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Continuity of the Outer Factorization and Mapping Properties with Applications

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***Abstract*— This paper investigates the smoothness behavior of the Poisson- and the conjugate Poisson integral on the closure of the unit disk. It gives sufficient and necessary conditions on the majorants of the data such that these integrals as well as the Hilbert- and Cauchy transform have always the same modulus of continuity as the data, provided that the data have no zeros on the unit circle. The results are applied to study the smoothness properties of the spectral factorization and Wiener filter.**

I. Introduction

In many areas of control theory and communications the integral operator

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\ \frac{e^{i\tau}+z}{e^{i\tau}-z}\ d\tau \quad \text{for} \quad |z| \leq 1. \tag{1}$$

of a function f plays an important role. For example, we consider two problems: Wiener filtering [1] and spectral factorization. In the Wiener filtering problem, the starting point is the well known formal solution [2], [3]: Given a real positive function $\Phi(t)$ and a complex function $\Psi(t)$ defined on $[-\pi,\pi)$, the transfer function of the Wiener filter is

$$H(z) = \frac{1}{\Phi_+(z)}\left[\frac{\Psi}{\Phi_-}\right]^+(z)\ . \tag{2}$$

In which the complex functions $\Phi_+(z)$ and $\Phi_-(z)$ are the *spectral factors* of Φ, defined by

$$\Phi(t) = \Phi_+(e^{it})\Phi_-(e^{it}), \quad \forall t \in [-\pi,\pi)$$

with the properties that $\Phi_+(z)$ is an analytic function without any zero ($\Phi_+(z) \neq 0$) for $|z| \leq 1$ and that $\Phi_-(z)$ is an analytic function without zero ($\Phi_-(z) \neq 0$) for $|z| \geq 1$. These spectral factors can be calculated from Φ by

$$\Phi_+(z) = [F_+(z)]^{1/2} \quad \text{and} \quad \Phi_-(z) = \overline{\Phi_+(1/\overline{z})}$$

in which $F_+(z)$ is the *outer function* of $\Phi(t)$ given by

$$F_+(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\Phi(\tau)\ \frac{e^{i\tau}+z}{e^{i\tau}-z}\ d\tau\right). \tag{3}$$

The plus-operator $[\cdot]^+$ in (2) is the Cauchy transform of the function inside the brackets. Let $\Gamma(t) := \Psi(t)/\Phi_-(e^{-it})$, then the right factor in (2) becomes

$$[\Gamma(t)]^+(z) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma(\tau)\frac{e^{i\tau}}{e^{i\tau}-z}\ d\tau\ . \tag{4}$$

Spectral factorization appears also in the following problem: Given a positive real function $\Phi(t)$, e.g. the magnitude response of a desired filter, find the transfer function $F_+(z)$ of a causal filter such that $\left|F_+(e^{it})\right| = \Phi(t)$ for all $t \in [-\pi,\pi)$ and such that $F_+(z)$ is an analytic function inside the unit disk. The solution of this problem is given by (3).

Thus, the solution of both problems is determined by the spectral factorization operator (3) and additionally (for the Wiener filter) by the plus-operator (4). Both operators are based on the integral transform (1). In practical applications the resulting filters H and F_+ have to be stable, i.e. the supremum norm $\|H\|_\infty := \sup_{|z|<1}|H(z)|$ of the transfer functions $H(z)$ and $F_+(z)$ have to be finite. Moreover, it may be desirable to approximate the transfer functions $H(z)$ and $F_+(z)$ by polynomials which yields simple finite impulse response (FIR) realization. How good such an approximation is possible depends on the smoothness of the transfer functions. Thus, the question is: Given the data $\Phi(t)$ and $\Psi(t)$ with a certain smoothness (measured by its modulus of continuity). Which conditions have these data to fulfill such that the resulting filters have the same smoothness as the data? The answer is non-trivial, since the operators (3) and (4) are non-linear (3) and non-continuous, and it can be shown that there exist continuous functions $\Phi(t)$ such that the corresponding integral transform (1) is unbounded [4], and that the outer function $F_+(z)$ is not continuous in the closure of the unit disk [5], [6].

The investigation of the properties of the spectral factorization operator in different spaces is an active field of mathematical research [7]–[10]. This article will considers the factorization in the space $\mathcal{C}_\omega$ of functions which modulus of continuity is bounded by a majorant ω (see Section II). Recently, a similar problem attracted some interest: In [11] conditions on the modulus of continuity were given, under which a function f has the same modulus of continuity as $|f|$ in the closure of the unit disk, and in [12] this result was extended to a much broader class of functions. In this paper we will use and extend these results to derive the necessary and sufficient conditions on a function, given on the unit circle, such that its analytic extension (1) into the unit disk has the same modulus of continuity as the data on the boundary. The investigation of the real- and imaginary part of (1) leads to the study of the Poisson and Hilbert integral, which both have a slightly different behavior.

