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A class of port-controlled Hamiltonian systems

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Abstract— **In this paper, we examine a particular class of port-controlled Hamiltonian systems for which the equations can be written in a form that is very similar to the equations of a linear passive system. We examine the passivity of such a system around an equilibrium point (u_0,x_0), which generates the output y_0. We show that under some mild assumptions, a new Hamiltonian can be found such that the system is again passive with respect to the new supply rate $(y-y_0)^T(u-u_0)$.**

I. INTRODUCTION

Port-controlled Hamiltonian systems on the state space $\mathbb{R}^n$ are, by definition, systems described by the equations

$$\dot{z} = A(z)\left(\frac{\partial H(z)}{\partial z}\right)^T + B(z)u, \quad (1)$$

$$y = B^T(z)\left(\frac{\partial H(z)}{\partial z}\right)^T, \quad (2)$$

where the state z, the input u and the output y are functions of $t \geq 0$, such that $z(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$. The continuous matrix-valued functions A, B are such that $A(z) \in \mathbb{R}^{n\times n}$, $A(z)+A^T(z) \leq 0$, $B(z) \in \mathbb{R}^{n\times m}$. The function $H \in \mathscr{C}^2(\mathbb{R}^n,\mathbb{R})$ is called the *Hamiltonian* of the system. We represent u, x and y as column vectors. In this paper, we treat $\frac{\partial H(z)}{\partial z}$ as a row vector, following the general convention for Jacobian matrices. Port-controlled Hamiltonian systems have been extensively studied, see, for example, [3], [5], [6].

In this paper we examine the particular class of port-controlled Hamiltonian systems for which the function $z \mapsto \left(\frac{\partial H(z)}{\partial z}\right)^T$ is left invertible, i.e., there exists a continuous function $F : \Omega \to \mathbb{R}^n$ such that

$$F\left(\left(\frac{\partial H(z)}{\partial z}\right)^T\right) = z \quad \forall z \in \mathbb{R}^n. \quad (3)$$

Here, $\Omega \subset \mathbb{R}^n$ is the set of all the vectors of the form $\left(\frac{\partial H(z)}{\partial z}\right)^T$, for some $z \in \mathbb{R}^n$. We claim that Ω *is open*. This is not a trivial fact. The reasoning goes as follows: The function $G : \mathbb{R}^n \to \Omega$ defined by

$$G(z) = \left(\frac{\partial H(z)}{\partial z}\right)^T \quad (4)$$

is of class $\mathscr{C}^1$, in particular, it is continuous. Its inverse F is (by assumption) also continuous. By Corollary 3.2 in [1, Chapter XVII], F maps any boundary point of Ω contained in Ω into a boundary point of $F(\Omega)$. But $F(\Omega) = \mathbb{R}^n$ has no boundary points, hence Ω contains none of its boundary points. Hence, it is open. Obviously, Ω is also simply connected.

In the sequel, we also assume that F is of class $\mathscr{C}^2$.

Typically, if a component of $x = \left(\frac{\partial H(z)}{\partial z}\right)^T$ represents a voltage on a capacitor (or a current through an inductor, or a force on a spring) then the corresponding component of $z = F(x)$ would be the electrical charge (or the flux, or the displacement) of the same storage element. If the storage elements of a system are non-linear, then of course F is non-linear. (See also the example treated at the end of this paper.)

Let us denote $\mathscr{H}(x) = H(F(x))$, then the definition of F implies that

$$\frac{\partial \mathscr{H}(x)}{\partial x} = x^T \frac{\partial F(x)}{\partial x}, \quad (5)$$

for all $x \in \Omega$. This implies that for all $x \in \Omega$, $\frac{\partial F(x)}{\partial x}$ is self-adjoint, to ensure that $\frac{\partial \mathscr{H}(x)}{\partial x}$ is a gradient, i.e., for any $i, j \in \{1,\ldots,n\}$,

$$\frac{\partial^2 \mathscr{H}(x)}{\partial x_i \partial x_j} = \frac{\partial^2 \mathscr{H}(x)}{\partial x_j \partial x_i}.$$

