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Problems in Decentralized Sensor-Actuator Networks

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Abstract— **There is a growing body of literature on networked control systems treating the relationship between network channel capacity and stability of the system's operation. In a very rough but intuitive sense, the main results in this area provide a quantitative understanding of the way in which restrictions on the rate of information exchange among system components in a real-time system will degrade the system's performance. Recent extensions of these results provide an understanding of how system performance will depend on the magnitude of noise and the degree of asynchronism in the operation of system components. A number of researchers have recently begun to look at the problem constraints on feedback channel capacity in decentralized feedback control structures. In the present paper, we examine the way in which decentralization magnifies the degradation of information due to noise and asynchronism among decentralized sensors and leads to instabilities even in cases where feedback channels have ample capacity for stable operation of a system with centralized components. Further, we discuss an approach to solving the observed problem by a novel source-coding strategy which is similar to but different from the well known Gray code.**

I. Introduction

More and more of today's control systems take advantage of digital sensors and actuators that can communicate with each other over wired or wireless networks. For these systems, it is important to understand the relationship between network channel capacity and stability of the system's operation. A major step towards this understanding is the well-known *Data-Rate Theorem* ([1], [7], [8], [9], etc.). Results showing how noise, asynchronism and fluctuation of communication capacities between sensors and actuators affect the system performance have also been reported in [3]-[5] in a somewhat centralized context. Designs that are efficient and robust with respect to those factors were proposed as well.

Recently, [6] studied the extension of the Data-Rate Theorem in decentralized linear feedback control systems. The main theorem of [6] states that: the decentralized linear system can be stabilized, as long as the communication interconnections between the decentralized sensors and decentralized controllers can be broken down and reassembled (algorithmatically) for each unstable mode, and provide enough channel capacity to satisfy the data rate requirement of each unstable mode as suggested by the Data-Rate Theorem. Novel techniques for breaking down and reassembling the channels that support the claim was also constructed. However, we have observed that: When the control is coded in a decentralized fashion, tiny amounts of noise and asynchronism of the decentralized sensors may dramatically degrade the control result under some circumstances. This has to do with the process by which knowledge about certain state variables is assembled from pieces of partial information provided by multiple sensors. Consequently, extra caution is needed to decide when the decentralized control design is applicable.

In the present paper, Section II formulates the decentralized control problem. Our formulation emphasizes that the communication within the sensor-actuator network can be designed in terms of a code $\mathcal{C}$, each digit of which corresponds to a unit-capacity communication channel between a sensor and an actuator. In this paper we consider a general scalar plant. A number of ways for decoupling the observation and control of the different modes of multidimensional linear plants have been provided by [6] and [4]. Section III discusses the data-rate problem in the decentralized control problem and briefly explains the results of [6], which are aimed at extending the Data-Rate theorem in centralized control to decentralized cases. The problem brought up by [6] led us to the observation that decentralization may magnify the degradation of the coded information due to noise and asynchronism among the sensors. Section IV proposes a robustness condition for decentralized code-based control implementation, which requires decentralized implementation to preserve the robustness property of the centralized design with respect to sensor noise. A corresponding condition in terms of the code $\mathcal{C}$ is also derived. The condition is closely related to the minimum Hamming distance property of the well-known *Gray code*. However, the Gray code can not be decoded by the decentralized actuators in general. Section V discusses special cases of sensor-actuator networks with certain particular topologies. These cases illustrate how the robustness requirement combined with decentralized decoding restrict the efficiency of the code being used, also how the Gray code may constitute a building block of such codes. In addition, we point out that: Although the decentralized systems we discussed require higher overall data-rates than the centralized systems, the data-rate requirement in terms of each individual sensor-actuator pair is still reduced by adopting the decentralized control structure. Section VI summarizess the results and suggets possible future research directions.

II. Problem formulation

In this paper, we are concerned with the decentralized control of the following scalar plant

$$x(k+1) = \Phi^{\Theta}(\ x(k),\ u(k)\ ), \tag{1}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$. $\Phi^t(x,u)$ is an unstable, time invariant, and continuous flow, and Θ is the sampling interval.

