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# Direct Adaptive Command Following and Disturbance Rejection for Minimum Phase Systems with Unknown Relative Degree

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***Abstract*— This paper considers parameter-monotonic direct adaptive command following and disturbance rejection for single-input, single-output minimum phase linear time-invariant systems with knowledge of the sign of the high-frequency gain and an upper bound on the magnitude of the high-frequency gain. We assume that the command and disturbance signals are generated by a linear system with known spectrum. Furthermore, we assume that the command signal is measured, but the disturbance signal is unmeasured.**

## 1. Introduction

Parameter-monotonic adaptive stabilization methods use simple adaptation laws and rely on a minimum phase assumption to attract poles to zeros under high gain [1–4]. Adaptive high-gain proportional feedback can stabilize square multi-input, multi-output systems that are minimum phase and relative degree one with known sign of the high-frequency gain [1]. This approach was extended to include systems where the sign of the high-frequency gain is unknown [5].

Generally, high-gain methods can stabilize systems with relative degree one. However, in [2], high-gain dynamic compensation is used to guarantee output convergence of single-input, single-output minimum phase systems with arbitrary-but-known relative degree. This result is surprising since classical roots locus is not high-gain stable for plants with relative degree exceeding two. However, in [4] it is shown that the results of [2] can fail when the relative degree of the plant exceeds four. Furthermore, in [4], the Fibonacci series is used to construct a direct adaptive stabilization algorithm for minimum phase systems with unknown-but-bounded relative degree.

In the present paper, we extend the Fibonacci-based adaptive stabilization controller presented in [4] to address the adaptive command following and disturbance rejection problems. We assume that the command and disturbance signals are generated by a linear system with known spectrum. However, the disturbance is unmeasured. Unlike direct model reference adaptive controllers, this adaptive controller does not require a bound on plant order or knowledge of the relative degree. Additionally, the method presented in this paper simultaneously addresses the command following and disturbance rejection problem, whereas model reference adaptive control is generally restricted to the command following problem.

## 2. Command Following and Disturbance Rejection

We consider the strictly proper single-input single-output linear time-invariant system

$$y = G(s)\,(u+w)\,, \quad G(s) \triangleq \delta\beta\frac{z(s)}{p(s)}, \tag{2.1}$$

where $z(s)$ and $p(s)$ are real monic polynomials, $\delta = \pm 1$ is the sign of the high-frequency gain, and $\beta > 0$ is the magnitude of the high-frequency gain. Define the notation

$$m \triangleq \deg\, z(s), \qquad n \triangleq \deg\, p(s), \qquad r \triangleq n - m. \tag{2.2}$$

Furthermore, we consider a command signal $y_{\rm r}(t)$ and a disturbance signal $w(t)$ that satisfy the exogenous dynamics

$$\dot{x}_{\rm r}(t) = A_{\rm r}x_{\rm r}(t), \;\; u_{\rm r}(t) = C_{\rm r}x_{\rm r}(t), \tag{2.3}$$

where $u_{\rm r}(t) \triangleq \begin{bmatrix} y_{\rm r}(t) \\ w(t) \end{bmatrix}$, $A_{\rm r} \in \mathbb{R}^{n_{\rm r}\times n_{\rm r}}$, $C_{\rm r} \in \mathbb{R}^{2\times n_{\rm r}}$, $(A_{\rm r}, C_{\rm r})$ is observable, and the characteristic polynomial of $A_{\rm r}$ is given by $p_{\rm r}(s)$ The eigenvalues of $A_{\rm r}$ are denoted by $\lambda_1, \ldots, \lambda_{n_{\rm r}}$. We assume that the eigenvalues of $A_{\rm r}$ are semisimple and on the imaginary axis, that is, for all $i = 1, \ldots, n_{\rm r}$, Re $\lambda_i = 0$. This assumption restricts our attention to command and disturbance signals that consist of steps and sinusoids.

In this paper, we address the adaptive command following and disturbance rejection problem for the system (2.1). The objective is to construct an adaptive controller that forces the plant output $y$ to asymptotically follow the command signal $y_{\rm r}$ while rejecting the unmeasured disturbance $w$. We make the following assumptions.

- (A1) $z(s)$ is a real monic Hurwitz polynomial but is otherwise unknown.
- (A2) $p(s)$ is a real monic polynomial but is otherwise unknown.
- (A3) $z(s)$ and $p(s)$ are coprime.
- (A4) The magnitude $\beta$ of the high-frequency gain satisfies $0 < \beta \le b_0$, where $b_0 \in \mathbb{R}$ is known.
- (A5) The sign $\delta = \pm 1$ of the high-frequency gain is known.
- (A6) The relative degree $r$ of $G(s)$ satisfies $0 < r \le \rho$, where $\rho$ is known, but $r$ is otherwise unknown.
- (A7) For all $\lambda \in {\rm spec}(A_{\rm r})$, Re $\lambda = 0$ and $\lambda$ is semisimple.
- (A8) The command signal $y_{\rm r}$ is measured, but the disturbance signal $w$ is unmeasured.
- (A9) The spectrum of the exogenous dynamics is known, that is, $p_{\rm r}(s)$ is known.

Next, we introduce parameter-dependent polynomials, transfer functions, and dynamic compensators. Let $c_k(s)$ and $d_k(s)$ be parameter-dependent polynomials, that is, polynomials in $s$ over the reals whose coefficients are functions of a parameter $k$. Furthermore, define the parameter-dependent transfer function $H_k(s) \triangleq \frac{c_k(s)}{d_k(s)}$. The polynomials $c_k(s)$ and $d_k(s)$ need not be coprime for all $k \in \mathbb{R}$.

**Definition 2.1.** *The parameter-dependent polynomial $d_k(s)$ is* high-gain Hurwitz *if there exists $k_{\rm s} > 0$ such that $d_k(s)$ is Hurwitz for all $k \ge k_{\rm s}$.*

**Definition 2.2.** *The parameter-dependent transfer function $H_k(s)$ is* high-gain stable *if, for all $k \in \mathbb{R}$, $H_k(s)$ can be expressed as the ratio of parameter-dependent polynomials $c_k(s)$ and $d_k(s)$, where the denominator polynomial $d_k(s)$ is high-gain Hurwitz.*

Now, consider the feedback controller

$$u = \hat{G}_k(s)y_e, \tag{2.4}$$

with the parameter-dependent dynamic compensator

$$\hat{G}_k(s) \triangleq \frac{\hat{z}_k(s)}{\hat{p}_k(s)}, \tag{2.5}$$

where the output error is $y_e \triangleq y_{\rm r} - y$. The polynomials $\hat{z}_k(s)$ and $\hat{p}_k(s)$ in $s$ over the reals are also functions of a scalar parameter $k$. For example, letting $\hat{z}_k(s) = \delta k$ and $\hat{p}_k(s) = 1$ yields $\hat{G}_k(s) = \delta k$, and the closed-loop poles can be determined by classical root locus.

