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Min-Max MPC using a tractable QP Problem

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Abstract— **Min-Max MPC (MMMPC) controllers [5] suffer from a great computational burden that is often circumvented by using upper bounds of the worst possible case of a performance index. These upper bounds are usually computed by means of linear matrix inequalities (LMI) techniques. In this paper a more efficient approach is shown. This paper proposes a computationally efficient MMMPC control strategy in which a close approximation of the solution of the min-max problem is computed using a quadratic programming problem. The overall computational burden is much lower than that of the min-max problem and the resulting control is shown to have guaranteed stability. Simulation examples are given in the paper.**

Keywords: Robust control; Uncertain systems; Min-max; Model predictive control

I. INTRODUCTION

In min-max model predictive control (MMMPC) controllers, the value of the control signal to be applied is found by minimizing the worst case of a performance index (usually quadratic) which is in turn computed by maximizing over the possible expected values of disturbances and uncertainty. Solving these problems can be very time consuming as they are of the NP-hard kind [18], [8], [17]. Thus, the implementation of this type of control is severely compromised leading to a lack of experimental results. For moderate fast dynamics the min-max problem can be solved numerically only when the number of extreme realizations of the uncertainty is relatively low. This is the case when the prediction horizon is small or when a complexity reduction strategy like that of [3] is used. When fast dynamics are to be controlled the min-max problem cannot be solved numerically, and approximate solutions have to be used [15], [16]. However, these techniques impose great rigidity in the controller parameters, as well as a certain degree of approximation error.

Often the computational burden issue is solved by using a bound of the worst case cost instead of computing it explicitly [1]. The upper bound can be efficiently computed by using linear matrix inequalities (LMI) techniques such as in [7], [6], [9], [19]. The LMI problems have a computational burden that cannot be neglected in certain applications, like those in which the sampling rates are measured in seconds.

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This paper proposes a different strategy in which the min-max problem is replaced by a quadratic programming (QP) problem that provides a close approximation to the solution of the original min-max problem. The computational burden is much lower than that of the min-max problem. Moreover, it can be easily implemented in almost any platform in which a constrained MPC can be implemented. On the other hand stability is guaranteed as it is shown in the paper.

The paper is organized as follows: section II presents the MMMPC controller and some of its properties. Section III presents the efficient approximation of the solution of the min-max problem based on a pair of QP problems. Robust stability of the proposed controller is shown in section IV. The strategy is illustrated by means of a simulation example in section V. Finally, section VI presents the conclusions.

II. MIN-MAX MPC WITH BOUNDED ADDITIVE UNCERTAINTIES

Consider the following state space model with bounded additive uncertainties [4]:

$$x(t+1) = Ax(t) + Bu(t) + D\theta(t+1) \tag{1}$$

with $x(t) \in \mathbb{R}^{dimx}$ the state vector, $u(t) \in \mathbb{R}^{dimu}$ the input vector and $\theta(t) \in \{\theta \in \mathbb{R}^{dim\theta} : \|\theta\|_\infty \leq \varepsilon\}$ the uncertainty, that is supposed to be bounded.

The system is subject to state and input time invariant constraints.

$$F_u u(t) + F_x x(t) \leq g.$$

The cost function is a quadratic performance index of the form:

$$\begin{aligned} V(x, \mathbf{v}, \boldsymbol{\theta}) = & \sum_{j=0}^{N-1} x(t+j|t)^T Q x(t+j|t) \\ & + \sum_{j=0}^{N-1} u(t+j|t)^T R u(t+j|t) \\ & + x(t+N|t)^T P x(t+N|t) \end{aligned} \tag{2}$$

where $x(t+j|t)$ is the prediction of the state for $t+j$ made at t. Note that this value depends on the future values of the uncertainty. Matrices $Q \in \mathbb{R}^{dimx \times dimx}$ and $R \in \mathbb{R}^{dimu \times dimu}$ are symmetric positive definite matrices used as weighting parameters.

Although the results presented in this paper are not valid in general for closed-loop MMMPC with a quadratic cost (see [10] and references therein), they are valid when using a semi-feedback approach in which the control input is given by

$$u(t) = -Kx(t) + v(t), \tag{3}$$

where the feedback matrix K is chosen to achieve some desired property such as nominal stability or LQR optimality without constraints. The MMMPC controller will compute the optimal sequence of correction control inputs $v(t)$. The state equation of system (1) can be rewritten as

$$x(t+1) = A_{CL}x(t) + Bv(t) + D\theta(t+1), \quad (4)$$

with $A_{CL} = (A - BK)$. In the following we will assume that the semi-feedback approach is used.

Min-Max MPC [5] is based on finding the control correction sequence $\mathbf{v} = [v(t|t) \cdots v(t+N-1|t)]^T$ that minimizes the cost function for the worst possible case of the predicted future evolution of the process state or output signal, while guaranteeing robust constraint satisfaction. This is accomplished through the solution of a min-max problem like:

$$\begin{aligned} \mathbf{v}^*(x) = \quad & \arg\min_{\mathbf{v}} \max_{\boldsymbol{\theta}\in\Theta} \quad V(x, \mathbf{v}, \boldsymbol{\theta}) \\ & \text{s.t.} \quad F_u u(t+j|t) + F_x x(t+j|t) \le g, \\ & \qquad j = 0, \ldots, N, \quad \forall \boldsymbol{\theta} \in \Theta, \\ & \quad x(t+N|t) \in \Omega, \\ & \qquad \forall \boldsymbol{\theta} \in \Theta, \end{aligned} \quad (5)$$

with $x(t|t) = x$, where $\boldsymbol{\theta} = [\theta(t+1) \cdots \theta(t+N)]^T$ is a sequence of future values of $\theta(t)$ over a prediction horizon N, and $\boldsymbol{\Theta} = \{\boldsymbol{\theta} \in \mathbb{R}^{N \cdot dim\theta} : \|\boldsymbol{\theta}\|_\infty \le \varepsilon\}$ is the set of possible uncertainty trajectories. A terminal region constraint $x(t+N|t) \in \Omega$, where Ω is a polyhedron, is included to assure stability of the control law.