Suppose that there are totally M decentralized sensors that observe and encode the state x in parallel. Assuming

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that each sensor encodes x into m q-ary digits. Possibly the encoding of the same state value by different sensors are different in order for the sensors to provide complementary information. Denote the subset of integers $\{1, 2, \ldots, n\}$ by $\mathbb{Z}_n$. For $i \in \mathbb{Z}_M,\ j \in \mathbb{Z}_m$, the jth digit generated by the ith sensor is

$$s_{i,j}(k) = h_{i,j}(\ x(k) + w_i(k)\ ) : \mathbb{R} \mapsto \mathbb{Z}_q, \tag{2}$$

where w_i is the noise that affects the ith sensor. We assume w_i is a bounded noise with uniform distribution. (The choice of particular distribution is not important here.) Specifically, there are q possible values for the jth digit of the output from the ith sensor. Let $S_{(i)}$ be the block of digits generated by the ith sensor, $S_{(i)} = (s_{i,1}, s_{i,2}, \ldots, s_{i,m})$. Its range — denote by $\mathcal{C}_{(i)}$ — is the code used by that sensor. Collectively, the matrix

$$S = \begin{pmatrix} S_{(1)} \\ S_{(2)} \\ \vdots \\ S_{(M)} \end{pmatrix}_{M\times m} \tag{3}$$

is a codeword generated by a surjective function

$$S = H(x, \mathbf{w}) : \mathbb{R} \times \mathbb{R}^M \mapsto \mathcal{C}, \tag{4}$$

where $\mathcal{C} \subset \{M\text{-by-}m \text{ matrices over } \mathbb{Z}_q\}$. $\mathcal{C}$ will be referred to as the *code* of the decentralized system.

On the other hand, suppose there are totally P decentralized actuators that contribute additively to the control input, i.e.,

$$u(k) = \sum_{r=1}^{P} u_r(k), \tag{5}$$

where u_r is provided by the rth actuator. Assume that each digit generated by a sensor is received by a unique actuator and each actuator receives at least one digit. Formally, let θ_r be the set (non-empty) of (i,j)'s such that the rth actuator receives the jth digit generated by the ith sensor, and consider the arrays of digits

$$S^{(r)} = (s_{i,j})_{(i,j)\in\theta_r}, \quad r \in \mathbb{Z}_P. \tag{6}$$

$\theta_r,\ r = 1, \ldots, P$ is a partition of $\mathbb{Z}_M \times \mathbb{Z}_m$ and so is $S^{(r)}$'s for the digits in the codeword S. In the decentralized control structure, each $S^{(r)}$ is likely to include only a small portion of the digits in S. The output of each decentralized actuator is

$$u_r(k) = g_r(S^{(r)}(k)). \tag{7}$$

From (5), (7) and the fact that $S^{(r)}$'s are digit blocks in S, the overall control input u is a function of the codeword S,

$$u = G(S) = \sum_{r=1}^{P} g_r(S^{(r)}(k)). \tag{8}$$

Note that the structure of $G(S)$, more specifically, how the digits of S are partitioned by the $S^{(r)}$'s, poses constraints on how the code $\mathcal{C}$ can be decoded. This is a very important feature of the problem under discussion. If the intended control law is

$$u = f(x) : \mathbb{R} \mapsto \mathcal{U} = \{\mu_1 < \mu_2 < \ldots < \mu_\lambda\},$$

then

$$G(\cdot) : \mathcal{C} \mapsto \mathcal{U}, \quad \text{and} \quad G \circ H(x, \mathbf{0}) = f(x).$$

For the intended control law $f(\cdot)$, let

$$X_l = \{x \in \mathbb{R} | f(x) = \mu_l\},\ l = 1, \ldots, \lambda.$$

Assume each X_l is a non-degenerate interval. Denote the minimum length of X_l's by ρ. Also assume that the elements in X_l are always smaller than the elements in X_{l+1}. Then X_l is adjacent to (having common boundary points with) only X_{l-1} and/or X_{l+1}. These are natural assumptions in the case of linear control systems. Cases where these assumptions are not valid, although may exist in particular applications, are beyond the scope of the present paper.

For arbitrary $q,\ M,\ m,\ P,\ \theta_r$'s and λ, the functions $G(\cdot)$ and $H(\cdot,\cdot)$ do not necessarily exist. $M,\ m,\ P$ and θ_r's are related to how much the system is decentralized. q can be used to represent the communication capacity between each sensor-actuator pair. λ reflects the richness of possible control actions. Thus the coding problem for $\mathcal{C}$ can be stated as "minimizing q with the other parameters given" or "maximizing λ with the other parameters given".