The single-input, single-output command following and disturbance rejection problem is shown in Figure 1. The closed-loop

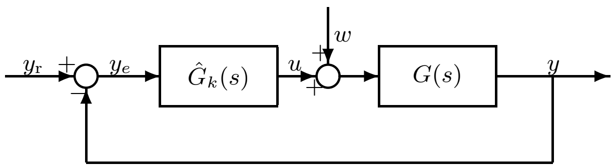


Fig. 1. Combined command following and disturbance rejection problem.

system (2.1) and (2.4)-(2.5) from the command $y_{\rm r}(t)$ and the disturbance $w(t)$ to the tracking error $y_e(t)$ is

$$y_e = \tilde{G}_k(s)u_{\rm r} = \left[\ \tilde{G}_{k,1}(s) \quad \tilde{G}_{k,2}(s)\ \right] \left[\begin{array}{c} y_{\rm r} \\ w \end{array}\right], \tag{2.6}$$

where

$$\tilde{G}_{k,1}(s) \triangleq \frac{1}{1+G(s)\hat{G}_k(s)} = \frac{\tilde{z}_{k,1}(s)}{\tilde{p}_k(s)}, \tag{2.7}$$

$$\tilde{G}_{k,2}(s) \triangleq \frac{-G(s)}{1+G(s)\hat{G}_k(s)} = \frac{\tilde{z}_{k,2}(s)}{\tilde{p}_k(s)}, \tag{2.8}$$

and

$$\tilde{z}_{k,1}(s) \triangleq p(s)\hat{p}_k(s), \tag{2.9}$$

$$\tilde{z}_{k,2}(s) \triangleq -\delta\beta z(s)\hat{p}_k(s), \tag{2.10}$$

$$\tilde{p}_k(s) \triangleq p(s)\hat{p}_k(s) + \delta\beta z(s)\hat{z}_k(s). \tag{2.11}$$

## 3. High-Gain Dynamic Compensation for Stabilization

In this section, a parameter-dependent dynamic compensator is used to high-gain stabilize (2.1). The controller construction utilizes the Fibonacci series. For all $j \geq 0$ let $F_j$ be the $j$th Fibonacci number, where $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, \ldots$. Define $f_{g,h} \triangleq F_{g+2} - F_{h+1}$, where $h$ satisfies $1 \leq h \leq g$.

Consider the parameter-dependent dynamic compensator

$$\hat{G}_{k,g}(s) \triangleq \frac{\delta k^{F_{g+2}}\hat{z}(s)}{s^g + k^{f_{g,g}}b_g s^{g-1} + \cdots + k^{f_{g,2}}b_2 s + k^{f_{g,1}}b_1}, \tag{3.1}$$

where $k \in \mathbb{R}$, $b_1, \ldots, b_g$ are real numbers, and $\hat{z}(s)$ is a degree $g-1$ monic polynomial.

Now, let $g$ be the upper bound on the relative degree of $G(s)$, that is, $g = \rho$. Let $\hat{G}_{k,\rho}(s)$ denote $\hat{G}_{k,g}(s)$ with $g = \rho$, and consider the feedback (2.4) with $\hat{G}_k(s) = \hat{G}_{k,\rho}(s)$. Then the closed-loop system (2.1), (2.4), and (3.1) is (2.6)-(2.8) where

$$\tilde{z}_{k,1}(s) \triangleq p(s)\left[s^\rho + k^{f_{\rho,\rho}}b_\rho s^{\rho-1} + k^{f_{\rho,\rho-1}}b_{\rho-1}s^{\rho-2} + \cdots + k^{f_{\rho,2}}b_2 s + k^{f_{\rho,1}}b_1\right], \tag{3.2}$$

$$\tilde{z}_{k,2}(s) \triangleq -\delta\beta z(s)\left[s^\rho + k^{f_{\rho,\rho}}b_\rho s^{\rho-1} + k^{f_{\rho,\rho-1}}b_{\rho-1}s^{\rho-2} + \cdots + k^{f_{\rho,2}}b_2 s + k^{f_{\rho,1}}b_1\right], \tag{3.3}$$

$$\tilde{p}_k(s) \triangleq p(s)s^\rho + k^{f_{\rho,\rho}}b_\rho p(s)s^{\rho-1} + k^{f_{\rho,\rho-1}}b_{\rho-1}p(s)s^{\rho-2} + \cdots + k^{f_{\rho,1}}b_1 p(s) + k^{F_{\rho+2}}\beta z(s)\hat{z}(s). \tag{3.4}$$

The following theorem provides the properties of $\tilde{p}_k(s)$ and thus $\tilde{G}_{k,1}(s)$ and $\tilde{G}_{k,2}(s)$ for sufficiently large $k$. The proof follows from examining the Hurwitz conditions of $\tilde{p}_k(s)$ for large $k$. For a complete proof of this result, see [4].

**Theorem 3.1.** *Consider the closed-loop system (2.6)-(2.8) and (3.2)-(3.4). Assume that the polynomials $\hat{z}(s)$, $B_{\rho-2}(s) \triangleq s^3 + b_\rho s^2 + b_{\rho-1}s + b_0$, and, for $i = 0, 1, \ldots, \rho-3$, $B_i(s) \triangleq b_{i+3}s^3 + b_{i+2}s^2 + b_{i+1}s + b_0$ are Hurwitz. Then $\tilde{p}_k(s)$ is high-gain Hurwitz and thus $\tilde{G}_{k,1}(s)$ and $\tilde{G}_{k,2}(s)$ are high-gain stable. Furthermore, as $k \to \infty$, $m + \rho - 1$ roots of $\tilde{p}_k(s)$ converge to the roots of $z(s)\hat{z}(s)$ and the real parts of the remaining $r+1$ roots approach $-\infty$.*

The parameter-dependent dynamic compensator $\hat{G}_{k,\rho}(s)$ is high-gain stabilizing for $G(s)$ under assumptions (A1)-(A6). However, the closed-loop system is not guaranteed to asymptotically follow the command signal or reject the disturbance. In fact, the closed-loop system will not generally follow the command signal or reject the disturbance since $\hat{G}_{k,\rho}(s)$ does not have an internal model of $p_{\rm r}(s)$ for all values of $k$. However, in the next section, we augment $\hat{G}_{k,\rho}(s)$ to incorporate an internal model of $p_{\rm r}(s)$.

## 4. High-Gain Dynamic Compensation for Command Following and Disturbance Rejection

In this section, we construct a high-gain dynamic compensator for command following and disturbance rejection by cascading an internal model of the exogenous dynamics $p_{\rm r}(s)$ with $\hat{G}_{k,g}(s)$, where the parameter $g$ is chosen to be an upper bound on the relative degree of an augmented system.