As usual in all predictive control schemes, the solution of problem (5) is applied in a feedback manner using a receding horizon strategy.

The predictions $x(t+j|t)$ and $u(t+j|t)$ depend linearly on x, $\mathbf{v}$ and $\boldsymbol{\theta}$. This means that it is possible to find a vector $d \in \mathbb{R}^p$ and matrices G_x, G_v and G_θ such that all the robust linear constraints of problem (5) can be rewritten as:

$$G_x^i x + G_v^i \mathbf{v} + G_\theta^i \boldsymbol{\theta} \le d_i, \quad i = 1\ldots, p, \ \forall \boldsymbol{\theta} \in \Theta,$$

where G_x^i, G_v^i, G_θ^i denote the i-th rows of G_x, G_v and G_θ respectively and d_i is the i-th component of $d \in \mathbb{R}^p$. Denote now $\|G_\theta^i\|_1$ the sum of the absolute values of row G_θ^i. Taking into account that $\max_{\boldsymbol{\theta}\in\Theta} G_\theta^i \boldsymbol{\theta} = \max_{\|\boldsymbol{\theta}\|_\infty \le \varepsilon} G_\theta^i \boldsymbol{\theta} = \varepsilon \|G_\theta^i\|_1$, the robust fulfillment of the constraints is satisfied if and only if: $G_x^i x + G_v^i \mathbf{v} + \varepsilon \|G_\theta^i\|_1 \le d_i, \quad i = 1\ldots, p$. Therefore, to guarantee robust constraint satisfaction, the following set of linear constraint must be satisfied:

$$G_x x + G_v \mathbf{v} \le d_\varepsilon.$$

where $d_\varepsilon \in \mathbb{R}^p$ is a vector whose i-th component is equal to $d_i - \varepsilon \|G_\theta^i\|_1$.

Taking into account (4),(3) and the quadratic nature of the performance index, the cost function can be evaluated as a quadratic function on the initial state, the control correction vector and the uncertainty trajectory.

$$\begin{aligned} V(x, \mathbf{v}, \boldsymbol{\theta}) = & \ \mathbf{v}^T M_{vv} \mathbf{v} + \boldsymbol{\theta}^T M_{\theta\theta} \boldsymbol{\theta} + 2\boldsymbol{\theta}^T M_{\theta v} \mathbf{v} \\ & + 2x^T M_{vf}^T \mathbf{v} + 2x^T M_{\theta f}^T \boldsymbol{\theta} + x^T M_{ff} x \end{aligned} \quad (6)$$

where the matrices can be obtained from the system and the control parameters, see for example [4].

It can be seen that $M_{\theta\theta}$ is a Gram matrix and therefore at least positive semidefinite. On the other hand, M_{vv} is positive definite as $R > 0$. Note that as $M_{\theta\theta} \ge 0$, $V(x, \mathbf{u}, \boldsymbol{\theta})$ is convex on $\boldsymbol{\theta}$, and because $M_{vv} > 0$ strictly convex on $\mathbf{v}$. Therefore the solution of (5) will be unique [4]. Moreover, due to the convexity on $\boldsymbol{\theta}$ the maximum is attained at least at one of the vertices of the hypercube described by Θ.

Taking this into account, problem (5) is equivalent to

$$\begin{aligned} \mathbf{v}^*(x) = \arg \quad & \min_{\mathbf{v}} \quad \max_{\boldsymbol{\theta}\in \text{vert}(\Theta)} V(x, \mathbf{v}, \boldsymbol{\theta}) \\ & \text{s.t.} \quad G_x x + G_v \mathbf{v} \le d_\varepsilon \end{aligned} \quad (7)$$

The terminal region Ω is assumed to satisfy the following conditions:

- **C1**: If $x \in \Omega$ then $A_{CL}x + D\theta \in \Omega$, for every $\theta \in \Theta$.
- **C2**: If $x \in \Omega$ then $-Kx \in U$, where $U \subseteq \mathbb{R}^{dimu}$ is a compact set that contains all admissible inputs.

Moreover, matrix P that characterizes the terminal cost is assumed to satisfy

- **C3**: $P - A_{CL}^T P A_{CL} > Q + K^T R K$.

The stability of A_{CL} guarantees the existence of a positive definite matrix P satisfying C3. Note that these conditions are standard in the literature and are also easy to met (see [2]).