For minimal coding in the feedback channel, $G(\cdot)$ needs to be bijective. For each μ_l, let σ_l denote the unique codeword in $\mathcal{C}$ such that $G(\sigma_l) = \mu_l$. Then,

$$\mathcal{C} = \{\sigma_1, \sigma_2, \ldots, \sigma_\lambda\}. \tag{9}$$

and the ith row of $H(x, \mathbf{w})$ satisfies

$$H_{(i)}(x, \mathbf{w}) = \sigma_l^{(i)} \quad \text{if} \quad x + w_i \in X_l. \tag{10}$$

To summarize (also see Figure 1),

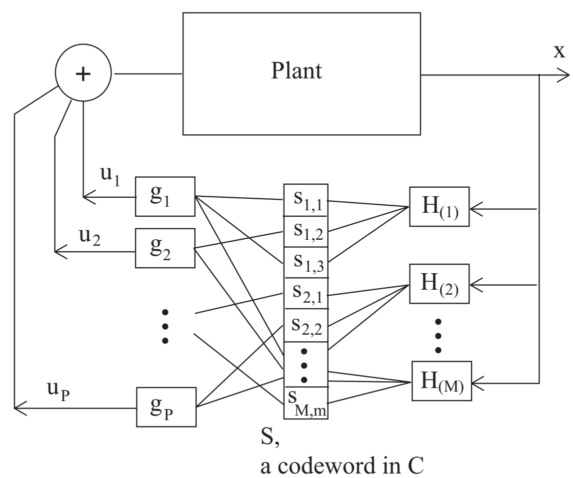

Fig. 1. Feedback loop implemented by sensor-actuator network.

$$\begin{cases} x(k+1) = \Phi^{\Theta}(x(k), u(k)), \\ S(k) = H(x(k), \mathbf{w}(k)) : \mathbb{R} \times \mathbb{R}^M \mapsto \mathcal{C}, \\ u(k) = G(S(k)) : \mathcal{C} \mapsto \mathcal{U} = \{\mu_1, \mu_2, \ldots, \mu_\lambda\}, \\ G \circ H(x, \mathbf{0}) = f(x). \end{cases} \tag{11}$$

Note that the digits of $S \in \mathcal{C}$ can be arranged in two ways. First S can be presented as in (3), in which case we say that "S is arranged by encoding units". Or, S can be presented by a concatenation of $S^{(r)},\ r = 1, \ldots, P$, see (6). In the second case, we say that "S is arranged by decoding units". In this paper, we discuss how the design of the code $\mathcal{C}$ affects the stability and robustness of the above system.

III. Data-rate and linear systems

The overall data-rate for completing the communications in system (11) is (in bits/unit time)

$$R = \frac{M \cdot m \cdot \log_2 q}{\Theta} = \frac{\sum_{r=1}^{P} |\theta_r| \cdot \log_2 q}{\Theta}, \tag{12}$$

where $|\cdot|$ denotes the cardinality of a set.

First, consider the linear plant

$$x(k+1) = e^{a\Theta} x(k) + (e^{a\Theta} - 1)u(k), \tag{13}$$

where $a > 0$. Results from the literature (see [8], etc.) shows that if the system is centralized ($M = P = 1$), then R — in this case $R = \log_2 q/\Theta$ — must be greater than $a \cdot \log_2 e$ for the system to be stable.

For the decentralized feedback implementation, assume that $\log_2 q/\Theta < a \cdot \log_2 e$, i.e., each individual pair of sensor-actuator does not possess enough communication capacity between them to stabilize the plant alone. Then, one question is, what is the minimum value for R, the aggregate data-rate of all sensor-actuator pairs (12), such that the system can be stabilized.

Recently, this problem has been studied in [6]. A multidimensional, open-loop unstable, linear plant was considered and contributions have been made both on decoupling the observation and control of the states, and on feedback coding schemes that are aimed at stabilizing the system with minimum communication. Here, we only consider a simplified, scalar version of the problem. (Readers are strongly recommended to read the original paper.) The discussion here will lead us to an important observation on the robustness issue in decentralized control.

Consider the system

$$\begin{cases} x(k+1) = e^{a\Theta} x(k) + (e^{a\Theta} - 1)(u_1(k) + u_2(k)), \\ s_i(k) = h_i(x(k) + w_i(k)), \\ u_i(k) = g_i(s_i(k)), \quad i = 1, 2. \end{cases} \tag{14}$$

Let R_i denote the interconnecting data-rate between the ith sensor and the ith controller. The code for the ith channel, $\mathcal{C}_i$ is the set of all possible values of s_i. Then

$$R_i = \frac{\log_2 |\mathcal{C}_i|}{\Theta}. \tag{15}$$

The overall data-rate of the system is $R = R_1 + R_2$. In terms of this system, the main results of [6] (Theorem 1 and 2) can be summarized as