Consider the feedback (2.4) with the strictly proper dynamic compensator $\hat{G}_k(s) \triangleq \hat{G}_{\rm r}(s)\hat{G}_{k,\bar{\rho}}(s)$, where $\hat{G}_{\rm r}(s) \triangleq \frac{\hat{z}_{\rm r}(s)}{p_{\rm r}(s)}$, $\hat{z}_{\rm r}(s)$ is a monic polynomial with $m_{\rm r} \triangleq \deg \hat{z}_{\rm r}(s) \leq n_{\rm r}$, and $\hat{G}_{k,\bar{\rho}}(s)$ is given by (3.1) with $g = \bar{\rho}$, where $\bar{\rho} \triangleq \rho + n_{\rm r} - m_{\rm r}$. Note that $\bar{\rho}$ is an upper bound on the relative degree of the cascaded system $G(s)\hat{G}_{\rm r}(s)$. Therefore, the parameter-dependent dynamic compensator is

$$\hat{G}_k(s) = \frac{\delta k^{F_{\bar{\rho}+2}}\hat{z}_{\rm r}(s)\hat{z}(s)}{p_{\rm r}(s)\left[s^{\bar{\rho}} + k^{f_{\bar{\rho},\bar{\rho}}}b_{\bar{\rho}}s^{\bar{\rho}-1} + \cdots + k^{f_{\bar{\rho},2}}b_2 s + k^{f_{\bar{\rho},1}}b_1\right]}, \tag{4.1}$$

where $k \in \mathbb{R}$, $b_1, \ldots, b_{\bar{\rho}}$ are real numbers, and $\hat{z}(s)$ is a degree $\bar{\rho}-1$ monic polynomial. Then the closed-loop system (2.1), (2.4), and (4.1) is (2.6)-(2.8) where

$$\tilde{z}_{k,1}(s) \triangleq p_{\rm r}(s)p(s)\left[s^{\bar{\rho}} + k^{f_{\bar{\rho},\bar{\rho}}}b_{\bar{\rho}}s^{\bar{\rho}-1} + k^{f_{\bar{\rho},\bar{\rho}-1}}b_{\bar{\rho}-1}s^{\bar{\rho}-2} + \cdots + k^{f_{\bar{\rho},2}}b_2 s + k^{f_{\bar{\rho},1}}b_1\right], \tag{4.2}$$

$$\tilde{z}_{k,2}(s) \triangleq -\delta\beta p_{\rm r}(s)z(s)\left[s^{\bar{\rho}} + k^{f_{\bar{\rho},\bar{\rho}}}b_{\bar{\rho}}s^{\bar{\rho}-1} + k^{f_{\bar{\rho},\bar{\rho}-1}}b_{\bar{\rho}-1}s^{\bar{\rho}-2} + \cdots + k^{f_{\bar{\rho},2}}b_2 s + k^{f_{\bar{\rho},1}}b_1\right], \tag{4.3}$$

$$\tilde{p}_k(s) \triangleq p_{\rm r}(s)p(s)s^{\bar{\rho}} + k^{f_{\bar{\rho},\bar{\rho}}}b_{\bar{\rho}}p_{\rm r}(s)p(s)s^{\bar{\rho}-1} + \cdots + k^{f_{\bar{\rho},2}}b_2 p_{\rm r}(s)p(s)s + k^{f_{\bar{\rho},1}}b_1 p_{\rm r}(s)p(s) + k^{F_{\bar{\rho}+2}}\beta z(s)\hat{z}_{\rm r}(s)\hat{z}(s). \tag{4.4}$$

**Theorem 4.1.** *Consider the closed-loop system (2.6)-(2.8) and (4.2)-(4.4). Assume that the dynamic compensators $\hat{G}_{\rm r}(s)$ and*

$\hat{G}_{k,\bar{\rho}}(s)$ *are minimum phase, that is, assume that the polynomials* $\hat{z}(s)$ *and* $\hat{z}_{\rm r}(s)$ *are Hurwitz. Furthermore, assume that the polynomials*

$$B_{\bar{\rho}-2}(s) \triangleq s^3 + b_{\bar{\rho}}s^2 + b_{\bar{\rho}-1}s + b_0, \tag{4.5}$$

*and, for* $i = 0, 1, \ldots, \bar{\rho} - 3$,

$$B_i(s) \triangleq b_{i+3}s^3 + b_{i+2}s^2 + b_{i+1}s + b_0, \tag{4.6}$$

*are Hurwitz. Then the following statements hold.*

*(i)* $\tilde{p}_k(s)$ *is high-gain Hurwitz and thus* $\tilde{G}_{k,1}(s)$ *and* $\tilde{G}_{k,2}(s)$ *are high-gain stable.*

*(ii) As* $k \to \infty$, $m + m_{\rm r} + \bar{\rho} - 1$ *roots of* $\tilde{p}_k(s)$ *converge to the roots of* $z(s)\hat{z}_{\rm r}(s)\hat{z}(s)$ *and the real parts of the remaining* $r + n_{\rm r} - m_{\rm r} + 1$ *roots approach* $-\infty$.

*(iii) There exists* $k_{\rm s} > 0$ *such that, for all* $k \geq k_{\rm s}$, $\lim_{t\to\infty} y_e(t) = 0$.

*Proof.* Statements *(i)* and *(ii)* follow from applying Theorem 3.1 to the cascade $G(s)\hat{G}_{\rm r}(s)$. Specifically, define $\bar{G}(s) \triangleq G(s)\hat{G}_{\rm r}(s)$. Since $\hat{z}_{\rm r}(s)$ is Hurwitz, it follows that $\bar{G}(s)$ satisfies assumptions (A1)-(A6) where $\bar{\rho}$ is an upper bound on the relative degree of $\bar{G}(s)$. Furthermore, $\tilde{p}_k(s)$ is the closed-loop parameter-dependent characteristic polynomial of $\bar{G}(s)$ connected in feedback with the controller $\hat{G}_{k,\bar{\rho}}(s)$. Then according to Theorem 3.1, $\tilde{p}_k(s)$ is high-gain Hurwitz, and, as $k \to \infty$, $m + m_{\rm r} + \bar{\rho} - 1$ roots of $\tilde{p}_k(s)$ converge to the roots of $z(s)\hat{z}_{\rm r}(s)\hat{z}(s)$ and the real parts of the remaining $r + n_{\rm r} - m_{\rm r} + 1$ roots approach $-\infty$.

