathrm{i} = 1, \cdots, \mathrm{n}-1$. Then, based on this output feedback controller, we will develop a scaled observer and controller to render the system (1) globally asymptotically stable under the polynomial growth condition (3).

A. Output Feedback Control of Nominal Nonlinear System

Theorem 3.1: Given a real number $\tau \geq 0$, there is a homogeneous output feedback controller of degree τ such that the nonlinear system (16) is global asymptotically stable.

Proof. The construction of the homogeneous output feedback controller is accomplished in three steps. First, by Theorem 2.1, a homogeneous state feedback stabilizer is constructed, then a homogeneous observer is designed, and lastly, we replace the unmeasurable states with the estimates. The closed-loop system can then be proven globally asymptotically stable by an appropriate observer gain. For simplicity, we again assume that r_i is odd. For general r_i, see Remark 2.1.

State Feedback Controller: For nonlinear system (16), A2.1 is automatically satisfied since $\phi_i(\cdot)$ is trivial. Hence, by Theorem 2.1, there is a homogeneous (with respect to the weight (4)) state feedback controller globally stabilizing (16). Specifically, for $k = 2, \cdots, n$ with $\xi_i = z_i^{\mu/r_1}$ and constants $\beta_1, \cdots, \beta_n > 0$, by defining

$$z_k^{*p_{k-1}} = -\xi_{k-1}^{r_k p_{k-1}/\mu}\beta_{k-1} \quad \xi_k = z_k^{\mu/r_k} - z_k^{*\mu/r_k}, \tag{17}$$

there exists $v^*(z) = -\beta_n \xi_n^{(r_n+\tau)/\mu}$, such that

$$\dot{V}_n \leq -\left(\xi_1^2 + \cdots + \xi_n^2\right) + \xi_n^{(2\mu-\tau-r_n)/\mu}(v - v^*(z)), \tag{18}$$

where V_n is a Lyapunov function of the form

$$V_n(z_1, \cdots, z_n) = \sum_{i=1}^{n} \int_{z_i^*}^{z_i} \left(s^{\frac{\mu}{r_i}} - z_i^{*\frac{\mu}{r_i}}\right)^{\frac{2\mu-\tau-r_i}{\mu}} ds.$$

Homogeneous Observer Design: Next, a homogeneous observer is constructed in the vein of [9], [12].

$$\dot{\eta}_k = -\ell_{k-1}\hat{z}_k^{p_{k-1}},\ \hat{z}_k^{p_{k-1}} = [\eta_k + \ell_{k-1}\hat{z}_{k-1}]^{\frac{r_k p_{k-1}}{r_{k-1}}}, \tag{19}$$

$k = 2, \ldots, n$, where $\hat{z}_1 = z_1$ and $l_i > 0,\ i = 1, \cdots, n-1$ are the gains to be determined in later steps. Based on the estimated states $\hat{z}_i$, we design an output feedback controller

$$v(\hat{z}) = \beta_n \left(\hat{z}_n^{\frac{\mu}{r_n}} + \cdots + \beta_2(\hat{z}_2^{\frac{\mu}{r_2}} + \beta_1 z_1^{\frac{\mu}{r_1}}) \cdots\right)^{\frac{r_n+\tau}{\mu}}. \tag{20}$$

For $i = 2, \cdots, n$, we designate

$$U_i = \int_{\gamma_i^{(2\mu-\tau-r_{i-1})/r_{i-1}}}^{z_i^{(2\mu-\tau-r_{i-1})/r_i}} \left(s^{\frac{r_{i-1}}{2\mu-\tau-r_{i-1}}} - \gamma_i\right) ds$$

denoting $\gamma_i = \eta_i + \ell_{i-1}z_{i-1}$. By construction, it can be verified that U_i is C^1. As a matter of fact, with a constant b_i, we have the following

$$\begin{aligned}
\frac{\partial U_i}{\partial z_i} &= b_i z_i^{(2\mu-\tau-r_{i-1}-r_i)/r_i}\left(z_i^{r_{i-1}/r_i} - \gamma_i\right), \\
\frac{\partial U_i}{\partial \eta_i} &= -\left(z_i^{(2\mu-\tau-r_{i-1})/r_i} - \gamma_i^{(2\mu-\tau-r_{i-1})/r_{i-1}}\right), \\
\frac{\partial U_i}{\partial z_{i-1}} &= -\ell_{i-1}\left(z_i^{(2\mu-\tau-r_{i-1})/r_i} - \gamma_i^{(2\mu-\tau-r_{i-1})/r_{i-1}}\right).
\end{aligned}$$

Hence, the derivative of U_i along (16)-(19) is

$$\begin{aligned}
\dot{U}_i = {}& z_{i+1}^{p_i} b_i z_i^{(2\mu-\tau-r_{i-1}-r_i)/r_i}\left(z_i^{r_{i-1}/r_i} - \gamma_i\right) \\
& -\ell_{i-1}(z_i^{p_{i-1}} - \hat{z}_i^{p_{i-1}})\left(z_i^{\frac{2\mu-\tau-r_{i-1}}{r_i}} - \gamma_i^{\frac{2\mu-\tau-r_{i-1}}{r_{i-1}}}\right).
\end{aligned}$$