The maximum cost for a given x and $\mathbf{v}$ is denoted as

$$V^*(x, \mathbf{v}) = \quad V(x, \mathbf{v}, 0) + \max_{\boldsymbol{\theta}\in \text{vert}(\Theta)} \boldsymbol{\theta}^T H \boldsymbol{\theta} + 2\boldsymbol{\theta}^T q(x, \mathbf{v}) \quad (8)$$

where $H = M_{\theta\theta}$, $q(x, \mathbf{v}) = M_{\theta v}\mathbf{v} + M_{\theta f}x$ and $V(x, \mathbf{v}, 0) = \mathbf{v}^T M_{vv} \mathbf{v} + 2x^T M_{vf}^T \mathbf{v} + x^T M_{ff} x$ is the part of the cost that does not depend on the uncertainty.

With this definition, problem (7) can be rewritten as

$$\begin{aligned} \mathbf{v}^*(x) = \quad & \arg\min_{\mathbf{v}} V^*(x, \mathbf{v}) \\ & \text{s.t.}\ G_x x + G_v \mathbf{v} \le d_\varepsilon, \end{aligned} \quad (9)$$

Note that the minimum of (9), i.e., $V^*(x, \mathbf{v}^*(x))$, is convex as both $V^*(x, \mathbf{v})$ and the constraints are convex. and the system is controlled by $K_{MPC}(x) = -Kx + v^*(t|t)$.

In order to evaluate $V^*(x, \mathbf{v})$ it is necessary to evaluate the function for all the vertices of Θ. Taking into account that the number of vertices is $2^{N*dim\theta}$ it is clear that the problem cannot be solved in real time beyond a certain dimension of $\boldsymbol{\theta}$ (because this is a well known NP-hard problem).

III. A QP APPROACH TO MIN-MAX MPC

In this section it is shown how the min-max problem (9) can be replaced by a tractable QP problem which provides a close approximation of the solution of the original problem. The strategy can be summarized in the following steps:

1) Obtain an initial guess of the solution of (9), denoted by $\tilde{\mathbf{v}}^*$. As seen later, this can be achieved solving a QP problem.
2) Using $\tilde{\mathbf{v}}^*$, obtain a quadratic function of $\mathbf{v}$ that bounds the worst case cost.

3) Compute the control law. This involves the solution of a QP problem.

In the following all these steps will be detailed.

A. Computing $\tilde{\mathbf{v}}^*$

Let T be a diagonal matrix computed as $T = \mathbf{diag}(T_1,\dots,T_n)$ where

$$T_i = \sum_{j=1}^{n} |H_{ij}|$$

Note that H_{ij} denotes the (i,j)-th component of matrix H (recall that matrix H is defined in equation (8)). Because of how matrix T is defined: $T \geq H$. Let $\tilde{V}(x,\mathbf{v},\boldsymbol{\theta})$ be defined as:

$$\tilde{V}(x,\mathbf{v},\boldsymbol{\theta}) = V(x,\mathbf{v},0) + \boldsymbol{\theta}^T T\boldsymbol{\theta} + 2q^T(x,\mathbf{v})\boldsymbol{\theta} \qquad (10)$$

From the inequality $T \geq H$ it is inferred that $\tilde{V}(x,\mathbf{v},\boldsymbol{\theta}) \geq V(x,\mathbf{v},\boldsymbol{\theta})$. The maximum of $\tilde{V}(x,\mathbf{v},\boldsymbol{\theta})$ can be computed as

$$\begin{aligned}\tilde{V}^*(x,\mathbf{v}) &= \max_{\boldsymbol{\theta}\in\Theta} \tilde{V}(x,\mathbf{v},\boldsymbol{\theta}) \\ &= V(x,\tilde{\mathbf{v}},0) + \mathrm{trace}(T)\varepsilon^2 + 2\varepsilon\,\|q(x,\mathbf{v})\|_1 \qquad (11)\end{aligned}$$

Then an initial guess of the solution of (9) can be obtained as

$$\begin{aligned}\tilde{\mathbf{v}}^*(x) = \arg\min_{\tilde{\mathbf{v}}} \quad & \tilde{V}^*(x,\tilde{\mathbf{v}}) \\ \text{s.t. } & G_x x + G_v \mathbf{v} \leq d_\varepsilon,\end{aligned} \qquad (12)$$

It is clear that this problem can be recast as a QP problem.

B. Obtaining an upper bound of the maximum

The upper-bound of the maximum will be obtained in two steps. In the first one we compute a set of parameters from $\tilde{\mathbf{v}}^*$ that allows us later, in the second step, to compute the bound as a quadratic function of $\mathbf{v}$.

1) Computing the parameters: Note that for $\mathbf{v} = \tilde{\mathbf{v}}^*$ the maximum is obtained from:

$$V^*(x,\tilde{\mathbf{v}}^*) = \max_{\boldsymbol{\theta}\in\mathrm{vert}(\Theta)} \begin{bmatrix} \boldsymbol{\theta} \\ 1 \end{bmatrix}^T M \begin{bmatrix} \boldsymbol{\theta} \\ 1 \end{bmatrix} = \max_{\|z\|_\infty \leq 1} z^T M z \qquad (13)$$

with $z = \begin{bmatrix} \boldsymbol{\theta} & 1 \end{bmatrix}^T$ and

$$M = \begin{bmatrix} \varepsilon^2 H & \varepsilon q(x,\tilde{\mathbf{v}}^*) \\ \varepsilon q^T(x,\tilde{\mathbf{v}}^*) & V(x,\tilde{\mathbf{v}}^*,0) \end{bmatrix}$$