Now, we show part *(iii)*. Define $\hat{p}_k(s) \triangleq s^{\bar{\rho}} + k^{f_{\bar{\rho},\bar{\rho}}}b_{\bar{\rho}}s^{\bar{\rho}-1} + \cdots + k^{f_{\bar{\rho},1}}b_1$. Letting $\mathcal{L}(\cdot)$ denote the Laplace operator, the final value theorem implies

$$\begin{aligned}
\lim_{t\to\infty} y_e(t) &= \lim_{s\to 0} s\mathcal{L}(y_e(t)) \\
&= \lim_{s\to 0} s \left[ \begin{array}{cc} \tilde{G}_{k,1}(s) & \tilde{G}_{k,1}(s) \end{array} \right] \left[ \begin{array}{c} \mathcal{L}(y_{\rm r}(t)) \\ \mathcal{L}(w(t)) \end{array} \right] \\
&= \lim_{s\to 0} s \frac{p_{\rm r}(s)p(s)\hat{p}_k(s)}{\tilde{p}_k(s)} \frac{z_{\rm r}(s)}{p_{\rm r}(s)} \\
&\quad + \lim_{s\to 0} s \frac{-\delta\beta p_{\rm r}(s)z(s)\hat{p}_k(s)}{\tilde{p}_k(s)} \frac{z_w(s)}{p_{\rm r}(s)} \\
&= \lim_{s\to 0} s \frac{[p(s)z_{\rm r}(s) - \delta\beta z(s)z_w(s)]\,\hat{p}_k(s)}{\tilde{p}_k(s)},
\end{aligned} \tag{4.7}$$

where $\mathcal{L}(y_{\rm r}(t)) = \frac{z_{\rm r}(s)}{p_{\rm r}(s)}$, $\mathcal{L}(w(t)) = \frac{z_w(s)}{p_{\rm r}(s)}$, and $z_{\rm r}(s)$ and $z_w(s)$ are polynomials. Since $\tilde{p}_k(s)$ is high-gain Hurwitz, there exists $k_{\rm s} > 0$ such that, for all $k \geq k_{\rm s}$, $\tilde{p}_k(s)$ is Hurwitz. Then (4.7) implies, for all $k \geq k_{\rm s}$, $\lim_{t\to\infty} y_e(t) = 0$. □

## 5. Parameter-Monotonic Adaptive Command Following and Disturbance Rejection

Although Theorem 4.1 guarantees the existence of a strictly proper parameter-dependent dynamic compensator (4.1) for asymptotic command following and disturbance rejection, the stabilizing threshold $k_{\rm s}$ is unknown. In this section, we introduce a parameter-monotonic adaptive law for the parameter $k$ and present our main result. First, we construct state space realizations for the open-loop system (2.1) and the compensator (2.4) and (4.1). Let the system (2.1) have the minimal state space realization

$$\dot{x} = Ax + B\,(u + w)\,, \quad y = Cx, \tag{5.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$.

Next, consider the parameter-dependent dynamic compensator $\hat{G}_k(s) = \hat{G}_{\rm r}(s)\hat{G}_{k,\bar{\rho}}(s)$ given by (2.4) and (4.1) and write $\hat{z}(s) = s^{\bar{\rho}-1} + \hat{z}_{\bar{\rho}-2}s^{\bar{\rho}-2} + \cdots + \hat{z}_1 s + \hat{z}_0$, so that $\hat{G}_k(s)$ has the state space realization

$$\dot{\hat{x}} = \hat{A}(k)\hat{x} + \hat{B}y_e, \quad u = \hat{C}(k)\hat{x}, \tag{5.2}$$

where $\hat{A}(k) \in \mathbb{R}^{(n_{\rm r}+\bar{\rho})\times(n_{\rm r}+\bar{\rho})}$, $\hat{B} \in \mathbb{R}^{(n_{\rm r}+\bar{\rho})\times 1}$, and $\hat{C} \in \mathbb{R}^{1\times(n_{\rm r}+\bar{\rho})}$ are given by

$$\hat{A}(k) \triangleq \left[ \begin{array}{cc} \hat{A}_{\rm r} & \hat{B}_{\rm r}\hat{C}_{\bar{\rho}}(k) \\ 0 & \hat{A}_{\bar{\rho}}(k) \end{array} \right], \quad \hat{B} \triangleq \left[ \begin{array}{c} 0 \\ \hat{B}_{\bar{\rho}} \end{array} \right], \tag{5.3}$$

$$\hat{C}(k) \triangleq \left[ \begin{array}{cc} \hat{C}_{\rm r} & \hat{D}_{\rm r}\hat{C}_{\bar{\rho}}(k) \end{array} \right], \tag{5.4}$$

where

$$\hat{A}_{\bar{\rho}}(k) \triangleq \left[ \begin{array}{cccc} -k^{f_{\bar{\rho},\bar{\rho}}}b_{\bar{\rho}} & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ -k^{f_{\bar{\rho},2}}b_2 & 0 & & 1 \\ -k^{f_{\bar{\rho},1}}b_1 & 0 & \cdots & 0 \end{array} \right], \quad \hat{B}_{\bar{\rho}} \triangleq \left[ \begin{array}{c} 1 \\ \hat{z}_{\bar{\rho}-2} \\ \vdots \\ \hat{z}_0 \end{array} \right], \tag{5.5}$$

$$\hat{C}_{\bar{\rho}}(k) \triangleq \left[ \begin{array}{cccc} \delta k^{F_{\bar{\rho}+2}} & 0 & \cdots & 0 \end{array} \right] \tag{5.6}$$

is a realization of $\hat{G}_{k,\bar{\rho}}(s)$ and $(\hat{A}_{\rm r}, \hat{B}_{\rm r}, \hat{C}_{\rm r}, \hat{D}_{\rm r})$ is a minimal realization of $\hat{G}_{\rm r}(s)$. Note that, for all nonzero $k \in \mathbb{R}$, $\left(\hat{A}_{\bar{\rho}}(k), \hat{C}_{\bar{\rho}}(k)\right)$ is observable. The closed-loop system (5.1) and (5.2)-(5.6) is

$$\dot{\tilde{x}} = \tilde{A}(k)\tilde{x} + \tilde{B}u_{\rm r}, \quad y_e = \tilde{C}\tilde{x} + \tilde{D}u_{\rm r}, \tag{5.7}$$

where $\tilde{x} \triangleq \left[ \begin{array}{c} x \\ \hat{x} \end{array} \right]$, $u_{\rm r} \triangleq \left[ \begin{array}{c} y_{\rm r} \\ w \end{array} \right]$,

$$\tilde{A}(k) \triangleq \left[ \begin{array}{cc} A & B\hat{C}(k) \\ -\hat{B}C & \hat{A}(k) \end{array} \right], \qquad \tilde{B} \triangleq \left[ \begin{array}{cc} 0 & B \\ \hat{B} & 0 \end{array} \right], \tag{5.8}$$

$$\tilde{C} \triangleq \left[ \begin{array}{cc} -C & 0 \end{array} \right], \qquad \tilde{D} \triangleq \left[ \begin{array}{cc} 1 & 0 \end{array} \right]. \tag{5.9}$$

Now we present the main result of this paper, namely direct adaptive command following and disturbance rejection for minimum phase systems with unknown-but-bounded relative degree.