Let $e_i = (z_i^{p_{i-1}} - \hat{z}_i^{p_{i-1}})^{\mu/r_i p_{i-1}}$. We now have

$$\begin{aligned}
\dot{U}_i = {}& z_{i+1}^{p_i} b_i z_i^{(2\mu-\tau-r_{i-1}-r_i)/r_i}\left(z_i^{r_{i-1}/r_i} - \gamma_i\right) \\
& -\ell_{i-1}e_i^{\frac{r_i p_{i-1}}{\mu}}\left(z_i^{\frac{2\mu-\tau-r_{i-1}}{r_i}} - \hat{z}_i^{\frac{2\mu-\tau-r_{i-1}}{r_i}}\right) \\
& -\ell_{i-1}e_i^{\frac{r_i p_{i-1}}{\mu}}\left(\hat{z}_i^{\frac{2\mu-\tau-r_{i-1}}{r_i}} - \gamma_i^{\frac{2\mu-\tau-r_{i-1}}{r_i}}\right),
\end{aligned} \tag{21}$$

$i = 2, \cdots, n$, where $z_{n+1} = v(\hat{z})$. Next, we estimate the terms in (21). By Lemma A.1, with constant m_i

$$\begin{aligned}
& -\ell_{i-1}e_i^{\frac{r_i p_{i-1}}{\mu}}\left((z_i^{p_{i-1}})^{\frac{2\mu-\tau-r_{i-1}}{r_i p_{i-1}}} - (\hat{z}_i^{p_{i-1}})^{\frac{2\mu-\tau-r_{i-1}}{r_i p_{i-1}}}\right) \\
\leq & -\ell_{i-1}m_i e_i^{\frac{r_i p_{i-1}}{\mu}}(z_i^{p_{i-1}} - \hat{z}_i^{p_{i-1}})^{\frac{2\mu-\tau-r_{i-1}}{r_i p_{i-1}}} = -\ell_{i-1}m_i e_i^{\frac{2\mu-\tau}{\mu}}
\end{aligned}$$

The remaining terms in (21) can be estimated using the following propositions whose proofs are in the Appendix.

Proposition 3.1: For $i = 2, \cdots, n-1$

$$\begin{aligned}
& z_{i+1}^{p_i} b_i z_i^{(2\mu-\tau-r_{i-1}-r_i)/r_i}\left(z_i^{r_{i-1}/r_i} - \gamma_i\right) \\
& \leq \frac{1}{12}\sum_{j=i-1}^{i+1}\xi_j^2 + \alpha_i e_i^2 + g_i(\ell_{i-1})e_{i-1}^2
\end{aligned} \tag{22}$$

with constant α_i, g_i a C^0 function of ℓ_{i-1}, and $g_2(\cdot) = 0$.

Proposition 3.2: For the controller $v(\hat{z})$, we have

$$\begin{aligned}
& v(\hat{z}) b_n z_i^{(2\mu-\tau-r_{n-1}-r_n)/r_n}\left(z_n^{r_{n-1}/r_n} - \gamma_n\right) \\
& \leq \frac{1}{8}\sum_{i=1}^{n}\xi_i^2 + \bar{\alpha}\sum_{i=2}^{n}e_i^2 + g_n(\ell_{n-1})e_{n-1}^2
\end{aligned} \tag{23}$$

for a constant $\bar{\alpha}$ and g_n a C^0 function of ℓ_{n-1}.

Proposition 3.3: For $i = 3, \cdots, n$

$$\begin{aligned}
& -\ell_{i-1}e_i^{r_i p_{i-1}/\mu}\left(\hat{z}_i^{(2\mu-\tau-r_{i-1})/r_i} - \gamma_i^{(2\mu-\tau-r_{i-1})/r_{i-1}}\right) \\
& \leq e_i^2 + \tfrac{\xi_i^2+\xi_{i-1}^2}{16} + h_i(l_{i-1})e_{i-1}^2
\end{aligned} \tag{24}$$

where $h_i(l_{i-1})$ is a continuous function. With the help of the previous propositions, the derivative of $U = \sum_{i=2}^{n} U_i$ is

$$\begin{aligned}\dot{U} \leq\ & \frac{1}{2}\sum_{i=1}^{n}\xi_i^2 + (-\ell_1 m_2 + \alpha_2 + \bar{\alpha} + g_3(\ell_2) \\ & + h_3(\ell_2))e_2^2 + \sum_{i=3}^{n-1}(-\ell_{i-1}m_i + \alpha_i + 1 + \bar{\alpha} \\ & + g_{i+1}(\ell_i) + h_{i+1}(\ell_i))e_i^2 + (-\ell_{n-1} + 1 + \bar{\alpha})e_n^2.\end{aligned} \quad (25)$$

Determination of Observer Gain ℓ_i: Due to the unmeasurable states, the controller $v = v(\hat{z})$ gives a redundant term in (18). To deal with this, we have the following.