Corollary 1: For the system (14), decentralized coder-controller can be constructed to stabilize the system if and only if

$$R_1 + R_2 > a \cdot \log_2 e. \tag{16}$$

The proof is trivial from the Theorem 1 and 2 of [6]. The coder design in [6] uses the following *Slepian-Wolf* coding. Suppose Γ is the apriori bound known for $|x|$. Expand the sample of x by

$$\frac{x}{2 \cdot \Gamma} + \frac{1}{2} = \sum_{j=1}^{\infty} \frac{\beta_j}{2^j}, \quad \beta_j \in \{0, 1\}. \tag{17}$$

Then $\{\beta_1, \beta_2, \ldots, \beta_l\}$ is the Slepian-Wolf code of the sample of length l. In terms of the simple system (14), the code is divided into two blocks and each block is sent to a controller. Further assume $e^{a\Theta}$ approaches 4 but is smaller than 4, and $R_1 = R_2 = 1/\Theta$. Imitating the algorithm constructed in [6], one can reach the following stabilizing coder-controller design: (See Figure 2.)

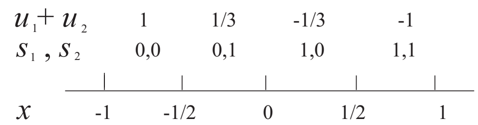

Fig. 2. Control coding for two sensor-actuator pair. Assume $\Gamma = 1$.

$$\begin{aligned}
h_1(x) &= \begin{cases} 0 & : x \le 0 \\ 1 & : x > 0 \end{cases} \\
h_2(x) &= \begin{cases} 0 & : x/\Gamma \in (-1, -0.5] \bigcup (0,\ 0.5] \\ 1 & : x/\Gamma \in (-0.5,\ 0] \bigcup (0.5,\ 1] \end{cases} \\
g_1(s) &= \begin{cases} 2\Gamma/3 & : s = 0 \\ -2\Gamma/3 & : s = 1 \end{cases} \\
g_2(s) &= \begin{cases} \Gamma/3 & : s = 0 \\ -\Gamma/3 & : s = 1 \end{cases}
\end{aligned} \tag{18}$$

Without sensor noise, the closed-loop system achieves asymptotic stability if Γ is updated properly. Recalling the approach in a slightly different context in [5], updating may occur as follows. Starting with $|x(0)| < \Gamma$, since the data-rate is higher than the minimum value, the state trajectory will be contained in an invariant domain $(-\beta\Gamma, \beta\Gamma)$, where $\beta < 1$ is a positive constant that only depends on how much the data-rate exceeds the minimum required value. Then Γ can be shrinked to $\beta\Gamma$. Repeat this process, Γ decreases exponentially. If Γ is not updated, then the magnitude of the system state will remain bounded by the a priori Γ. As is emphasized in [5], any control law that produces bounded response (such as the one given by (18)) can be rendered asymptotically stable by the addition of a low-bandwidth side channel. Thus our primary concern is whether a bounded response is possible in the case $\Gamma \equiv 1$.

The communication data-rate in this system is close to the theoretical minimum. However, for the decentralized control, we next show that small amount of noise may significantly increase the required data-rate!

Assume that $\Gamma \equiv 1$ is never updated. (Asymptotic stability is then not possible.) Consider the decentralized control with sensor noise. Assume $w_1(k)$ and $w_2(k)$ are uniformly distributed random noise in the neighborhood $(-\epsilon, \epsilon)$. The coder-controller uses (18). When operating near the data-rate limit, the trajectory of such a system is dense in $(-\Gamma, \Gamma)$ unless it leaves this region. Then, there is always a point in time when $-w_1(k) < x(k) < -w_2(k)$ (assuming $w_1(k) > w_2(k)$ at this instant). When such a situation happens, combining (14) and (18) gives

$$\begin{aligned} s_1(k) &= 1, & s_2(k) &= 1 \\ u_1(k) &= -2/3, & u_2(k) &= -1/3 \end{aligned} \tag{19}$$

Thus, the controllers collectively believes the state is in the interval coded by $(s_1(k), s_2(k)) = (1, 1)$, while the actual state is near the boundary of the intervals coded by $(s_1(k), s_2(k)) = (1, 0)$ and $(s_1(k), s_2(k)) = (0, 1)$. Consequently, the aggregated control action $u_1(k)+u_2(k) = -1$ will drive the state out of the interval $|x| < \Gamma = 1$. Since Γ is assumed a constant, the state will escape exponentially to infinity. To maintain stability by increasing data-rate, the data-rate must increase above twice the theoretical minimum (Θ reduces by half). Note that tiny asynchronism among decentralized sensors (possibly one analog sensor output observed by multiple digital processors) may cause similar problems.