**Theorem 5.1.** *Consider the closed-loop system (5.7)-(5.9) consisting of the open-loop system (5.1) with unknown relative degree* $r$ *satisfying* $0 < r \leq \rho$, *and the feedback controller (5.2)-(5.6). Furthermore, consider the parameter-monotonic adaptive law*

$$\dot{k}(t) = \gamma e^{-\alpha k(t)}y_e^2(t), \tag{5.10}$$

*where* $\gamma > 0$ *and* $\alpha > 0$. *Assume that the dynamic compensators* $\hat{G}_{\rm r}(s)$ *and* $\hat{G}_{k,\bar{\rho}}(s)$ *are minimum phase, that is, assume that the polynomials* $\hat{z}(s)$ *and* $\hat{z}_{\rm r}(s)$ *are Hurwitz. Furthermore, assume that the polynomials* $B_0(s), \ldots, B_{\bar{\rho}-2}(s)$ *given by (4.5)-(4.6) are Hurwitz. Then, for all initial conditions* $\tilde{x}(0)$ *and* $k(0) > 0$, $k(t)$ *converges and* $\lim_{t\to\infty} y_e(t) = 0$.

*Proof.* The closed-loop system (5.7)-(5.9) with the inputs $y_{\rm r}$ and $w$ generated by the linear system (2.3) can be written as

$$\dot{x}_c(t) = A_c(k)x_c(t), \tag{5.11}$$

$$y_e(t) = C_c x_c(t), \tag{5.12}$$

where $x_c(t) \triangleq \left[ \begin{array}{c} \tilde{x}(t) \\ x_{\rm r}(t) \end{array} \right]$,

$$A_c(k) \triangleq \left[ \begin{array}{cc} \tilde{A}(k) & \tilde{B}C_{\rm r} \\ 0 & A_{\rm r} \end{array} \right], \quad C_c \triangleq \left[ \begin{array}{cc} \tilde{C} & \tilde{D}C_{\rm r} \end{array} \right]. \tag{5.13}$$

We first show that $k(t)$ converges. Theorem 4.1 implies that there exists $k_\mathrm{s} > 0$, such that for all $k \geq k_\mathrm{s}$, $\tilde{A}(k)$ is asymptotically stable and $\lim_{t\to\infty} y_e(t) = 0$. Since, for all $k \geq k_\mathrm{s}$, $\tilde{A}(k)$ is asymptotically stable and $\lim_{t\to\infty} y_e(t) = 0$, it follows from Lemma A.2 that there exists $P : \mathbb{R} \to \mathbb{R}^{(n+2n_\mathrm{r}+\bar{\rho})\times(n+2n_\mathrm{r}+\bar{\rho})}$ and $Q : \mathbb{R} \to \mathbb{R}^{(n+2n_\mathrm{r}+\bar{\rho})\times(n+2n_\mathrm{r}+\bar{\rho})}$ such that the entries of $P$ and $Q$ are real rational functions, and for all $k \geq k_\mathrm{s}$, $P(k)$ is positive definite, $Q(k)$ is positive semidefinite, and $A_c^\mathrm{T}(k)P(k) + P(k)A_c(k) = -Q(k) - \gamma C_c^\mathrm{T} C_c$. For all $k \geq k_\mathrm{s}$, define $V_0(x_c, k) \triangleq e^{-\alpha k(t)} x_c^\mathrm{T} P(k) x_c$. Taking the derivative of $V_0(x_c, k)$ along trajectories of (5.11)-(5.12) yields

$$\dot{V}_0(x_c,k) = -e^{-\alpha k} x_c^\mathrm{T} Q(k) x_c - \gamma e^{-\alpha k} x_c^\mathrm{T} C_c^\mathrm{T} C_c x_c - \dot{k} e^{-\alpha k} x_c^\mathrm{T} \left[\alpha P(k) - \frac{\partial P(k)}{\partial k}\right] x_c. \qquad (5.14)$$

Lemma A.3 implies that there exists $k_2 \geq k_\mathrm{s}$ such that, for all $k \geq k_2$, $\alpha P(k) > \frac{\partial P(k)}{\partial k}$. Therefore, for all $k \geq k_2$, $\dot{V}_0(x_c,k) \leq -e^{-\alpha k} x_c^\mathrm{T} Q(k) x_c - \gamma e^{-\alpha k} y_e^2 \leq -\gamma e^{-\alpha k} y_e^2$, which implies

$$\dot{V}_0(x_c,k) \leq -\dot{k}. \qquad (5.15)$$

Next, we show that if $x_c(t)$ escapes at finite time $t_e$, then $k(t)$ also escapes at finite time $t_e$. Assume that $x_c(t)$ escapes at finite time $t_e$ whereas $k(t)$ does not escape at finite time $t_e$. Then (5.11) is a linear time-varying differential equation, whose dynamics matrix $A_c(k(t))$ is continuous in $t$. The solution to the linear time-varying system, where $A(t)$ is continuous in $t$, exists and is unique on all finite intervals [6]. Therefore, $x_c(t)$ does not escape at finite time $t_e$. Hence, if $x_c(t)$ escapes at finite time $t_e$, then $k(t)$ also escapes at finite time $t_e$.

Since (5.10)-(5.12) is locally Lipschitz, it follows that the solution to (5.10)-(5.12) exists and is unique locally, that is, there exists $t_e > 0$ such that $(x_c(t), k(t))$ exists on the interval $[0, t_e)$. Now suppose that $k(t)$ diverges to infinity at $t_e$. Then, there exists $t_2 < t_e$ such that $k(t_2) = k_2$. Integrating (5.15) from $t_2$ to $t < t_e$ and solving for $k(t)$ yields

$$\begin{aligned} k(t) &\leq V_0(x_c(t_2), k_2) + k_2 - e^{-\alpha k(t)} x_c^\mathrm{T}(t) P(k(t)) x_c(t) \\ &\leq V_0(x_c(t_2), k_2) + k_2, \end{aligned} \qquad (5.16)$$

for $t \in [t_2, t_e)$. Hence, $k(\cdot)$ is bounded on $[0, t_e)$, which is a contradiction. Therefore, the solution to (5.10)-(5.12) exists and is unique on all finite intervals. Then integrating (5.15) from $t_2$ to $t$ yields (5.16) for $t \in [t_2, \infty)$. Therefore, $k(\cdot)$ is bounded on $[0, \infty)$. Since $k(t)$ is non-decreasing, $k_\infty \triangleq \lim_{t\to\infty} k(t)$ exists.

Since for all $t > 0$, $k(t) < k_\infty$, it follows that

$$\gamma e^{-\alpha k_\infty} \int_0^t y_e^2(\tau) d\tau \leq \gamma \int_0^t e^{-\alpha k(\tau)} y_e^2(\tau) d\tau < k_\infty - k(0), \qquad (5.17)$$

and thus $y_e(\cdot)$ is square integrable on $[0, \infty)$. This property will be used later.