Proposition 3.4: There is a constant $\tilde{\alpha} \geq 0$ such that

$$\xi_n^{\frac{2\mu-\tau-r_n}{\mu}}(v(\hat{z}) - v^*(z)) \leq \frac{1}{4}\sum_{i=1}^{n}\xi_i^2 + \tilde{\alpha}\sum_{i=2}^{n}e_i^2. \quad (26)$$

Combining (25), (18) and (26) together yields

$$\begin{aligned}\dot{T} \leq\ & -\frac{1}{4}\left(\xi_1^2 + \cdots + \xi_{n-1}^2 + \xi_n^2\right) \\ & + (-\ell_1 m_1 + \alpha_2 + \tilde{\alpha} + \bar{\alpha} + g_3(\ell_2) + h_3(\ell_2))e_2^2 \\ & + \sum_{i=3}^{n-1}(-\ell_{i-1}m_i + \alpha_i + 1 + \tilde{\alpha} + \bar{\alpha} + g_{i+1}(\ell_i) + \\ & + h_{i+1}(\ell_i))e_i^2 + (-\ell_{n-1} + 1 + \tilde{\alpha} + \bar{\alpha})e_n^2\end{aligned} \quad (27)$$

for the Lyapunov function $T = V_n + U$. Clearly, by choosing

$$\ell_{n-1} = \frac{1}{4} + 1 + \tilde{\alpha} + \bar{\alpha},$$

$$\ell_{i-1} = m_i^{-1}\left[\frac{1}{4} + \alpha_i + 1 + \tilde{\alpha} + \bar{\alpha} + g_{i+1}(\ell_i) + h_{i+1}(\ell_i)\right],$$

$$\ell_1 = m_2^{-1}\left[\frac{1}{4} + \alpha_2 + \tilde{\alpha} + \bar{\alpha} + g_3(\ell_2) + h_3(\ell_2)\right],$$

$i = n-1, \cdots, 3$, (27) becomes

$$\dot{T} \leq -\frac{1}{4}\left(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 + e_2^2 + e_3^2 + \cdots + e_n^2\right). \quad (28)$$

Note that from the construction of T, it is easily verified that T is positive definite and proper with respect to

$$(z_1, \cdots, z_n, \eta_2, \cdots, \eta_n)^T =: \mathcal{Z}. \quad (29)$$

Similarly, the right hand side of (28) is negative definite. Therefore, the closed-loop system is globally asymptotically stable. Denoting $f_{n+1} = \dot{\eta}_2, f_{n+2} = \dot{\eta}_3, \cdots, f_{2n-1} = \dot{\eta}_n$, it is straightforward to verify that the closed-loop system (16)-(19)-(20), which can be rewritten in the following form

$$\begin{aligned}\dot{\mathcal{Z}} &= F(\mathcal{Z}) \quad (30)\\ &= (z_2^{p_1}, \cdots, z_n^{p_{n-1}}, v(z_1, \eta_2, \cdots, \eta_n), f_{n+1}, \cdots, f_{2n-1})^T\end{aligned}$$

is homogeneous. In fact, by choosing the dilation weight

$$\Delta = (\underbrace{r_1,\ r_2, \cdots,\ r_n,}_{\text{for } z_1, \cdots, z_n}\ \underbrace{r_1,\ r_2, \cdots,\ r_{n-1}}_{\text{for } \eta_2, \cdots, \eta_n}), \quad (31)$$

it can be shown that (30) is homogeneous of degree τ. In addition, T is homogeneous of degree $2\mu - \tau$ and the right hand side of (28) is homogeneous of degree 2μ. ∎

Remark 3.1: Note that the right hand side of (28) is negative definite and homogenous of degree 2μ. Hence, it can be shown that there is a constant $c_1 > 0$ so $\frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}}F(\mathcal{Z}) \leq -c_1\|\mathcal{Z}\|_\Delta^{2\mu}$ where $\|\mathcal{Z}\|_\Delta = \sqrt{\sum_{i=1}^{2n-1}\|\mathcal{Z}_i\|^{2/r_i}}$.

B. Global Output Feedback Stabilization for System (1)

Utilization of the homogeneous controller and observer established in the preceding sections enable the next theorem.

Theorem 3.2: Under A2.1, the *inherently* nonlinear system (1) can be globally stabilized by output feedback.