Taking the above issue into account, the data-rate requirement for the linear systems under decentralized feedback control needs to be further examined.

For nonlinear plants that are unstable (possibly in a local sense), whether the closed-loop system can be stabilized depends on the total data-rate $R = R_1+R_2$ as well. Although the derivations of the minimum stabilizing data-rate can only be discussed case-by-case, the same robustness issues always exist and the discussion of these issues by the present paper is still applicable.

IV. Robustness condition and code design for decentralized sensor-actuator network

In the present paper, we are concerned with robustness of the decentralized, code-based control system with respect to noise in the following sense.

Definition 1: The decentralized system (11) is said to be robust to noise if for all $x \in \mathbb{R}$, there is an $\epsilon_1 > 0$, such that for all $0 < \epsilon < \epsilon_1$ and all $\| \mathbf{w} \|_\infty \leq \epsilon$,

$$G \circ H(x, \mathbf{w}) = f(x + \epsilon) \text{ or } f(x - \epsilon). \tag{20}$$

Remarks 1:
1) Since $f(\cdot)$ is a quantized control law and each quantization level has nonzero length, $f(x)$ equals either $f(x + \epsilon)$ or $f(x - \epsilon)$ when ϵ is small enough.
2) Recalling that $G \circ H(x, \mathbf{0}) = f(x)$, the above definition of robustness says that the difference in control value due to sensors' noise should depend only on the noise magnitude, not the specific pattern of the noise vector $\mathbf{w}$. In consequence, the decentralized system should have, qualitatively, the same robustness as a centralized system with the control law $f(\cdot)$.
3) If the the system is robust in the above sense, then infinitesimal noise will only lead to infinitesimal change of the critical data-rate. Otherwise, infinitesimal noise may lead to finite change of the critical data-rate. This is especially likely when $f(\cdot)$ is an efficient control design in terms of data-rate requirement. Section III has illustrated such situations.

Theorem 1: System (11) is robust in the sense of (20) if and only if for each $l = 1, \ldots, \lambda - 1$, the codewords (matrices) σ_l and σ_{l+1} differ by only one row. (The digits of the codewords are arranged by encoding units, see Section II.)

Proof: For necessity, suppose for a certain $l < \lambda$, σ_l and σ_{l+1} differ in both the $i$the and jth rows ($i \neq j$). I.e., $\sigma_{l,(i)} \neq \sigma_{l+1,(i)}$ and $\sigma_{l,(j)} \neq \sigma_{l+1,(j)}$. Since $l < \lambda$, X_l must have a common boundary point with X_{l+1}. Let the common boundary point be ξ. For each $\epsilon \in (0, \rho/2)$ (recall that ρ is the minimum length of X_l's), consider the state value $x \in (\xi - \epsilon, \xi + \epsilon)$, and the noise vector $\mathbf{w}$ in which $w_i = \epsilon,\ w_j = -\epsilon$. Then $x + w_i \in X_{l+1}$ and $x + w_j \in X_l$. Thus $H_{(i)}(x, \mathbf{w}) = \sigma_{l+1,(i)}$ and $H_{(j)}(x, \mathbf{w}) = \sigma_{l,(j)}$. (See (10).) Then $H(x, \mathbf{w}) \neq \sigma_l$ and $H(x, \mathbf{w}) \neq \sigma_{l+1}$. (The noise will cause the sensors to give contradicting readings.) Hence $G \circ H(x, \mathbf{w})$ does not equal either $f(x - \epsilon) = \mu_l$ or $f(x + \epsilon) = \mu_{l+1}$.

For sufficiency, for each $\epsilon \in (0, \rho/2)$ and any given state x, $x-\epsilon$ and $x+\epsilon$ are either in the same X_l, or in X_l and X_{l+1} respectively, for some l. The former case is trivial. For the latter, given $\mathbf{w}$ such that $\| \mathbf{w} \|_\infty < \epsilon$, $x+w_i \in (x-\epsilon, x+\epsilon) \in X_l \bigcup X_{l+1},\ i = 1, \ldots, M$. If σ_l and σ_{l+1} are only different by one row, say, the i^*th row, then $H_{(i)}(x, \mathbf{w}) = \sigma_{l,(i)} = \sigma_{l+1,(i)}$ except for $i = i^*$. Since $H_{(i^*)}(x, \mathbf{w})$ equals either $\sigma_{l,(i^*)}$ or $\sigma_{l+1,(i^*)}$, $H(x, \mathbf{w}) = \sigma_l$ or σ_{l+1} and $G \circ H(x, \mathbf{w}) = \mu_l$ or $\mu_{l+1} = f(x - \epsilon)$ or $f(x + \epsilon)$. ■

Codes that satisfy the condition in Theorem 1 already exist. Indeed, the famous *Gray code* is a family of such codes [2]. An α-ary Gray code of length n has α^n codewords that are arranged in a sequence in which adjacent codewords differ by only one digit. If the system (11) has only one actuator, then applying a q^m-ary Gray code, in which the digits generated by the same sensor are lumped into one q^m-ary letter of the Gray code solves the problem.