Next, we show that, for all $k > 0$, the pair $\left(\tilde{A}(k), \tilde{C}\right)$ is detectable. Let $\lambda$ be an element of the closed right half plane. Then

$$\begin{aligned} \mathrm{rank} \begin{bmatrix} \tilde{A}(k) - \lambda I \\ \tilde{C} \end{bmatrix} &= \mathrm{rank} \begin{bmatrix} A - \lambda I & B\hat{C}(k) \\ C & 0 \\ 0 & \hat{A}(k) - \lambda I \end{bmatrix} \\ &= \mathrm{rank}\ \Omega \begin{bmatrix} I_n & 0 \\ 0 & \hat{C}(k) \\ 0 & \hat{A}(k) - \lambda I \end{bmatrix}. \end{aligned} \qquad (5.18)$$

Since $(A, B, C)$ is a minimal realization of the minimum phase plant (2.1), it follows that $\Omega \triangleq \begin{bmatrix} A - \lambda I & B & 0 \\ C & 0 & 0 \\ 0 & 0 & I_{n_\mathrm{r}+\bar{\rho}} \end{bmatrix}$ is nonsingular. Thus

$$\begin{aligned} \mathrm{rank} \begin{bmatrix} \tilde{A}(k) - \lambda I \\ \tilde{C} \end{bmatrix} &= \mathrm{rank} \begin{bmatrix} I_n & 0 \\ 0 & \hat{C}(k) \\ 0 & \hat{A}(k) - \lambda I \end{bmatrix} \\ &= \mathrm{rank} \begin{bmatrix} I_n & 0 & 0 \\ 0 & \hat{A}_\mathrm{r} - \lambda I & \hat{B}_\mathrm{r}\hat{C}_{\bar{\rho}}(k) \\ 0 & \hat{C}_\mathrm{r} & \hat{D}_\mathrm{r}\hat{C}_{\bar{\rho}}(k) \\ 0 & 0 & \hat{A}_{\bar{\rho}}(k) - \lambda I \end{bmatrix} \\ &= \mathrm{rank}\ \Gamma \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{n_\mathrm{r}} & 0 \\ 0 & 0 & \hat{C}_{\bar{\rho}}(k) \\ 0 & 0 & \hat{A}_{\bar{\rho}}(k) - \lambda I \end{bmatrix}. \end{aligned} \qquad (5.19)$$

Since $(\hat{A}_\mathrm{r}, \hat{B}_\mathrm{r}, \hat{C}_\mathrm{r}, \hat{D}_\mathrm{r})$ is a minimal realization of the minimum phase compensator $\hat{G}_\mathrm{r}(s)$, it follows that $\Gamma \triangleq \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & \hat{A}_\mathrm{r} - \lambda I & \hat{B}_\mathrm{r} & 0 \\ 0 & \hat{C}_\mathrm{r} & \hat{D}_\mathrm{r} & 0 \\ 0 & 0 & 0 & I_{\bar{\rho}} \end{bmatrix}$ is nonsingular for all $\lambda$ in the closed right half plane. Thus

$$\mathrm{rank} \begin{bmatrix} \tilde{A}(k) - \lambda I \\ \tilde{C} \end{bmatrix} = \mathrm{rank} \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{n_\mathrm{r}} & 0 \\ 0 & 0 & \hat{C}_{\bar{\rho}}(k) \\ 0 & 0 & \hat{A}_{\bar{\rho}}(k) - \lambda I \end{bmatrix}. \qquad (5.20)$$

Since, for all $k > 0$, $\left(\hat{A}_{\bar{\rho}}(k), \hat{C}_{\bar{\rho}}(k)\right)$ is observable, it follows that, for all $k > 0$, $\mathrm{rank} \begin{bmatrix} \tilde{A}(k) - \lambda I \\ \tilde{C} \end{bmatrix} = n + n_\mathrm{r} + \bar{\rho}$. Therefore, for all $k > 0$, $\left(\tilde{A}(k), \tilde{C}\right)$ is detectable.

Next, we show that $\lim_{t\to\infty} y_e(t) = 0$. Define $A_\infty \triangleq \tilde{A}(k_\infty)$. Since $(A_\infty, \tilde{C})$ is detectable, it follows that there exists $L \in \mathbb{R}^{(n+n_\mathrm{r}+\bar{\rho})\times 1}$ such that $A_\mathrm{s} \triangleq A_\infty + L\tilde{C}$ is asymptotically stable. Then adding and subtracting $A_\mathrm{s}$ and $L\tilde{D}u_\mathrm{r}$ from (5.7) implies

$$\dot{\tilde{x}}(t) = A_\mathrm{s}\tilde{x}(t) + \Delta(t)\tilde{x}(t) + Ju_\mathrm{r}(t) - Ly_e(t), \qquad (5.21)$$

where $\Delta(t) \triangleq A(k(t)) - A_\infty$, and $J \triangleq \tilde{B} + L\tilde{D}$. Since $A_\mathrm{s}$ is asymptotically stable, $\Delta(\cdot)$ is continuous, $\lim_{t\to\infty}\Delta(t) = 0$, $u_\mathrm{r}(\cdot)$ is bounded on $[0, \infty)$, and $y_e(\cdot)$ is square integrable on $[0, \infty)$, it follows from Lemma A.4 that $\tilde{x}(\cdot)$ is bounded on $[0, \infty)$.

Next, since $\tilde{A}(\cdot)$ is bounded, $\tilde{x}(\cdot)$ is bounded, and $u_\mathrm{r}(\cdot)$ is bounded, it follows from (5.7) that $\dot{\tilde{x}}(\cdot)$ is bounded. Since $\tilde{x}(\cdot)$, $\dot{\tilde{x}}(\cdot)$, $u_\mathrm{r}(\cdot)$, and $\dot{u}_\mathrm{r}(\cdot)$ are bounded, it follows from (5.7) that $y_e(\cdot)$ and $\dot{y}_e(\cdot)$ are bounded. Therefore, $\frac{d}{dt}\left(y_e^2(t)\right) = 2\dot{y}_e(t)y_e(t)$ is bounded, and thus $y_e^2(t)$ is uniformly continuous. Since $y_e^2(t)$ is uniformly continuous and $\lim_{t\to\infty}\int_0^t y_e^2(\tau)d\tau$ exists, Barbalat's lemma implies that $\lim_{t\to\infty} y_e(t) = 0$. □

Figure 2 illustrates the adaptive controller presented in Theorem 5.1.

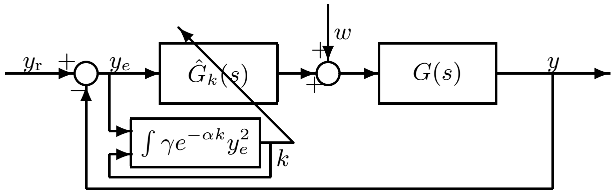


Fig. 2. Adaptive controller for the command following and disturbance rejection problem.