Now suppose Γ a diagonal matrix such that $\Gamma \geq M$, then:

$$z^T M z \leq z^T \Gamma z = \sum \Gamma_{ii} z_i^2 \leq trace(\Gamma)\|z\|_\infty^2 \leq trace(\Gamma)$$

thus $V^*(x,\tilde{\mathbf{v}}^*) \leq \mathrm{trace}(\Gamma)$. Therefore, a conservative upper bound of $V^*(x,\tilde{\mathbf{v}}^*)$ can be found solving the following problem:

$$\begin{aligned}\xi^* = \quad \min \quad & \mathrm{trace}(\Gamma) \\ s.t. \quad & \\ & \Gamma \geq M \\ & \Gamma \quad \text{diagonal}\end{aligned} \qquad (14)$$

This upper bound satisfies [11]:

$$V^*(x,\tilde{\mathbf{v}}^*) \leq \xi^* \leq \frac{\pi}{2} V^*(x,\tilde{\mathbf{v}}^*) \qquad (15)$$

provided that $M \geq 0$[1]. The main purpose of this section is to obtain a quadratic function of $\mathbf{v}$ that bounds the worst case cost. As in the bound obtained in (14) using LMIs, the approach presented here relies in a diagonalization of matrix M.

The goal here is to find a diagonal matrix Γ such that $\Gamma > M$, while trying to keep its trace small. The strategy is to obtain a diagonal matrix adding to M $n-1$ positive definite matrices of the form $c_i c_i^T$:

$$M + c_1c_1^T + c_2c_2^T + c_3c_3^T + \cdots + c_{n-1}c_{n-1}^T = M_f$$

where M_f is a diagonal matrix. Thus the problem is to find c_i, $i = 1,\cdots,n-1$ such that M_f is diagonal and the conservativeness of the bound is kept as low as possible. Suppose that $M = \begin{bmatrix} a & b^T \\ b & M_r \end{bmatrix}$, $a \in \mathbb{R}$ and that we want to add $c_1c_1^T$ in such a way that:

$$\begin{bmatrix} a & b^T \\ b & M_r \end{bmatrix} + c_1c_1^T = \begin{bmatrix} d & 0 \\ 0 & \hat{M}_r \end{bmatrix}, \qquad d \in \mathbb{R} \qquad (16)$$

Once c_1 is found, the process continues by choosing c_2 such that $\hat{M}_r$ is also partially diagonalized and so on. If c_1 is chosen to be $\begin{bmatrix} \alpha_1 & e^T \end{bmatrix}^T$ then $c_1c_1^T$ becomes:

$$\begin{bmatrix} \alpha_1 \\ e \end{bmatrix} \begin{bmatrix} \alpha_1 & e^T \end{bmatrix} = \begin{bmatrix} \alpha_1^2 & \alpha_1 e^T \\ \alpha_1 e & ee^T \end{bmatrix}$$

with $\alpha_1 > 0$. This implies that $\alpha_1 e = -b$ thus $e = \frac{-b}{\alpha_1}$, $d = a + \alpha_1^2$ and $\hat{M}_r = M_r + \frac{bb^T}{\alpha_1^2}$. Such α_1 is the first of the parameters that will allow us later to express the bound as a function of $\mathbf{v}$.

The parameter α_1 should be chosen to minimize the error introduced by the diagonalization in the original augmented maximization problem. This error is:

$$z^T c_1 c_1^T z = z^T \begin{bmatrix} \alpha_1 \\ -\frac{b}{\alpha_1} \end{bmatrix} \begin{bmatrix} \alpha_1 & -\frac{b^T}{\alpha_1} \end{bmatrix} z$$

The error is maximum when z turns out to be:

$$z^* = \mathrm{sign} \begin{bmatrix} \alpha_1 \\ -\frac{b}{\alpha_1} \end{bmatrix}$$

(and also when it is of opposite sign). Taking into account that

$$\begin{bmatrix} \alpha_1 & -\frac{b^T}{\alpha_1} \end{bmatrix} z^* = \left\| \begin{matrix} \alpha_1 \\ -\frac{b}{\alpha_1} \end{matrix} \right\|_1$$

(where $\|x\|_1$ is the 1-norm equal to the sum of the absolute values of the components of x), the maximum error is

$$\left\| \begin{matrix} \alpha_1 \\ -\frac{b}{\alpha_1} \end{matrix} \right\|_1^2$$

The value of α_1 that minimizes the maximum error can easily be computed by finding the value that makes the derivative of

$$\left\| \begin{matrix} \alpha_1 \\ -\frac{b}{\alpha_1} \end{matrix} \right\|_1 = \alpha_1 + \frac{1}{\alpha_1}\|b\|_1$$

[1]Due to the fact that $H \geq 0$ and $V(x,\mathbf{v},\theta) \geq 0$, $\forall\theta$ it is easy to show that $M \geq 0$.

equal to zero. Such value is:

$$\alpha_1 = \sqrt{\|b\|_1} \tag{17}$$

The following procedure summarizes the steps to compute the sequence α_k, $k = 1 \dots n-1$.

Procedure 1: Computation of α_k for a given M.