However, the Gray code cannot be decoded in a decentralized fashion in general. In terms of equations (6) (7) and (8), given a partition of the digits $\{\theta_r\}_{r=1,\ldots,P}$, there may not exist a function $G(\cdot)$ that decodes a α-ary, length n Gray code into α^n real numbers. Nevertheless, Gray code may still be used for the blocks of digits for $\mathcal{C}$. We will discuss this topic by some interesting special cases in the next section.

V. Special cases

A. Sensor-actuator pairs are mutually exlusive

Consider the case where each sensor sends all the digits to a unique actuator, and that actuator receives messages only from that sensor. Without lose of generality, assume $m = 1$. Omitting the indices whose range becomes a singleton, $\theta_r = \{r\}$.

Theorem 2: If $m = 1$ and $\theta_r = \{r\}$ for $r = 1, \ldots, M$ (then $P = M$), then there exists a code $\mathcal{C}$ such that the system (11) satisfies the robustness condition in Theorem 1 if and only if

$$(q - 1)M + 1 \geq \lambda. \tag{21}$$

Proof: Because of the one-to-one correspondence between the sensors and the actuators, we will index both the sensors and actuators by $i = 1, \ldots, M$.

Necessity: For the overall system to provide λ control values, the following set of λ equations must be satisfied:

$$\sum_{i=1}^{M} g_i(\sigma_l^{(i)}) \ = \ \mu_l, \quad l = 1, \ldots, \lambda. \tag{22}$$

Subtracting each of the last $\lambda - 1$ equations in (22) by its previous one produces

$$\sum_{i=1}^{M} (g_i(\sigma_{l+1}^{(i)}) - g_i(\sigma_l^{(i)})) = \mu_{l+1} - \mu_l, \quad l = 1, \ldots, \lambda - 1. \tag{23}$$

From the robustness condition provided by Theorem 1, there is a unique i for each l (denote by i_l) in (23) such that $\sigma_l^{(i)}$

and $\sigma_{l+1}^{(i)}$ are different. So,

$$\begin{aligned} g_{i_l}(\sigma_{l+1}^{(i_l)}) - g_{i_l}(\sigma_l^{(i_l)}) &= \mu_{l+1} - \mu_l, \\ g_i(\sigma_{l+1}^{(i)}) - g_i(\sigma_l^{(i)}) &= 0, \text{ for } i \neq i_l, \\ & l = 1, \ldots \lambda - 1. \end{aligned} \tag{24}$$

Recall that μ_{l+1} is always greater than μ_l. Then for all i,

$$g_i(\sigma_1^{(i)}) \leq g_i(\sigma_2^{(i)}) \leq \ldots \leq g_i(\sigma_\lambda^{(i)}). \tag{25}$$

Notice that this need not to be true if the robustness condition were not enforced!

From the hypothesis of this theorem, each message communicated within a sensor-actuator pair is a single q-ary digit, hence for each i, the set $\{\sigma_1^{(i)}, \ldots, \sigma_\lambda^{(i)}\}$ contains at most q distinct symbols. With these as the input, each $g_i(\cdot)$ can only produce q different values. Notice that this is because the actuation is decentralized, individual actuators do not have access to all the sensor readings.

Let $\gamma_{i,j} = g_i(\sigma_{j+1}^{(i)}) - g_i(\sigma_j^{(i)}),\ i \in \mathbb{Z}_M,\ j \in \mathbb{Z}_{\lambda-1}$. For each i, because of (25) and that $g_i(\cdot)$ can only produce q different values, $\gamma_{i,j}$ is nonzero for only $q-1$ values of j. Thus, $\gamma_{i,j}$'s form a M-by-$(\lambda-1)$ matrix in which each column has exactly one nonzero element and each row has at most $q-1$ nonzero elements. This matrix has at most $(q-1)M$ columns, thus λ is at most $(q-1)M+1$.