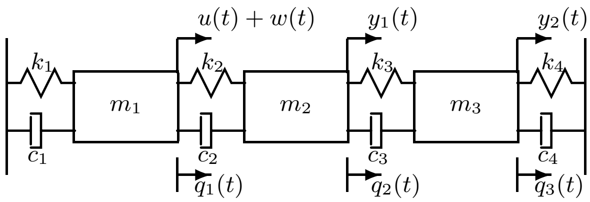


Fig. 3. Three-mass serially connected spring-mass-damper system.

## 6. Serially Connected Spring-Mass-Damper

Consider the three-mass serially connected spring-mass-damper system shown in Figure 3. The dynamics of the system are given by

$$M\ddot{q} + C\dot{q} + Kq = b\,(u + w)\,, \tag{6.1}$$

where

$$M \triangleq \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}, \quad b \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{6.2}$$

$$C \triangleq \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix}, \tag{6.3}$$

$$K \triangleq \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}, \tag{6.4}$$

$$q \triangleq \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^{\mathrm{T}}. \tag{6.5}$$

The masses are $m_1 = 1$ kg, $m_2 = 0.5$ kg, and $m_3 = 1$ kg; the damping coefficients are $c_1 = c_2 = c_3 = c_4 = 2$ kg/sec; and the spring constants are $k_1 = 2$ kg/sec$^2$, $k_2 = 4$ kg/sec$^2$, $k_3 = 1$ kg/sec$^2$, and $k_4 = 3$ kg/sec$^2$.

Our objective is to design an adaptive controller so that all single-input, single-output (SISO) force-to-position transfer functions of the system (6.1)-(6.5) can track a sinusoid of $\omega_1 = 11$ rad/sec and a step, while rejecting a sinusoid of $\omega_2 = 8$ rad/sec and a constant disturbance. Thus, the dynamics for tracking and disturbance rejection are given by the characteristic polynomial

$$p_{\mathrm{r}}(s) = s\left(s^2 + \omega_1^2\right)\left(s^2 + \omega_2^2\right). \tag{6.6}$$

All SISO force-to-position transfer functions of a serially connected structure are known to be minimum phase [7]. Furthermore, [7] shows that the relative degree of a SISO force-to-position transfer function for a serially connected structure is equal to the number of intervening masses plus two. For a three mass system, all force-to-position transfer functions have relative degree not exceeding four. Therefor, $\rho = 4$ is an upper bound on the relative degree of the force-to-position transfer functions for a three mass system. For this example, all SISO force-to-position transfer functions have a positive high-frequency gain, so let $\delta = 1$. Next, let us assume that the upper bound on the magnitude of the high-frequency gain is $b_0 = 10$. Then all SISO force-to-position transfer functions satisfy assumptions (A1)-(A6).

Next, consider the parameter-dependent transfer function (4.1) where $\bar{\rho} = 4$

$$\hat{G}_k(s) = \frac{k^8 \hat{z}_{\mathrm{r}}(s)\hat{z}(s)}{p_{\mathrm{r}}(s)\left[s^4 + k^3 b_4 s^3 + k^5 b_3 s^2 + k^6 b_2 s + k^7 b_1\right]}. \tag{6.7}$$

To satisfy the assumptions of Theorem 4.1 the design parameters are chosen to be

$$\hat{z}_{\mathrm{r}}(s) = (s+2)\,(s+4)\,(s+6)\,(s+8)\,(s+10)\,, \tag{6.8}$$

$$\hat{z}(s) = (s+15)\,(s+20)\,(s+25)\,, \tag{6.9}$$

$$b_4 = 4, \quad b_3 = 4, \quad b_2 = 12, \quad b_1 = 4. \tag{6.10}$$

Then, the adaptive controller considered in Theorem 5.1 is given by the adaptive law

$$\dot{k}(t) = \gamma e^{-\alpha k(t)} y_e^2(t), \tag{6.11}$$

and (5.2), where

$$\hat{A}_{\mathrm{r}}(k) \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -7744 & 0 & -185 & 0 \end{bmatrix}, \quad \hat{B}_{\mathrm{r}} \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{6.12}$$

$$\hat{C}_{\mathrm{r}}(k) \triangleq \begin{bmatrix} 3840 & -3360 & 1800 & 155 & 30 \end{bmatrix}, \quad \hat{D}_{\mathrm{r}} \triangleq 1, \tag{6.13}$$

$$\hat{A}_{\bar{\rho}}(k) \triangleq \begin{bmatrix} -4k^3 & 1 & 0 & 0 \\ -4k^5 & 0 & 1 & 0 \\ -12k^6 & 0 & 0 & 1 \\ -4k^7 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_{\bar{\rho}} \triangleq \begin{bmatrix} 1 \\ 60 \\ 1175 \\ 7500 \end{bmatrix}, \tag{6.14}$$

$$\hat{C}_{\bar{\rho}}(k) \triangleq \begin{bmatrix} k^8 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma = 1, \quad \alpha = 0.1. \tag{6.15}$$

Now, we assume that the sensor is placed so that the position of $m_2$ is the output of the force-to-position system we are trying to control. This system is

$$y_1 = G_1(s)(u + w), \tag{6.16}$$

where

$$G_1(s) \triangleq \frac{4s^3 + 24s^2 + 48s + 32}{s^6 + 16s^5 + 84s^4 + 224s^3 + 330s^2 + 280s + 100}. \tag{6.17}$$

Furthermore, let us assume that the reference and disturbance signals are

$$y_{\mathrm{r}}(t) = 10\sin(\omega_1 t) + 5, \tag{6.18}$$

$$w(t) = 7\cos(\omega_2 t) - 8. \tag{6.19}$$

The spring-mass-damper system system (6.16)-(6.17) is simulated with the initial conditions $q(0) = \begin{bmatrix} -0.5 & 0.25 & 1.0 \end{bmatrix}^{\mathrm{T}}$ m and $\dot{q}(0) = \begin{bmatrix} 0.1 & -0.2 & 0.3 \end{bmatrix}^{\mathrm{T}}$ m/s. The adaptive controller (5.2) and (6.11)-(6.15) is implemented in the feedback loop with $y_e(t) = y_{\mathrm{r}}(t) - y_1(t)$ and initial conditions $\hat{x}(0) = 0$ and $k(0) = 25$. Figure 4 shows that $y_1(t)$ asymptotically tracks $y_{\mathrm{r}}(t)$, that is, $y_e(t)$ converges to zero, and $k(t)$ converges to approximately 42.2.

Now let us assume that the position sensor is placed on the third mass instead of the second mass. Then, we are trying to

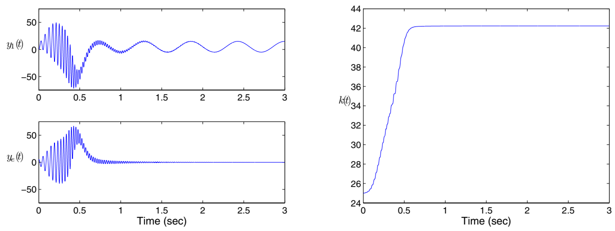


Fig. 4. The output $y_1(t)$ asymptotically tracks the reference $y_{\rm r}(t)$, so $y_{\rm e}(t)$ converges to zero (left). The adaptive parameter $k(t)$ converges to approximately 42.2 (right).