Remark 1.1: The equation (5) implies that if F is given, we can compute $\mathscr{H}$ (up to an additive constant). For any $a, b \in \mathbb{R}^n$, we denote by $[a,b]$ the closed straight line segment from a to b. Let Ω_0 be the set of all those $x \in \Omega$ for which $[0,x] \subset \Omega$. If $\mathscr{H}$ is given, we can compute $F(x)$ for $x \in \Omega_0$ (up to an additive constant), by integrating (5) on $[0,x]$.

Using F and denoting $x = \left(\frac{\partial H(z)}{\partial z}\right)^T$, the standard port-controlled Hamiltonian equations (1)–(2) can be rewritten as

$$\dot{F}(x) = \mathscr{A}(x)x + \mathscr{B}(x)u, \quad (6)$$

$$y = \mathscr{B}^T(x)x, \quad (7)$$

where $x(t) \in \Omega$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, $\mathscr{A}(x) = A(F(x))$ and $\mathscr{B}(x) = B(F(x))$ (for all $x \in \Omega$).

For the system described by (6)–(7), it is particularly easy to check passivity, i.e., $\dot{\mathscr{H}}(x) \leq y^T u$.

In this paper we show that this special class of port-controlled Hamiltonian systems, under some additional assumptions, has the following desirable property:

Property A. If $(u_0, x_0) \in \mathbb{R}^m \times \Omega$ is an *equilibrium point* of the system (6)–(7), generating the output y_0, i.e.,

$$0 = \mathscr{A}(x_0)x_0 + \mathscr{B}(x_0)u_0, \quad (8)$$

$$y_0 = \mathscr{B}^T(x_0)x_0, \quad (9)$$

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(the corresponding equilibrium state for (1)–(2) is, of course, $z_0 = F(x_0)$), then the same system is again passive with respect to a suitable new Hamiltonian $\mathscr{H}_0$ and the new supply rate $(y-y_0)^T(u-u_0)$, i.e.,

$$\dot{\mathscr{H}}_0(x) \leq (y-y_0)^T(u-u_0). \tag{10}$$

Property **A** is natural in LTI systems. Suppose that an LTI system **P** with state z, input u and output y, is passive with respect to the quadratic storage function H. Let (u_0, z_0) be an equilibrium point of **P**, generating the output y_0. Then, the same LTI system **P** is also passive with respect to the quadratic storage function $z \mapsto H(z-z_0)$ and the supply rate $(y-y_0)^T(u-u_0)$, i.e., (10) holds.

One application of Property **A** is the PI controller-based tracking control of a constant reference signal for such a system, see [2].

II. MAIN RESULT

Notation and definitions. Throughout this paper, $\mathbb{R}_+ = [0,\infty)$, for $x \in \mathbb{R}^n$ we use the norm $\|x\| = \left(\sum_n |x_n|^2\right)^{\frac{1}{2}}$ and the space $\mathscr{C}^q(\mathbb{R}^l, \mathbb{R}^p)$ consists of q times continuously differentiable functions $f : \mathbb{R}^l \to \mathbb{R}^p$.

Let $\Omega \subset \mathbb{R}^n$ be open. A function $f : \Omega \to \mathbb{R}^n$ is called *monotone and non-decreasing* if it satisfies

$$(f(a)-f(b))^T(a-b) \geq 0,$$

for any $a, b \in \Omega$ and it is called *strictly monotone increasing* if it satisfies

$$(f(a)-f(b))^T(a-b) > 0,$$

for any $a \neq b$, $a, b \in \Omega$, see, for example, [7].

H is called *proper* if $H(z) \to \infty$ whenever $\|z\| \to \infty$.