1) Let $\Gamma = M \in \mathbb{R}^{n\times n}$.
2) for $k = 1$ to $n-1$
3) Let $M_{sub} = [\Gamma_{ij}]$ for $i, j = k \cdots n$.
4) Compute α_k for $M_{sub} = \begin{bmatrix} a & b \\ b^T & M_r \end{bmatrix}$ from (17).
5) Make $c_k^T = \begin{bmatrix} \alpha_k & \frac{-b^T}{\alpha_k} \end{bmatrix}$.
6) Make $c_e^T = \begin{bmatrix} 0 & \cdots & 0 & c_k^T \end{bmatrix} \in \mathbb{R}^n$
7) Update Γ by making $\Gamma = \Gamma + c_e c_e^T$.
8) endfor

Note that, although not necessary, the bound of the maximum for $\mathbf{v} = \tilde{\mathbf{v}}^*$ would be computed as $\sum_{i=1}^{n} \Gamma_{ii}$. On the other hand, this upper bound results to be a good approximation to the maximum as shown in [13], [14].

2) Obtaining the bound as a function of $\mathbf{v}$*:* Once the sequence $\alpha_1, \dots, \alpha_{n-1}$ has been computed for $\mathbf{v} = \tilde{\mathbf{v}}^*$ the diagonalization process shown in III-B.1 can be repeated using such $\alpha_1, \dots, \alpha_{n-1}$ to obtain a bound of the maximum that can be computed as a quadratic function of $\mathbf{v}$. Consider problem (13) in which M is expressed as a matrix in which some of its elements depend on $\mathbf{v}$, i.e.

$$M(x,\mathbf{v}) = \begin{bmatrix} \varepsilon^2 H_{11} & \varepsilon^2 H_{1r}^T & \varepsilon q_1(x,\mathbf{v}) \\ \varepsilon^2 H_{1r} & \varepsilon^2 H_{rr} & \varepsilon q_r(x,\mathbf{v}) \\ \varepsilon q_1(x,\mathbf{v}) & \varepsilon q_r^T(x,\mathbf{v}) & V(x,\mathbf{v},0) \end{bmatrix}$$

where H_{11}, $q_1(x,\mathbf{v})$ and $V(x,\mathbf{v},0) \in \mathbb{R}$, $q_r(x,\mathbf{v}) \in \mathbb{R}^{(N\cdot dim\theta)-1}$ and $H_{rr} \in \mathbb{R}^{\{(N\cdot dim\theta)-1\}\times\{(N\cdot dim\theta)-1\}}$. Note that $q_1(x,\mathbf{v})$ and $q_r(x,\mathbf{v})$ have an affine dependence on $\mathbf{v}$ whereas $V(x,\mathbf{v},0)$ is a quadratic function. Using the parameter α_1, M can be partially diagonalizated as in (16) adding a term of the form cc^T such as

$$\begin{bmatrix} \varepsilon^2 H_{11} & \varepsilon^2 H_{1r}^T & \varepsilon q_1(x,\mathbf{v}) \\ \varepsilon^2 H_{1r} & \varepsilon^2 H_{rr} & \varepsilon q_r(x,\mathbf{v}) \\ \varepsilon q_1(x,\mathbf{v}) & \varepsilon q_r^T(x,\mathbf{v}) & V(x,\mathbf{v},0) \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ -\frac{\varepsilon^2 H_{1r}}{\alpha_1} \\ -\frac{\varepsilon q_1(x,\mathbf{v})}{\alpha_1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ -\frac{\varepsilon^2 H_{1r}}{\alpha_1} \\ -\frac{\varepsilon q_1(x,\mathbf{v})}{\alpha_1} \end{bmatrix}^T$$

which yields

$$\begin{bmatrix} \varepsilon^2 H_{11} + \alpha_1^2 & 0 & 0 \\ 0 & \varepsilon^2 H_{rr} + \frac{\varepsilon^4 H_{1r} H_{1r}^T}{\alpha_1^2} & \varepsilon q_r(x,\mathbf{v}) + \frac{\varepsilon^3 H_{1r} q_1(x,\mathbf{v})}{\alpha_1^2} \\ 0 & \varepsilon q_r^T(x,\mathbf{v}) + \frac{\varepsilon^3 H_{1r}^T q_1(x,\mathbf{v})}{\alpha_1^2} & V(x,\mathbf{v},0) + \frac{\varepsilon^2 q_1^2(x,\mathbf{v})}{\alpha_1^2} \end{bmatrix}$$

Note that the sub-matrix

$$M_r(x,\mathbf{v}) = \begin{bmatrix} \varepsilon^2 H_{rr} + \frac{\varepsilon^4 H_{1r} H_{1r}^T}{\alpha_1^2} & \varepsilon q_r(x,\mathbf{v}) + \frac{\varepsilon^3 H_{1r} q_1(x,\mathbf{v})}{\alpha_1^2} \\ \varepsilon q_r^T(x,\mathbf{v}) + \frac{\varepsilon^3 H_{1r}^T q_1(x,\mathbf{v})}{\alpha_1^2} & V(x,\mathbf{v},0) + \frac{\varepsilon^2 q_1^2(x,\mathbf{v})}{\alpha_1^2} \end{bmatrix}$$

has the same structure as $M(x,\mathbf{v})$, that is, the last element is a quadratic function of $\mathbf{v}$, the remaining elements of the first row and column are affine functions of $\mathbf{v}$ and all the other elements are constants. Thus the diagonalization process can continue adding terms as in (III-B.2) in a procedure which is almost the same as procedure 1 but without having to compute the sequence $\alpha_1, \dots, \alpha_{n-1}$ (which were computed previously). At the end of this process a diagonal matrix $\hat{\Gamma}$ is obtained in which all the elements are constants (i.e., they do not depend on $\mathbf{v}$) except the last which has the form

$$\hat{\Gamma}_{nn}(\mathbf{v}) = V(x,\mathbf{v},0) + \frac{\varepsilon^2 q_1^2(x,\mathbf{v})}{\alpha_1^2} + \cdots \tag{18}$$

Once $\hat{\Gamma}$ has been obtained, the bound of the maximum can be computed as

$$\hat{V}^*(x,\mathbf{v}) = \hat{\Gamma}_{nn}(\mathbf{v}) + \sum_{i=1}^{n-1} \hat{\Gamma}_{ii} \geq V^*(x,\mathbf{v}) \tag{19}$$

which taking into account (18) is clearly a quadratic function of $\mathbf{v}$.