Proof: Under the new coordinates, denoting $\kappa_1 = 0$ and $\kappa_i = \frac{\kappa_{i-1}+1}{p_{i-1}}$ for $i = 2, \cdots, n$,

$$z_1 = x_1,\ z_i = x_i/L^{\kappa_i},\ v = u/(L^{\kappa_n+1}) \quad (32)$$

with $L > 1$, the system (1) can be rewritten as

$$\dot{z}_{i-1} = Lz_i^{p_{i-1}} + \frac{\phi_{i-1}(\cdot)}{L^{\kappa_{i-1}}}, \quad \dot{z}_n = Lv + \frac{\phi_n(\cdot)}{L^{\kappa_n}}. \quad (33)$$

Next, we construct an observer with the scaling gain L

$$\dot{\eta}_k = -L\ell_{k-1}\hat{z}_k^{p_{k-1}},\ \hat{z}_k^{p_{k-1}} = [\eta_k + \ell_{k-1}\hat{z}_{k-1}]^{\frac{r_k p_{k-1}}{r_{k-1}}} \quad (34)$$

$k = 2, \ldots, n$, where $\hat{z}_1 = z_1$ and ℓ_i, $i = 1, \cdots, n-1$ are the gains selected by (27) in Theorem 3.1. Using the same notations (29) and (30), the closed-loop system (33)-(34)-(20) can be written as

$$\dot{\mathcal{Z}} = LF(\mathcal{Z}) + (\phi_1(\cdot), \frac{\phi_2(\cdot)}{L^{\kappa_2}}, \cdots, \frac{\phi_n(\cdot)}{L^{\kappa_n}}, 0 \cdots, 0)^T. \quad (35)$$

Note that the $F(\mathcal{Z})$ in (35) has the exact same structure as (30) due to the use of same gains ℓ_i and β_i. Hence, adopting the same Lyapunov function $T(\mathcal{Z})$ used in preceding subsection, it can be concluded from Remark 3.1 that

$$\begin{aligned}\dot{T} &= L\frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}}F(\mathcal{Z}) + \frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}}(\phi_1(\cdot), \frac{\phi_2(\cdot)}{L^{\kappa_2}}, \cdots, \frac{\phi_n(\cdot)}{L^{\kappa_n}}, \bar{0})^T \\ &\leq -Lc_1\|\mathcal{Z}\|_\Delta^{2\mu} + \frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}}(\phi_1(\cdot), \frac{\phi_2(\cdot)}{L^{\kappa_2}}, \cdots, \frac{\phi_n(\cdot)}{L^{\kappa_n}}, \bar{0})^T,\end{aligned} \quad (36)$$

where $\bar{0}$ is a vector of zeros of suitable dimension. Under the change of coordinates (32), we deduce from A2.1 and the nomenclature of $L > 1$ that

$$\left|\frac{\phi_i(t,x,u)}{L^{\kappa_i}}\right| \leq cL^{1-\nu_i}\sum_{j=1}^{i}|z_j|^{\frac{r_{i+1}p_i}{r_j}} = cL^{1-\nu_i}\sum_{j=1}^{i}|z_j|^{\frac{r_i+\tau}{r_j}} \quad (37)$$

for some constant $\nu_i > 0$. Recall that for $i = 1, \cdots, n-1$, $\partial T/\partial \mathcal{Z}_i$ is homogeneous of degree $2\mu - \tau - r_i$. Then

$$\left|\frac{\partial T}{\partial \mathcal{Z}_i}\right|\left(|z_1|^{\frac{r_i+\tau}{r_1}} + |z_2|^{\frac{r_i+\tau}{r_2}} + \cdots + |z_i|^{\frac{r_i+\tau}{r_i}}\right) \quad (38)$$

is homogeneous of degree 2μ. With (37) and (38) in mind, we can find a constant ρ_i such that

$$\frac{\partial T}{\partial \mathcal{Z}_i}\frac{\phi_i(\cdot)}{L^{\kappa_i}} \leq \rho_i L^{1-\nu_i}\|\mathcal{Z}\|_\Delta^{2\mu}. \quad (39)$$

Substituting (39) into (36) yields

$$\dot{T}|_{(33)-(34)-(20)} \leq -L(c_1 - \sum_{i=1}^{n}\rho_i L^{-\nu_i})\|\mathcal{Z}\|_\Delta^{2\mu}. \quad (40)$$

Apparently, when L is large enough the right hand side of the (40) is negative definite. Consequently, the closed-loop system is globally asymptotically stable. ∎

The following corollaries demonstrate the generalized framework of the methodology described in the preceding sections. When $\tau = 0$, A2.1 reduces to the bound described in [12], where $r_1 = 1,\ r_2 = 1/p_1, \ldots,\ r_n = 1/(p_1 p_2 \cdots p_{n-1})$; and when $p_i = p$, $p \geq 1$ an odd integer, by selecting $\tau = p - 1$, it is apparent that the Lipschitz-like growth condition of [14] is contained in A2.1.

Corollary 3.1: [12] Under A2.1, with $\tau = 0$, there is an output feedback controller of the form (34)-(20) which achieves global asymptotic stabilization of system (1).