Theorem 2.1: Suppose that, for the port-controlled Hamiltonian system described by (1)–(2) with Hamiltonian function H, there exists a function $F \in \mathscr{C}^2(\Omega, \mathbb{R}^n)$ that satisfies (3), where Ω is the set described after (3). Let $\mathscr{A}(x) = A(F(x))$ and $\mathscr{B}(x) = B(F(x))$ (for all $x \in \Omega$), so that the system is described by (6)–(7). Assume that the function $x \mapsto -\mathscr{A}(x)x$ is monotone and non-decreasing, and $\mathscr{B}$ is constant.

Then, the system (6)–(7) has Property **A**. More precisely, for any equilibrium point $(u_0, x_0) \in \mathbb{R}^m \times \Omega$ of (6)–(7) generating the output y_0, we define $\mathscr{H}_0 \in \mathscr{C}^2(\Omega, \mathbb{R})$ (up to an additive constant) by

$$\frac{\partial \mathscr{H}_0(x)}{\partial x} = (x-x_0)^T \frac{\partial F(x)}{\partial x}. \tag{11}$$

Then, the same system is passive w.r.t. $\mathscr{H}_0$ and supply rate $(y-y_0)^T(u-u_0)$, i.e., (10) holds.

Proof: The first step is to prove that for every point $x \in \Omega$, the matrix $\frac{\partial F(x)}{\partial x}$ is invertible. Indeed, using the function G from (4), by (3), $F(G(z)) = z$ for all $z \in \mathbb{R}^n$, and hence,

$$\left.\frac{\partial F(x)}{\partial x}\right|_{x=G(z)} . \frac{\partial G(z)}{\partial z} = I.$$

This shows that $\frac{\partial F(x)}{\partial x}$ is invertible at $x = G(z)$.

Define

$$H_0(z) = \mathscr{H}_0(G(z)). \tag{12}$$

The second step is to show that for any $x \in \Omega$,

$$\left.\frac{\partial H_0(z)}{\partial z}\right|_{z=F(x)} = (x-x_0)^T. \tag{13}$$

We see from (12), taking $z = F(x)$, that

$$\mathscr{H}_0(x) = H_0(F(x)).$$

Hence, by the chain rule,

$$\frac{\partial \mathscr{H}_0(x)}{\partial x} = \left.\frac{\partial H_0(z)}{\partial z}\right|_{z=F(x)} \frac{\partial F(x)}{\partial x}.$$

Using (11), we obtain

$$(x-x_0)^T \frac{\partial F(x)}{\partial x} = \left.\frac{\partial H_0(z)}{\partial z}\right|_{z=F(x)} \frac{\partial F(x)}{\partial x}. \tag{14}$$

According to the result from the first step, we can eliminate $\frac{\partial F(x)}{\partial x}$ appearing on both sides, which yields (13).

The final step is to show that, for any equilibrium point (u_0, x_0) generating the output y_0, (10) holds.

We rewrite the system (6)–(7) with constant $\mathscr{B}$ as follows:

$$\begin{aligned} \dot{F}(x) &= \mathscr{A}(x)x + \mathscr{B}u_0 + \mathscr{B}(u-u_0), && (15)\\ y - y_0 &= \mathscr{B}^T(x-x_0). && (16) \end{aligned}$$

Using (8)–(16), $\dot{\mathscr{H}}_0(x)$ is given by

$$\begin{aligned} \dot{\mathscr{H}}_0(x) &= \left.\frac{\partial H_0(z)}{\partial z}\right|_{z=F(x)} \dot{F}(x) \\ &= (x-x_0)^T \mathscr{A}(x)x + (x-x_0)^T \mathscr{B}u_0 \\ &\quad + (x-x_0)^T \mathscr{B}(u-u_0) \\ &= (x-x_0)^T (\mathscr{A}(x)x - \mathscr{A}(x_0)x_0) \\ &\quad + (y-y_0)^T(u-u_0) \\ &\leq (y-y_0)^T(u-u_0), \end{aligned}$$