C. Computing the control law

The value of the control signal is obtained solving the following QP optimization problem

$$\begin{aligned} \hat{\mathbf{v}}^*(x) = \arg\min_{\hat{\mathbf{v}}} \quad & \hat{V}^*(x,\hat{\mathbf{v}}) \\ & \text{s.t. } G_x x + G_v \mathbf{v} \leq d_\varepsilon, \end{aligned} \tag{20}$$

and the system is controlled by $\hat{K}_{MPC}(x(t)) = -Kx(t) + \hat{\mathbf{v}}^*(t|t)$. Note that the constant terms which are added to obtain the bound are not needed when solving the optimization problem (20) as we are only interested in the minimizer, not the minimum itself. As found by the authors in many simulation examples, the performance of the proposed strategy is very close to that of the original min-max problem (see section V).

IV. Stability of the Proposed Control Law

In this section the stability properties of the control $\hat{K}_{MPC}(x(t))$ are shown. First some properties are presented and then stability is proved. Recall that $\mathbf{v}^*$, $\tilde{\mathbf{v}}^*$ and $\hat{\mathbf{v}}^*$ are the solutions of (9), (12) and (20) respectively. Denote also $J(x) = V^*(x,\mathbf{v}^*)$, $\tilde{J}(x) = \tilde{V}^*(x,\tilde{\mathbf{v}}^*)$ and $\hat{J}(x) = \hat{V}^*(x,\hat{\mathbf{v}}^*)$. Note that the optimization problems (9), (12) and (20) have the same feasibility region as the constraints are the same. Also, recall that $J(x)$ is convex. The following property will be used to proof the stability of the control law.

Property 1:

$$\hat{J}(x) \leq \tilde{J}(x)$$

Proof:

Taking into account the definition of $\hat{J}(x)$ and that $\hat{\mathbf{v}}^*$ is the minimizer of $\hat{V}^*(x,\mathbf{v})$ it is evident that

$$\hat{V}^*(x,\tilde{\mathbf{v}}^*) \geq \hat{J}(x) \tag{21}$$

Thus, in order to prove that $\hat{J}(x) \leq \tilde{J}(x)$ it suffices to show that $\tilde{V}^*(x,\tilde{\mathbf{v}}^*) \geq \hat{V}^*(x,\tilde{\mathbf{v}}^*)$. First, note that taking into account that $V(x,v,0) \geq 0$:

$$\begin{aligned}\tilde{V}^*(x,\tilde{\mathbf{v}}^*) &= \|M\|_1 = \left\| \begin{bmatrix} \varepsilon^2 H & \varepsilon q(x,\tilde{\mathbf{v}}^*) \\ \varepsilon q^T(x,\tilde{\mathbf{v}}^*) & V(x,\tilde{\mathbf{v}}^*,0) \end{bmatrix} \right\|_1 \quad (22)\\ &= \left\| \begin{bmatrix} a & b^T \\ b & M_r \end{bmatrix} \right\|_1 \\ &= |a| + 2\|b\|_1 + \|M_r\|_1\end{aligned}$$

On the other hand $\hat{V}^*(x,\tilde{\mathbf{v}}^*)$ is equal to $\sum_{i=1}^n \hat{\Gamma}_{ii}$, that is the sum of the elements of the diagonal matrix computed in procedure 1 which also is equal to $\|\hat{\Gamma}\|_1$ as $\hat{\Gamma} \geq 0$. The initial value of $\hat{\Gamma}$ is $\hat{\Gamma} = M$, thus its 1-norm is equal to $\tilde{V}^*(x,\tilde{\mathbf{v}}^*)$. Taking into account (17) the 1-norm of $\hat{\Gamma}$ after the first diagonalization step is

$$\left\| \begin{matrix} a + \|b\|_1 & 0 \\ 0 & M_r + \frac{bb^T}{\|b\|_1} \end{matrix} \right\|_1 \leq |a| + \|b\|_1 + \|M_r\|_1 + \left\| \frac{bb^T}{\|b\|_1} \right\|_1 \quad (23)$$

Taking into account that $\left\| \frac{bb^T}{\|b\|_1} \right\|_1 = \|b\|_1$ it follows that

$$\left\| \begin{matrix} a + \|b\|_1 & 0 \\ 0 & M_r + \frac{bb^T}{\|b\|_1} \end{matrix} \right\|_1 \leq \|M\|_1 \quad (24)$$

and thus every diagonalization step decreases $\|\hat{\Gamma}\|_1$. This proves that:

$$\hat{V}^*(x,\tilde{\mathbf{v}}^*) \leq \tilde{V}^*(x,\tilde{\mathbf{v}}^*) \quad (25)$$

and this completes the proof. □

It is clear that the optimal solution $\hat{\mathbf{v}}^*$ of problem (20) is a suboptimal feasible solution for problem (9). As it is claimed in the following property, the difference between the optimal value of the original objective function and the value obtained with $\hat{\mathbf{v}}^*$ is bounded by $\mathrm{trace}(T)\varepsilon^2$.