Corollary 3.2: [14] When $p_i = p$, $p \geq 1$ an odd integer, under A2.1, with $\tau = p - 1$, global stability is achieved for (1) by an output feedback controller of the form (34)-(20).

For the case when $p_i = 1$, A2.1 reduces to the assumption in [9], and (1) becomes a linear chain of integrators perturbed by a nonlinear vector field, which is the system covered in the main result of [9].

Corollary 3.3: [9] When $p_i = 1$, under A2.1, there is an output feedback controller of the form (34)-(20) which achieves global asymptotic stability of the system (1).

Remark 3.2: Note that system (2) can now be stabilized by output feedback with $q = 2$. Under A2.1, ϕ_1 is trivial and $\phi_2 = x_2^2$, therefore $p_1 = 3,\ \tau = 1/2,\ r_1 = 1,\ r_2 = 1/2$, and $\mu = 3/2$. By the form of τ the controller structure of Remark 2.1 is used and the output feedback controller is $\dot{\eta}_2 = -L\ell_1 \mathrm{sign}(\eta_2 + \ell_1 y)|\eta_2 + \ell_1 y|^{3/2}$ and $u = -L^{4/3}\beta_2 \mathrm{sign}\left(\mathrm{sign}(\eta_2 + \ell_1 y)|\eta_2 + \ell_1 y|^{3/2} + \beta_1 \mathrm{sign}(y)|y|^{3/2}\right)$ $\left|\mathrm{sign}(\eta_2 + \ell_1 y)|\eta_2 + \ell_1 y|^{3/2} + \beta_1 \mathrm{sign}(y)|y|^{3/2}\right|^{2/3}$, where $\beta_1,\ \beta_2,\ \ell_1$, and L are appropriate positive constants.

APPENDIX

A. Useful Inequalities The next three lemmas, given without proof, were used for the implicit tool of adding a power integrator [10], and proved therein.

Lemma A.1: For $x \in \mathbb{R}, \mathrm{y} \in \mathbb{R}$, $p \geq 1$ is a constant, the following inequalities hold:

$$|x+y|^p \leq 2^{p-1}|x^p + y^p|, \tag{A.1}$$

$$(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}}(|x| + |y|)^{\frac{1}{p}}. \tag{A.2}$$

If $p \in \mathbb{R}^{\geq 1}_{\mathrm{odd}}$, then

$$|x-y|^p \leq 2^{p-1}|x^p - y^p| \quad \text{and} \quad |x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{\frac{p-1}{p}}|x-y|^{\frac{1}{p}}. \tag{A.3}$$

Lemma A.2: Let c, d be positive constants. Given any positive number $\gamma > 0$, the following inequality holds:

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}. \tag{A.4}$$

Lemma A.3: Let $p \in \mathbb{R}^{\geq 1}_{\mathrm{odd}}$ and $x,\ y$ be real-valued functions. Then, for a constant $c > 0$

$$\begin{aligned} |x^p - y^p| &\leq p|x-y|(x^{p-1} + y^{p-1}) &\text{(A.5)}\\ &\leq c|x-y||(x-y)^{p-1} + y^{p-1}|. &\text{(A.6)}\end{aligned}$$

B. Proof of Propositions This part of the appendix contains the technical details of the proofs. Herein we use a generic constant c which exemplifies any finite positive constant value and may be implicitly changed in various places. Nevertheless, the constant c is always independent of ℓ_i.

Proof of Proposition 2.1: First, for $l = 1, \cdots, k-1$

$$\begin{aligned} \left|\frac{\partial W_k}{\partial x_l}\dot{x}_l\right| &\leq c|x_k - x_k^*||\xi_k|^{\frac{2\mu-\tau-r_k-\mu}{\mu}}\left|\frac{\partial x_k^{*\mu/r_k}}{\partial x_l}\dot{x}_l\right| \\ &\leq c|\xi_k|^{\frac{2\mu-\tau-\mu}{\mu}}\left|\frac{\partial x_k^{*\mu/r_k}}{\partial x_l}\dot{x}_l\right| \end{aligned} \tag{B.1}$$

where the last inequality is from (A.3) with $p = \frac{\mu}{r_k} > 1$. By definition of x_k^* and (A.2),

$$\frac{\partial x_k^{*\mu/r_k}}{\partial x_l} = \frac{\partial(\bar{\beta}_{k-1}\xi_{k-1})}{\partial x_l} \leq c(|\xi_{l-1}|^{\frac{\mu-r_l}{\mu}} + |\xi_l|^{\frac{\mu-r_l}{\mu}}). \tag{B.2}$$