where the last inequality is due to the monotonicity of $-\mathscr{A}(x)x$. □

Assume now that $0 \in \Omega$ (this implies that $(0,0)$ is an equilibrium point of the system (6)–(7)). Let $x \in \Omega_0$ (Ω_0 has been defined in Remark 1.1). If we integrate (5) on $[0,x]$ and denote $\mathrm{d}x = x\mathrm{d}\lambda$, where $\lambda \in [0,1]$, we get

$$\begin{aligned} \mathrm{d}\mathscr{H}(x) &= x^T \mathrm{d}F(x) = x^T \frac{\partial F(x)}{\partial x} \mathrm{d}x \\ &= x^T \frac{\partial F(x)}{\partial x} x \mathrm{d}\lambda. && (17) \end{aligned}$$

Thus, if $\frac{\partial F(x)}{\partial x} > 0$ for all $x \in \Omega_0$, (17) implies that $\mathscr{H}(x)$ is radially increasing in Ω_0 and has its minimum over the set Ω_0 at $x = 0$.

Let Ω_{x_0} be the set of those $x \in \Omega$ for which $[x_0, x] \subset \Omega$. If we integrate (11) on $[x_0, x]$ and denote $\mathrm{d}x = (x-x_0)\mathrm{d}\lambda$, where $\lambda \in [0,1]$, we have

$$\begin{aligned} \mathrm{d}\mathscr{H}_0(x) &= (x-x_0)^T \mathrm{d}F(x) = (x-x_0)^T \frac{\partial F(x)}{\partial x} \mathrm{d}x \\ &= (x-x_0)^T \frac{\partial F(x)}{\partial x} (x-x_0) \mathrm{d}\lambda. && (18) \end{aligned}$$

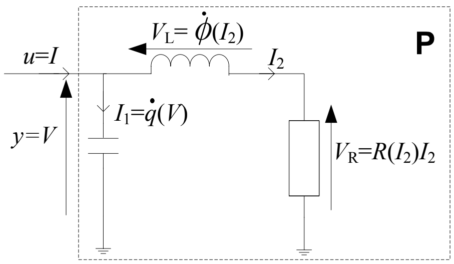

Fig. 1. A nonlinear electrical circuit which consists of a nonlinear resistor, a nonlinear inductor and a nonlinear capacitor.

Then, if $\frac{\partial F(x)}{\partial x} > 0$ for all $x \in \Omega_{x_0}$, (18) implies that $\mathscr{H}_0(x)$ is radially increasing in Ω_{x_0} and has its minimum over the set Ω_{x_0} at x_0.

Note that if $\Omega = \mathbb{R}^n$ and $\frac{\partial F(x)}{\partial x} \geq \varepsilon > 0$ for all $x \in \mathbb{R}^n$, then we can also conclude from the above argument that $\mathscr{H}$ has a global minimum at 0 and it is proper, i.e., $\lim_{\|x\|\to\infty} \mathscr{H}(x) = \infty$, and similarly for $\mathscr{H}_0(x)$.

III. EXAMPLE

Consider the electrical circuit diagram in Fig. 1. The nonlinear system **P** in Fig. 1 can be described in state space as follows:

$$\begin{bmatrix} \dot{\phi}(I_2) \\ \dot{q}(V) \end{bmatrix} = \begin{bmatrix} -R(I_2) & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} I_2 \\ V \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad (19)$$

$$y = V, \qquad (20)$$

where $x(t) \equiv \begin{bmatrix} I_2(t) \\ V(t) \end{bmatrix} \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$ is the input current, $y(t) \in \mathbb{R}$ is the output voltage, $\phi \in \mathscr{C}^2(\mathbb{R},\mathbb{R})$ such that $\phi(0) = 0$, $(\phi(b) - \phi(a))(b-a) > 0$ for any $a \neq b$, $q \in \mathscr{C}^2(\mathbb{R},\mathbb{R})$ such that $q(0) = 0$, $(q(b) - q(a))(b-a) > 0$ for any $a \neq b$ and $R \in \mathscr{C}(\mathbb{R},\mathbb{R}_+)$ such that $R(I_2)I_2$ is strictly monotone increasing, i.e.,