Property 2: It holds that:

$$V^*(x,\hat{\mathbf{v}}^*) - \sigma\varepsilon^2 \leq J(x),$$

where $\sigma = \mathrm{trace}(T)$.

Proof: Note that $J(x) = V^*(x,\mathbf{v}^*)$. On the other hand:

$$\tilde{V}(x,\mathbf{v},\boldsymbol{\theta}) = V(x,\mathbf{v},\boldsymbol{\theta}) + \boldsymbol{\theta}^T(T-H)\boldsymbol{\theta}.$$

Taking into account that $T \geq H \geq 0$, $\|\boldsymbol{\theta}\|_\infty \leq \varepsilon$ and that T is diagonal

$$\tilde{V}(x,\mathbf{v},\boldsymbol{\theta}) \leq V(x,\mathbf{v},\boldsymbol{\theta}) + \boldsymbol{\theta}^T T\boldsymbol{\theta} \leq V(x,\mathbf{v},\boldsymbol{\theta}) + \mathrm{trace}(T)\varepsilon^2$$

thus it can be inferred that $V^*(x,\mathbf{v}^*) \geq \tilde{V}^*(x,\mathbf{v}^*) - \sigma\varepsilon^2$ with $\sigma = \mathrm{trace}(T)$. As $\tilde{\mathbf{v}}^*$ is the minimizer of $\tilde{V}^*(x,\tilde{\mathbf{v}})$, then it holds that $V^*(x,\mathbf{v}^*) \geq \tilde{V}^*(x,\tilde{\mathbf{v}}^*) - \sigma\varepsilon^2$ which in turn can be rewritten as $J(x) \geq \tilde{J}(x) - \sigma\varepsilon^2$. Recall that from property 1: $\hat{J}(x) \leq \tilde{J}(x)$; thus $J(x) \geq \hat{V}^*(x,\hat{\mathbf{v}}^*) - \sigma\varepsilon^2$. Furthermore by construction $\hat{V}^*(x,\mathbf{v}) \geq V^*(x,\mathbf{v})$ thus $J(x) \geq V^*(x,\hat{\mathbf{v}}^*) - \sigma\varepsilon^2$ and this completes the proof. □

The following property, which is proved in [2] will be used in the proof of the stability of the proposed approach (see theorem 1 below).

Property 3: Consider that assumptions C1, C2 and C3 are satisfied. Let $\mathbf{v} = [v_0 v_1 \cdots v_{n-1}]^T$ and $\mathbf{v}_s$ a shifted version of $\mathbf{v}$ computed as $\mathbf{v}_s = [v_1 v_2 \cdots v_{n-1} 0]^T$. If $\mathbf{v}$ is feasible for problem (9) at $x(t)$ then $\mathbf{v}_s$ is also feasible at $x(t+1)$ and there is $\gamma > 0$ such that for every feasible sequence $\mathbf{v}$:

$$V^*(x(t+1),\mathbf{v}_s) \leq V^*(x(t),\mathbf{v}) - x(t)^T Q x(t) + \gamma\varepsilon^2$$

Proof: See [2] for a proof. □

Theorem 1: Under assumptions C1, C2 and C3, the control law given by $\mathbf{u}(x) = -Kx + \hat{\mathbf{v}}^*(t|t)$ stabilizes system (1).

Proof: Let $\hat{\mathbf{v}}_s^*$ the shifted version (as in property 3) of $\hat{\mathbf{v}}^*$. Due to non optimality of $\hat{\mathbf{v}}_s^*$ for problem (9) it holds that

$$J(x(t+1)) \leq V^*(x(t+1),\hat{\mathbf{v}}_s^*) \quad (26)$$

Note that $\hat{\mathbf{v}}_s^*$ is feasible for both (20) and (9), thus by property 3

$$V^*(x(t+1),\hat{\mathbf{v}}_s^*) \leq V^*(x(t),\hat{\mathbf{v}}^*) - x(t)^T Q x(t) + \gamma\varepsilon^2 \quad (27)$$

Furthermore, by property (2) $V^*(x(t),\hat{\mathbf{v}}^*) \leq J(x(t)) + \sigma\varepsilon^2$, thus taking into account this in (27) and using (26) $J(x(t+1)) \leq J(x(t)) - x(t)^T Q x(t) + (\gamma+\sigma)\varepsilon^2$ which can be rewritten

$$J(x(t+1)) - J(x(t)) \leq -x(t)^T Q x(t) + (\gamma+\sigma)\varepsilon^2 \quad (28)$$

Define

$$\Phi_\varepsilon = \left\{ x \in \mathbb{R}^n : (9) \text{ is feasible and } x(t)^T Q x(t) \leq (\gamma+\sigma)\varepsilon^2 \right\}$$