This, together with (13) gives

$$\begin{aligned} \left|\frac{\partial x_k^{*\mu/r_k}}{\partial x_l}\dot{x}_l\right| &\leq c\sum_{i=l-1}^{l}|\xi_i|^{\frac{\mu-r_l}{\mu}}\left(|x_{l+1}|^{p_l} + \sum_{j=1}^{l}|\xi_j|^{\frac{r_{l+1}p_l}{\mu}}\right) \\ &\leq c\sum_{i=l-1}^{l}|\xi_i|^{\frac{\mu-r_l}{\mu}}\left(|\xi_{l+1}|^{\frac{r_{l+1}p_l}{\mu}} + |\xi_l|^{\frac{r_{l+1}p_l}{\mu}} + \sum_{j=1}^{l}|\xi_j|^{\frac{r_{l+1}p_l}{\mu}}\right) \end{aligned}$$

By Lemma A.2 and the fact that $r_{l+1}p_l = \tau + r_l$, we have

$$\left|\frac{\partial x_k^{*\mu/r_k}}{\partial x_l}\dot{x}_l\right| \leq c\sum_{i=1}^{l+1}|\xi_i|^{\frac{\mu-r_l+r_{l+1}p_l}{\mu}} = c\sum_{i=1}^{l+1}|\xi_i|^{\frac{\mu+\tau}{\mu}} \tag{B.3}$$

for $l = 1, \cdots, k-1$. Clearly, Proposition 2.1 follows from (B.1) and (B.3). ∎

Proof of Proposition 3.1: By the definition of $\hat{z}_i$, it can be shown that, with $q_i := 2\mu - \tau - r_{i-1} - r_i$

$$\begin{aligned} &z_{i+1}^{p_i} b_i z_i^{q_i/r_i}\left(z_i^{r_{i-1}/r_i} - (\eta_i + \ell_{i-1}z_{i-1})\right) \\ &= z_{i+1}^{p_i} b_i z_i^{\frac{q_i}{r_i}}\Bigg((z_i^{p_{i-1}})^{\frac{r_{i-1}}{r_i p_{i-1}}} - (\hat{z}_i^{p_{i-1}})^{\frac{r_{i-1}}{r_i p_{i-1}}} \\ &\qquad - \ell_{i-1}(z_{i-1}^{p_{i-2}/p_{i-2}} - \hat{z}_{i-1}^{p_{i-2}/p_{i-2}})\Bigg). \end{aligned} \tag{B.4}$$

Note that $r_{i-1}/(r_i p_{i-1}) \leq 1$. By (A.3) with $p = r_{i-1}/(r_i p_{i-1})$,

$$\left|(z_i^{p_{i-1}})^{\frac{r_{i-1}}{r_i p_{i-1}}} - (\hat{z}_i^{p_{i-1}})^{\frac{r_{i-1}}{r_i p_{i-1}}}\right| \leq 2^{1-\frac{r_{i-1}}{r_i p_{i-1}}}|e_i|^{\frac{r_{i-1}}{\mu}} \tag{B.5}$$

$$(z_{i-1}^{p_{i-2}/p_{i-2}} - \hat{z}_{i-1}^{p_{i-2}/p_{i-2}}) \leq 2^{1-\frac{1}{p_{i-2}}}|e_{i-1}|^{\frac{r_{i-1}}{\mu}}, \tag{B.6}$$

which is similar to (B.5). By (17), we know that

$$|z_i| \leq c\left(|\xi_i|^{\frac{r_i}{\mu}} + |\xi_{i-1}|^{\frac{r_i}{\mu}}\right),\ |z_{i+1}| \leq c\left(|\xi_{i+1}|^{\frac{r_{i+1}}{\mu}} + |\xi_i|^{\frac{r_{i+1}}{\mu}}\right).$$

With this in mind, by Young's inequality, utilizing $r_{i+1}p_i = \tau + r_i$ and (A.1), the following holds

$$\begin{aligned} z_{i+1}^{p_i} z_i^{\frac{q_i}{r_i}} &\leq c\left[|\xi_{i+1}|^{\frac{r_{i+1}p_i}{\mu}} + |\xi_i|^{\frac{r_{i+1}p_i}{\mu}}\right]\left[|\xi_i|^{\frac{q_i}{\mu}} + |\xi_{i-1}|^{\frac{q_i}{\mu}}\right] \\ &\leq c(|\xi_{i-1}|^{\frac{2\mu-r_{i-1}}{\mu}} + |\xi_i|^{\frac{2\mu-r_{i-1}}{\mu}} + |\xi_{i+1}|^{\frac{2\mu-r_{i-1}}{\mu}}) \end{aligned} \tag{B.7}$$

Applying (B.5), (B.6), and (B.7) to (B.4) yields,

$$\begin{aligned} &z_{i+1}^{p_i} b_i z_i^{\frac{q_i}{r_i}}\left(z_i^{\frac{r_{i-1}}{r_i}} - (\eta_i + \ell_{i-1}z_{i-1})\right) \leq c\sum_{j=i-1}^{i+1}|\xi_j|^{\frac{2\mu-r_{i-1}}{\mu}} \times \\ &\left(2^{1-\frac{r_{i-1}}{r_i p_{i-1}}}|e_i|^{\frac{r_{i-1}}{\mu}} + \ell_{i-1}2^{1-\frac{1}{p_{i-2}}}|e_{i-1}|^{\frac{r_{i-1}}{\mu}}\right) \end{aligned} \tag{B.8}$$