$$(R(b)b - R(a)a)(b-a) > 0, \qquad (21)$$

for any $a \neq b$. I_2 is the current flowing through the inductor and $\phi(I_2)$ denotes magnetic flux of the inductor. The voltage across inductor is given by $V_L = \dot{\phi}(I_2)$. The voltage of the capacitor is equal to V and $q(V)$ denotes electric charge in the capacitor. The current flowing through the capacitor is given by $I_1 = \dot{q}(V)$. Note that for a linear resistor, $\alpha(I_2) = RI_2$, where $R > 0$ is the resistance of the element, for a linear inductor, $\phi(I_2) = LI_2$, where $L > 0$ denotes the inductance and for a linear capacitor, $q(V) = CV$, where $C > 0$ denotes the capacitance.

It can be seen that the system described by (19) and (20) is of the form (6)–(7) with $F(x) = \begin{bmatrix} \phi(I_2) \\ q(V) \end{bmatrix}$. Since $\phi(I_2)$ and $q(V)$ are strictly monotone, it follows that $\frac{\partial F(x)}{\partial x} > 0$. The storage function $\mathscr{H}(x)$ (up to an additive constant) can be found by integrating (5) on a straight line from 0 to x. More precisely,

$$\begin{aligned} \mathscr{H}(x) - \mathscr{H}(0) &= q(V)V - \int_0^V q(\lambda)\mathrm{d}\lambda \\ &\quad + \phi(I_2)I_2 - \int_0^{I_2} \phi(\lambda)\mathrm{d}\lambda, \end{aligned} \qquad (22)$$

where $\mathscr{H}(0) \in \mathbb{R}$ is an additive constant. $\mathscr{H}(x)$ is radially increasing and has global minimum at 0. It is easy to see that **P** is passive with input u and output y using $\mathscr{H}(x)$ in (22), i.e. $\dot{\mathscr{H}} \leq y^T u$.

For any $y_0 \in \mathbb{R}$, let I_{20} be the solution to $R(I_{20})I_{20} = y_0$ (it is unique by (21)), let $V_0 = y_0$ and let $u_0 = I_{20}$. It can be checked that ($u_0, x_0 = \begin{bmatrix} I_{20} \\ V_0 \end{bmatrix}$) is the equilibrium point of (19)–(20) and the corresponding output is y_0. Thus, for any $y_0 \in \mathbb{R}$, there is a unique (u_0, x_0) which gives the equilibrium point of **P**.

For any set of equilibrium point (u_0, x_0) with the corresponding output y_0, Theorem 2.1 shows that the electrical circuit P is also passive with input $u - u_0$ and output $y - y_0$, where the storage function $\mathscr{H}_0(x)$ (up to an additive constant) is computed by integrating (11) from x_0 to x. More precisely,

$$\begin{aligned} \mathscr{H}_0(x) - \mathscr{H}_0(x_0) &= q(V)(V - V_0) - \int_{V_0}^V q(\lambda)\mathrm{d}\lambda \\ &\quad + \phi(I_2)(I_2 - I_{20}) - \int_{I_{20}}^{I_2} \phi(\lambda)\mathrm{d}\lambda, \end{aligned}$$

where $\mathscr{H}_0(x_0) \in \mathbb{R}$ is an additive constant. $\mathscr{H}_0(x)$ satisfies (11), radially increasing and has global minimum at x_0. Indeed, $\dot{\mathscr{H}}_0(x) \leq (y - y_0)^T(u - u_0)$.

IV. CONCLUSIONS

A particular class of port-controlled Hamiltonian systems has been presented, for which the equations can be written in a form that is very similar to the equations of a linear passive system. For such a system, it is interesting to note that around an equilibrium point (u_0, x_0), which generates the output y_0, and under some mild assumptions, a new Hamiltonian can be found such that the system is again passive with respect to the new supply rate $(y - y_0)^T(u - u_0)$.

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