Then the system evolves into set $\Omega_\beta = \{x \in \mathbb{R}^n : V^*(x,\mathbf{v}^*) \leq \beta(\varepsilon)\}$ where $\beta(\varepsilon) = \max_{x \in \Phi_\varepsilon} V^*(x,\mathbf{v}^*) + (\gamma+\sigma)\varepsilon^2$. Thus the state system is ultimately bounded which means that the system is stabilized by the control law $\hat{K}_{MPC}(x(t)) = -Kx(t) + \hat{\mathbf{v}}^*(t|t)$. □

V. Example

To illustrate the results presented in this paper, consider the two-tank network example given in chapter 20 of [12]. Using the parameters given in [3] the following continuous time state-space model can be obtained:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -\frac{0.5}{3} & \frac{0.2}{3} \\ \frac{0.5}{2} & -\frac{0.5}{2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} u \quad (29)\\ y &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x\end{aligned}$$

Constraints are imposed on both states and control actions such that $\|x(k)\|_\infty \leq 1.5$ and $\|u(k)\|_\infty \leq 0.4$. A discrete time model has been obtained from (29) sampling at 0.2 minutes using a zero-order holder. Figure 1 shows the results of the proposed controller applied to the two-tank model. The set-point for the liquid level of each tank was 1 m and 0.7 m respectively. The prediction and control horizons were $N = Nu = 7$. Identity matrices were chosen as Q and R. An uncertainty of ± 0.025 meters is considered to affect both liquid levels. In the simulation a random noise of ± 0.01 meters has been added to both levels and an unexpected loss of liquid in tank 1 is introduced at sampling time $t = 60$.

The absolute deviation of the solution of (20) from that of (9) (computed as $\hat{\mathbf{v}}^*(x) - \mathbf{v}^*(x)$) is also shown in figure 1. It can be seen that it is very small throughout the simulation. This conclusion is supported by figure 2. In it the worst

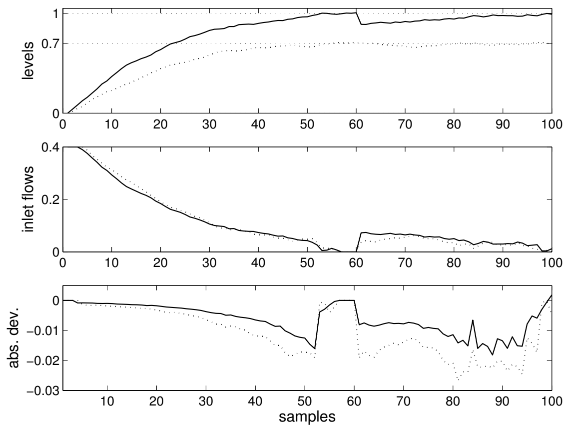

Fig. 1. Liquid levels, inlet flows and absolute deviation (from the exact MMMPC) of the proposed strategy (tank 1 solid plot, tank 2 dotted plot).

TABLE I

MEAN FLOPS FOR THE ORIGINAL MIN-MAX MPC AND THE PROPOSED STRATEGY FOR DIFFERENT VALUES OF THE PREDICTION AND CONTROL HORIZON (N) IN THE SIMULATION EXAMPLE OF SECTION V.

N	Avg. flops (min-max)	Avg. flops (prop.)
4	4.77×10^6	4.28×10^4
5	3.73×10^7	7.6×10^4
6	3.43×10^8	1.28×10^5
7	1.84×10^9	1.42×10^5

case cost when $\mathbf{v} = \hat{\mathbf{v}}^*(x)$ is plotted along with its deviation (percentage) from the optimal cost when $\mathbf{v} = \mathbf{v}^*(x)$. Note that at worst this deviation is under 4%.

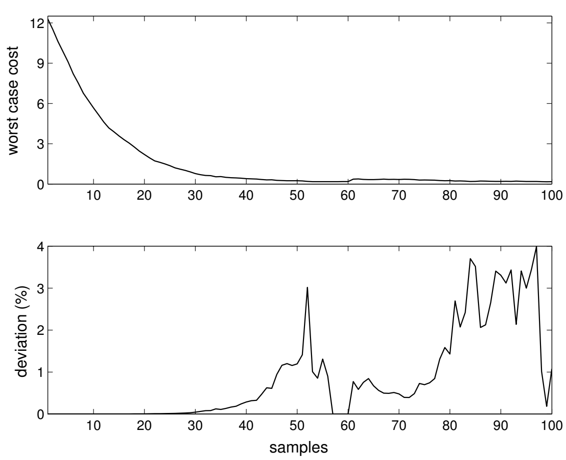

Fig. 2. Worst case cost using $\hat{\mathbf{v}}^*(x)$ (top) and deviation from the worst case cost of $\mathbf{v}^*(x)$ (bottom).

Finally, the lower computational burden of the proposed strategy is illustrated in table I in which are listed the average flops that problem (9) took to be solved as well as the flops needed to compute $\hat{\mathbf{v}}^*(x)$ (including those needed to compute $\tilde{\mathbf{v}}^*(x)$, the sequence $\alpha_1, \ldots, \alpha_{n-1}$ and $\Gamma_{ii}(\mathbf{v})$). The computational burden is, thus, much lower in the proposed strategy and the gap broadens exponentially as prediction horizon grows.

VI. CONCLUSIONS

An MMMPC based on an tractable QP problem has been presented in this paper. This QP problem has a much lower computational burden than the original min-max problem whereas its solution is close to that of the min-max problem. As it is based on a QP problem, it can be implemented almost in whatever industrial hardware capable to run a constrained MPC controller. Thus it extends very much the fields of application of MMMPC controllers.

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