Applying Young's inequality to each term in (B.8) will lead to (22). In the case when $i = 2$, $e_1 = 0$, so $g_2 := 0$. ∎

Proof of Proposition 3.2: Similar to (B.4), (B.5), and (B.6), we have, with $q_n := 2\mu - \tau - r_{n-1} - r_n$

$$v(\hat{z}) b_n z_n^{\frac{q_n}{r_n}} \left(z_n^{\frac{r_{n-1}}{r_n}} - (\eta_n + \ell_{n-1} z_{n-1}) \right) \leq c |v(\hat{z})| \sum_{j=n-1}^{n} |\xi_j|^{\frac{q_n}{\mu}} \times \left(2^{1-\frac{r_{n-1}}{r_n}} |e_n|^{\frac{r_{n-1}}{\mu}} + \ell_{n-1} 2^{1-\frac{1}{p_{n-2}}} |e_{n-1}|^{\frac{r_{n-1}}{\mu}} \right) \quad \text{(B.9)}$$

By the homogeneity of v, $|v(\hat{z})| \leq c \|\hat{z}\|_{\Delta_z}^{r_n+\tau}$, where $\|\hat{z}\|_{\Delta_z} = \left(\sum_{i=1}^{n} |\hat{z}_i|^{2/r_i} \right)^{1/2}$, $\Delta_z = (r_1, \cdots, r_n)$. So, by the definition of the homogeneous norm, we have

$$\|\hat{z}\|_{\Delta_z} = \left(\sum_{i=1}^{n} |\hat{z}_i|^{\frac{2}{r_i}} \right)^{1/2} = \left(\sum_{i=1}^{n} \left| z_i^{p_{i-1}} - e_i^{\frac{r_i p_{i-1}}{\mu}} \right|^{\frac{2}{r_i p_{i-1}}} \right)^{1/2} \leq c \sum_{i=1}^{n} |z_i|^{\frac{1}{r_i}} + c \sum_{i=1}^{n} |e_i|^{\frac{1}{\mu}}. \quad \text{(B.10)}$$

Likewise, using (17) to replace ξ_i for z_i, together with (B.10),

$$|v(\hat{z})| \leq c \sum_{i=1}^{n} |\xi_i|^{\frac{r_n+\tau}{\mu}} + c \sum_{i=2}^{n} |e_i|^{\frac{r_n+\tau}{\mu}}. \quad \text{(B.11)}$$

Applying (B.11) to (B.9) yields,

$$\begin{aligned} & v(\hat{z}) b_n z_n^{q_n/r_n} \left(z_n^{r_{n-1}/r_n} - (\eta_n + \ell_{n-1} z_{n-1}) \right) \\ & \leq c \left(\sum_{i=1}^{n} |\xi_i|^{\frac{r_n+\tau}{\mu}} + \sum_{i=2}^{n} |e_i|^{\frac{r_n+\tau}{\mu}} \right) \left(|\xi_n|^{\frac{q_n}{\mu}} + |\xi_{n+1}|^{\frac{q_n}{\mu}} \right) \times \\ & \qquad \left(2^{1-\frac{r_{n-1}}{r_n}} |e_n|^{\frac{r_{n-1}}{\mu}} + \ell_{n-1} 2^{1-\frac{1}{p_{n-2}}} |e_{n-1}|^{\frac{r_{n-1}}{\mu}} \right) \\ & \leq \frac{1}{8} \sum_{i=1}^{n} \xi_i^2 + \bar{\alpha} \sum_{i=2}^{n} e_i^2 + g_n(\ell_{n-1}) e_{n-1}^2, \end{aligned}$$

for a constant $\bar{\alpha} > 0$. The last relation is obtained by applying Lemma A.2 to each term in the above inequality. ∎

Proof of Proposition 3.3: By definition of $\hat{z}_i$, γ_i, and Lemma A.3 ($p = q_i/r_{i-1} := (2\mu - \tau - r_{i-1})/r_{i-1} > 1$) we have,

$$\begin{aligned} & -\ell_{i-1} e_i^{r_i p_{i-1}/\mu} \left(\hat{z}_i^{q_i/r_i} - \gamma_i^{q_i/r_{i-1}} \right) \\ & \leq c \ell_{i-1} |e_i|^{\frac{r_i p_{i-1}}{\mu}} |\eta_i + \ell_{i-1} \hat{z}_{i-1} - (\eta_i + \ell_{i-1} z_{i-1})| \times \\ & \qquad \left| (\eta_i + \ell_{i-1} \hat{z}_{i-1})^{\frac{q_i}{r_{i-1}} - 1} - \gamma_i^{\frac{q_i}{r_{i-1}} - 1} \right| \end{aligned}$$

using (A.3) with $p = p_{i-2}$,

$$\leq c \ell_{i-1}^{1+\frac{1}{p_{i-2}}} |e_i|^{\frac{r_i p_{i-1}}{\mu}} |e_{i-1}|^{\frac{r_{i-1}}{\mu}} \left| \hat{z}_i^{\frac{q_i - r_{i-1}}{r_i}} + |\ell_{i-1} e_{i-1}|^{\frac{q_i - r_{i-1}}{\mu}} \right|.$$