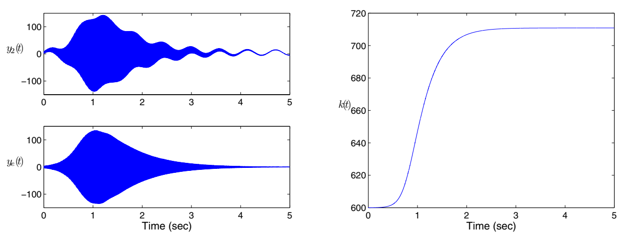


Fig. 5. The output $y_2(t)$ asymptotically tracks the reference $y_{\rm r}(t)$, so $y_{\rm e}(t)$ converges to zero (left). The adaptive parameter $k(t)$ converges to approximately 711 (right).

control the force-to-position system

$$y_2 = G_2(s)(u+w), \tag{6.20}$$

where

$$G_2(s) \triangleq \frac{8s^2+20s+8}{s^6+16s^5+84s^4+224s^3+330s^2+280s+100}. \tag{6.21}$$

Note that $G_2(s)$ has relative degree 4 instead of 3. As before, the reference and disturbance signals are given by (6.18)-(6.19). The spring-mass-damper system system (6.20)-(6.21) is simulated with the initial conditions $q(0) = \left[\begin{array}{ccc} -0.5 & 0.25 & 1.0 \end{array}\right]^{\rm T}$ m and $\dot{q}(0) = \left[\begin{array}{ccc} 0.1 & -0.2 & 0.3 \end{array}\right]^{\rm T}$ m/s. The adaptive controller (5.2) and (6.11)-(6.15) is implemented in the feedback loop with $y_{\rm e}(t) = y_{\rm r}(t) - y_2(t)$ and initial conditions $\hat{x}(0) = 0$ and $k(0) = 600$. Figure 5 shows that $y_{\rm e}(t)$ converges to zero and $k(t)$ converges to approximately 711.

## Appendix A: Preliminary Results for Analyzing Gain-Monotonic Adaptive Systems

In this appendix, we present several preliminary results useful for analyzing gain-monotonic adaptive systems. The proofs have been omitted due to space considerations. In this section, we consider the system

$$\dot{x} = A(k)x, \tag{A.1}$$
$$y = C(k)x, \tag{A.2}$$

where $A(k) \in \mathbb{R}^{l\times l}$ and $C(k) \in \mathbb{R}^{d\times l}$ have entries that are polynomials in $k$.

The first two results concern the solution to a Lyapunov equation for the system (A.1)-(A.2).

**Lemma A.1.** *Assume that there exists $k_{\rm s} > 0$ such that, for all $k \ge k_{\rm s}$, $A(k)$ is asymptotically stable. Let $Q(k) \in \mathbb{R}^{l\times l}$ have entries that are polynomial functions of $k$, where, for all $k \ge k_{\rm s}$, $Q(k)$ is positive definite. Then there exists $P : \mathbb{R} \to \mathbb{R}^{l\times l}$ such that each entry of $P$ is a real rational function, and for all $k \ge k_{\rm s}$, $P(k)$ is positive definite and satisfies*

$$A^{\rm T}(k)P(k) + P(k)A(k) = -Q(k). \tag{A.3}$$

**Lemma A.2.** *Consider the system (A.1)-(A.2), and assume that*

$$A(k) \triangleq \left[\begin{array}{cc} A_1(k) & A_3(k) \\ 0 & A_2 \end{array}\right], \tag{A.4}$$

$$C(k) \triangleq \left[\begin{array}{cc} C_1(k) & C_2(k) \end{array}\right], \tag{A.5}$$

*where $A_1(k) \in \mathbb{R}^{l_1\times l_1}$, $A_3(k) \in \mathbb{R}^{l_1\times l_2}$, $C_1(k) \in \mathbb{R}^{d\times l_1}$, and $C_2(k) \in \mathbb{R}^{d\times l_2}$ have entries that are polynomials in $k$, and $A_2 \in \mathbb{R}^{l_2\times l_2}$. For all $\lambda \in \operatorname{spec}(A_2)$, assume that $\lambda$ is semisimple and Re $\lambda = 0$. Furthermore, assume that there exists $k_{\rm s} > 0$ such that, for all $k \ge k_{\rm s}$, $A_1(k)$ is asymptotically stable and $\lim_{t\to\infty} y(t) = 0$. Let $\gamma > 0$. Then there exist $P : \mathbb{R} \to \mathbb{R}^{(l_1+l_2)\times(l_1+l_2)}$ and $Q : \mathbb{R} \to \mathbb{R}^{(l_1+l_2)\times(l_1+l_2)}$ such that the entries of $P$ and $Q$ are real rational functions, and for all $k \ge k_{\rm s}$, $P(k)$ is positive definite, $Q(k)$ is positive semidefinite, and they satisfy*

$$A^{\rm T}(k)P(k) + P(k)A(k) = -Q(k) - \gamma C^{\rm T}(k)C(k). \tag{A.6}$$

The next result concerns the derivative of a positive-definite matrix whose entries are real rational functions of a single parameter.

**Lemma A.3.** *Let $P : \mathbb{R} \to \mathbb{R}^{l\times l}$, where each entry of $P$ is a real rational function. Assume that there exists $k_{\rm s} > 0$ such that, for all $k \ge k_{\rm s}$, $P(k)$ is symmetric positive definite. Then, for all $\alpha > 0$, there exists $k_2 \ge k_{\rm s}$ such that, for all $k \ge k_2$, $\frac{dP(k)}{dk} < \alpha P(k)$.*

The final result of this section is integral to the proof of asymptotic command following and disturbance rejection for the adaptive controller presented in this paper.

**Lemma A.4.** *Consider the nonhomogeneous linear time-varying system*

$$\dot{\zeta}(t) = A_{\rm s}\zeta(t) + \Delta(t)\zeta(t) + L\phi(t) + D\omega(t), \tag{A.7}$$

*where $\zeta \in \mathbb{R}^{l_\zeta}$, $\phi : [0,\infty) \to \mathbb{R}^{l_\phi}$, $\omega : [0,\infty) \to \mathbb{R}^{l_\omega}$, and $\Delta : [0,\infty) \to \mathbb{R}^{l_\zeta\times l_\zeta}$. Assume that $A_{\rm s}$ is asymptotically stable, $\Delta(\cdot)$ is continuous, $\lim_{t\to\infty}\Delta(t) = 0$, $\phi(\cdot)$ is square integrable on $[0,\infty)$, and $\omega(\cdot)$ is bounded on $[0,\infty)$. Then, for all $\zeta(0)$, $\zeta(\cdot)$ is bounded on $[0,\infty)$.*

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