Sufficiency: Use the following code (let the q-ary letters be $0, 1, \ldots, q-1$)

$$\sigma_l^{(i)} = \begin{cases} 0 & : \quad l \leq (i-1)(q-1)+1, \\ q-1 & : \quad l \geq i(q-1)+1, \\ l-(i-1)(q-1)-1 & : \text{ otherwise.} \end{cases} \tag{26}$$

I.e.,

$$\begin{aligned} &[\ \sigma_1,\ \sigma_2,\ \sigma_3, \ldots, \sigma_q,\ \sigma_{q+1}, \ldots, \sigma_{2q}, \ldots, \sigma_{\lambda-q}, \ldots, \sigma_\lambda\ ] \\ &= \\ &\begin{bmatrix} 0 & 1 & 2 & \cdots & q & q & & q & & q & & q \\ 0 & 0 & 0 & & 0 & 1 & & q & & q & & q \\ 0 & 0 & 0 & & 0 & 0 & & 0 & & q & & q \\ \vdots, & \vdots, & \vdots & ,\ldots, & \vdots, & \vdots & ,\ldots, & \vdots & ,\ldots, & \vdots & ,\ldots, & \vdots \\ 0 & 0 & 0 & & 0 & 0 & & 0 & & q & & q \\ 0 & 0 & 0 & & 0 & 0 & & 0 & & 0 & & q \end{bmatrix} \end{aligned} \tag{27}$$

Since each sensor only generates one digit, here we omit the corresponding indices and write $s_i = h_i(x)$ for the message from the ith sensor. Then, let the ith sensor send out $s_i = h_i(x) = \sigma_l^{(i)}$ if $x \in X_l$, and design the ith actuator to produce the following output when the message s_i reads $\sigma_l^{(i)}$.

$$g_i(\sigma_1^{(i)}) = \mu_1/M,$$

and for $l > 1$,

$$g_i(\sigma_l^{(i)}) = \begin{cases} g_i(\sigma_{l-1}^{(i)}) & : \sigma_l^{(i)} = \sigma_{l-1}^{(i)} \\ g_i(\sigma_{l-1}^{(i)}) + (\mu_l - \mu_{l-1}) & : \text{otherwise,} \end{cases} \tag{28}$$

The above design proves the sufficiency. ■

Note that the code design that proves the above theorem is not unique. However, for all such codes, if one browse through the code alphabet, the change of each digit from one codeword to the next follows a unique sequence. For instance, if the first digits of $\sigma_1, \ldots, \sigma_q$ are $1, \ldots, q$, then the first digit of all the other codewords must remain q. For this reason, we call this a *unique sequence code*.

Example 1 (Complete Decentralization): Assume that $q = 2,\ m = 1,\ P = M$ and $\theta_r = (r, 1)$, i.e., the ith sensor only sends out one binary digit to the ith actuator. We call this case "complete decentralization" because every single digit of the feedback is generated in a decentralized fashion.

According to Theorem 2, the largest feasible value for λ equals $M+1$. The following unique sequence code can be applied to the decentralized control system as shown in the proof of Theorem 2.

$$\begin{aligned} &[\ \sigma_1,\ \sigma_2,\ \sigma_3, \ldots, \sigma_{M+1}] \\ &= \\ &\begin{bmatrix} 0 & 1 & 1 & & 1 \\ 0 & 0 & 1 & & 1 \\ \vdots, & \vdots, & \vdots & ,\ldots, & \vdots \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & & 1 \end{bmatrix} \end{aligned} \tag{29}$$

B. Each sensor communicates to a unique actuator

In this case, each sensor sends all the digits it generates to a single actuator, but each actuator receives messages from p sensors, $p > 1$. (See Figure 3, (a).)

Without loss of generality, assume that each sensor only generates one digit, i.e., $m = 1$. Here, θ_r's partition the set of sensor indices, hence the group of sensors, into P disjoint sets, each having p elements. (Assume M/P is an integer.) Within each of such sensor groups, the Gray code can be applied to the p digits generated, since all these digits are read by a single actuator. Then, treat each of these blocks of digits as a q^p-ary digit. Apply the unique sequence code.

To check whether such a coding satisfies Theorem 1: By applying the unique sequence code, adjacent codewords in $\mathcal{C}$ are only different by one q^p-ary digit (the lumped digit that comprises a block of digits as partitioned by θ_r's). Further, because the change of each digit of the unique sequence code follows an order, we can assume this order coincides with the order in the Gray code which is applied to the block of q-ary digits that comprised by this q^p-ary digit. Then, only one digit in this block changes. Thus the condition in Theorem 1 is satisfied and the decentralized control system with the feedback coded as such is robust in the sense of (20).