On the other hand, by (A.3) with $p = p_{i-1}$, $\hat{z}_i \leq c(|e_i|^{r_i/\mu} + |\xi_i|^{r_i/\mu} + \tilde{\beta}_{i-1} |\xi_{i-1}|^{r_i/\mu})$. Thus,

$$\begin{aligned} & -\ell_{i-1} e_i^{\frac{r_i p_{i-1}}{\mu}} \left(\hat{z}_i^{\frac{q_i}{r_i}} - \gamma_i^{\frac{q_i}{r_{i-1}}} \right) \\ & \leq c \ell_{i-1}^{1+\frac{1}{p_{i-2}}} |e_i|^{\frac{r_i p_{i-1}}{\mu}} |e_{i-1}|^{\frac{r_{i-1}}{\mu}} \left(|\xi_i|^{\frac{q_i - r_{i-1}}{\mu}} \right. \\ & \qquad \left. + \hat{\beta}_{i-1} |\xi_{i-1}|^{\frac{q_i - r_{i-1}}{\mu}} + |e_i|^{\frac{q_i - r_{i-1}}{\mu}} + |\ell_{i-1} e_{i-1}|^{\frac{q_i - r_{i-1}}{\mu}} \right). \end{aligned}$$

By using Young's inequality to each term in above relation, the desired result can be proven for a function $h_i(\ell_{i-1})$. ∎

Proof of Proposition 3.4: Firstly, let $w(\cdot) = v(\cdot)^{\mu/(r_n+\tau)}$. By (A.3) of Lemma A.1, we have

$$|v(\hat{z}) - v^*(z)| \leq c \, |w(\hat{z}) - w^*(z)|^{(r_n+\tau)/\mu}$$

Now, because w is at least C^1, we expand this function as (with $\chi_i := (z_i^{p_{i-1}} - \lambda e_i^{r_i p_{i-1}/\mu})^{1/p_{i-1}}$)

$$|w(\hat{z}) - w^*(z)| \leq c \sum_{i=2}^{n} |e_i|^{\frac{r_i}{\mu}} \int_0^1 \left. \frac{\partial w(X)}{\partial X_i} \right|_{X=\chi} d\lambda.$$

By the homogeneity of $w^*(z)$ whose degree is μ, $\partial w(X)/\partial X_i$ is homogeneous of degree $\mu - r_i$. Hence,

$$\begin{aligned} \left. \frac{\partial w(X)}{\partial X_i} \right|_{\chi} & \leq c \|(z_i^{p_{i-1}} - \lambda e_i^{r_i p_{i-1}/\mu})^{1/p_{i-1}}\|_{\Delta_z}^{\mu - r_i} \\ & \leq c \left(\sum_{i=1}^{n} |z_i|^{\frac{\mu - r_i}{r_i}} \right) + c \left(\sum_{i=2}^{n} |e_i|^{\frac{\mu - r_i}{\mu}} \right), \end{aligned}$$

for $\lambda \in [0, 1]$. Noting (B.11), we have

$$\left. \frac{\partial w(X)}{\partial X_i} \right|_{\chi} \leq c \left(\sum_{i=1}^{n} |\xi_i|^{\frac{\mu - r_i}{\mu}} \right) + c \left(\sum_{i=2}^{n} |e_i|^{\frac{\mu - r_i}{\mu}} \right).$$

Therefore,

$$\begin{aligned} & \xi_n^{\frac{2\mu - \tau - r_n}{\mu}} (v(\hat{z}) - v^*(z)) \leq c |\xi_n^{\frac{2\mu - \tau - r_n}{\mu}}| \, |w(\hat{z}) - w^*(z)|^{\frac{r_n+\tau}{\mu}} \\ & \leq c |\xi_n^{\frac{2\mu - \tau - r_n}{\mu}}| \sum_{i=1}^{n} \left(\left| e_i^{\frac{r_i}{\mu}} \xi_i^{\frac{\mu - r_i}{\mu}} \right|^{\frac{r_n+\tau}{\mu}} + \left| e_i^{\frac{r_i}{\mu}} e_i^{\frac{\mu - r_i}{\mu}} \right|^{\frac{r_n+\tau}{\mu}} \right) \\ & \leq \frac{1}{4} \sum_{i=1}^{n} \xi_i^2 + \tilde{\alpha} \sum_{i=2}^{n} e_i^2, \text{ for a constant } \tilde{\alpha} \geq 0. \end{aligned}$$ ∎

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