One can show that the above feedback coding is the most efficient feedback coding scheme for the decentralized system to be robust, and it satisfies

$$\lambda = (q^p - 1)P + 1 = (q^p - 1)M/p + 1. \tag{30}$$

Notice that both here and in the previous case, each sensor is assumed to communicate one q-ary digit, and thus the code $\mathcal{C}$ is of length M. Then comparing (30) and (21) shows that with the same amount of communication, more control actions can be encoded here than in the previous case. This is because the actuation is somewhat more centralized here.

C. Each actuator receives messages from a unique sensor

In this case, assume each actuator only receives a single digit, but each sensor sends out messages to m actuators, $m = P/M > 1$ is an integer. See Figure 3, (b).

If the intended control values (μ_l's) are evenly spaced, then with a little modification, the Slepian-Wolf coding adopted in

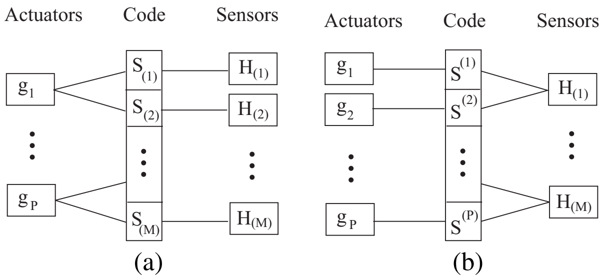

Fig. 3. (a) Each sensor communicates to a unique actuator. (b) Each actuator receives messages from a unique sensor.

[6] can be applied to each digit block $S_{(i)}$, which is generated by the ith sensor alone. Treating each $S_{(i)}$ as a q^m-ary digit, apply the unique sequence code to get $\mathcal{C}$. This is also the most efficient code construction that makes the decentralized control system robust in the sense of (20), and

$$\lambda = (q^m - 1)M + 1 = (q^m - 1)P/m + 1. \tag{31}$$

Compare to the previous cases. Here, each actuator receives one q-ary digit, and thus $\mathcal{C}$ is of length P. (Here P plays the same role as M in the previous case.) With the same amount of communication, the number of control actions encoded here is the same as that of Case V-B if $m = p$, and is more than that of Case V-A.

The decentralized sensor-actuator network may take many other topologies, some of which may allow more efficient encoding of control actions (more control actions with the same amount of communication). This is an active subject of our current research.

We close this section by revisiting the communication data-rate requirement for the decentralized systems. For stabilizing the unstable linear plant (13) with the decentralized feedback loop (11), the necessary and sufficient condition is

$$\lambda > e^{a\Theta}. \tag{32}$$

Then, for Case V-A, the condition becomes

$$(q-1)M + 1 > e^{a\Theta} \;\Rightarrow$$
$$\frac{\log_2((q-1)M+1)}{\Theta} > a \cdot \log_2 e. \tag{33}$$

As stated in (12), noticing $m = 1$ in this case, the overall data-rate of the sensor-actuator network is

$$R = \frac{M \cdot \log_2 q}{\Theta}. \tag{34}$$

Combining (33) and (34),

$$R > M \cdot \frac{\log_2 q}{\log_2((q-1)M+1)} \cdot a \cdot \log_2 e. \tag{35}$$

Clearly, this is much higher than $a \cdot \log_2 e$, the data-rate requirement for the centralized control structure. However, (35) implies that the data-rate requirement for each sensor-actuator pair is

$$\frac{\log_2 q}{\log_2((q-1)M+1)} \cdot a \cdot \log_2 e,$$

which is smaller than $a \cdot \log_2 e$. Recall that one of the main reasons for using decentralized control structure is to stabilize unstable plants using feedback control provided by multiple sensor-actuator pairs, each of which does not possess enough communication capacity to stabilize the plant alone. This goal is achieved. Similar arguments apply to cases V-B and V-C.

VI. Conclusion

In decentralized systems, the degradation of information due to noise and asynchronism can be more severe than in centralized systems. In this paper, we discussed the conditions for communication designs in decentralized sensor-actuator network that are robust with respect to sensor noise and asynchronism. Specific designs that are robust were proposed for particular cases. The implication of such designs in terms of data-rate requirements for the whole sensor-actuator network and for individual sensor-actuator pairs is also examined.

For decentralized control systems with the same number of sensor-actuator pairs and each with the same communication capacity, some interconnection topologies may allow the encoding of richer sets of control actions than others. In the future, it would be interesting to understand the extent to which the interconnection topology affects the highest possible efficiency of decentralized control coding/decoding.

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