II. Integral Transforms and Majorants

In the following, we investigate the integral transform (1). It is assumed that $f(t)$ is a real function, defined on the

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interval $[-\pi,\pi)$ with $f(t)>0$ for all $t\in[-\pi,\pi)$. If F in (1) is written as $F(z)=u(z)+i\cdot v(z)$ then $u(z)$ and $v(z)$ are given by *Poisson* and by the *conjugate Poisson* integral, respectively. For $z=re^{it}$ and $r<1$, we have

$$u(re^{it}) \quad = (Pf)(re^{it}) := \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\ \mathcal{P}_r\,(t-\tau)\,d\tau \quad (5)$$

$$v(re^{it}) \quad = (Qf)(re^{it}) := \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\ \mathcal{Q}_r\,(t-\tau)\,d\tau \quad (6)$$

in which the kernels are defined by

$$\begin{aligned}\mathcal{P}_r\,(\tau) &= \Re\left\{\frac{e^{i\tau}-z}{e^{i\tau}+z}\right\} = \frac{1-r^2}{1-2r\cos(\tau)+r^2}\\ \mathcal{Q}_r\,(\tau) &= \Im\left\{\frac{e^{i\tau}-z}{e^{i\tau}+z}\right\} = \frac{2r\sin(\tau)}{1-2r\cos(\tau)+r^2}\ .\end{aligned}$$

F is analytic in the unit disk $\mathcal{D} := \{z\in\mathbb{C}: |z|<1\}$, and u and v are harmonic in $\mathcal{D}$. The boundary values of u are uniquely determined almost everywhere on the unit circle $\partial\mathcal{D} := \{z\in\mathbb{C}: |z|=1\}$. Therefore, we can define $u(e^{it}) := \lim_{r\to1} u(re^{it})$ and for continuous functions f the boundary values $u(e^{it})$ are equal to $f(t)$ for all $t\in[-\pi,\pi)$. The boundary function $\lim_{r\to1} v(re^{it})$ of the imaginary part is determined by the *Hilbert transform* of f. From (6) it becomes

$$\widetilde{f}(t) := \lim_{r\to1} v(re^{it}) = v(e^{it}) = (Hf)(t), \quad t\in[-\pi,\pi)$$

with the Hilbert transform $(Hf)(t)=\lim_{\epsilon\to0}(H_\epsilon f)(t)$ and

$$(H_\epsilon f)(t) = \frac{1}{2\pi}\int_{\epsilon<|\tau|\le\pi} \frac{f(\tau+t)}{\tan(\tau/2)}\ d\tau. \qquad (7)$$

However, the Hilbert transform may not converge for arbitrary functions f because of the singularity of its kernel at $\tau=0$. The function $\widetilde{f}(t)$ is called the *conjugate* to $f(t)$ and $H_\epsilon f$ is the *truncated Hilbert transform* of f. Note that the imaginary part v of F can also be written as the Poisson integral of the conjugate function $\widetilde{f}$: $v=P\widetilde{f}$.

Sometimes it is advantageous to separate the singular integral of the Hilbert transform into a regular part and into a simpler singular part. Therefore the kernel of the Hilbert transform (7) is written as

$$\frac{1}{2\tan(\tau/2)} = \frac{1}{\tau} + \mathcal{K}(\tau)\ ;\ \mathcal{K}(\tau) = \frac{\tau - 2\tan(\tau/2)}{2\tau\tan(\tau/2)}\ . \quad (8)$$

It is easily verified that $\mathcal{K}(\tau)$ is a regular function at $\tau=0$. Therewith, the Hilbert transform (7) becomes

$$(Hf)(t) = \lim_{\epsilon\to0}\frac{1}{\pi}\int_{\epsilon<|\tau|\le\pi}\frac{f(\tau+t)}{\tau}\ d\tau + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau+t)\mathcal{K}(\tau)\ d\tau\ .$$

Let $\Omega\subset\mathbb{C}$ be a compact set in the complex plane, and let $f:\Omega\to\mathbb{C}$ be a function. The function $\omega_f(\delta)$ defined by

$$\omega_f(\delta) := \sup_{|t_1-t_2|\le\delta} |f(t_1)-f(t_2)| \quad \text{for} \quad \forall t_1,t_2\in\Omega$$

is called the *modulus of continuity* of f.

A continuous, increasing, real valued function $\omega(t)$ defined on the interval $[0,\pi]$ is called a *majorant* if $\omega(0)=0$ and if the function $\omega(t)/t$ is non increasing. A majorant ω is called *regular*, if there exists a constant C such that

$$\int_0^x \frac{\omega(\tau)}{\tau}\ d\tau + x\int_x^\pi \frac{\omega(\tau)}{\tau^2}\ d\tau \le C\ \omega(x), \quad 0<x<1.$$

This well known definition (cf. [11]) of a regular majorant consists of two terms. This paper investigates how these two parts influence the continuity behavior of the Poisson- and the conjugate Poisson integral. Therefore, we introduce additionally the following two classes of majorants

Definition: A majorant ω is said to be *weak regular of type 1*, if there exists a constant C such that

$$\textstyle\int_0^x \frac{\omega(\tau)}{\tau}\ d\tau \le C\ \omega(x), \quad 0<x<1 \qquad (9)$$

and ω is said to be *weak regular of type 2*, if there exists a constant C such that

$$\textstyle x\int_x^\pi \frac{\omega(\tau)}{\tau^2}\ d\tau \le C\ \omega(x), \quad 0<x<1\ . \qquad (10)$$

Clearly, every regular majorant is also weak regular. Given a majorant ω and a bounded domain $\Omega\subset\mathbb{C}$. The set of all functions $f:\Omega\to\mathbb{C}$ which modulus of continuity is bounded by ω is denoted by $\mathcal{C}_\omega(\Omega)$ with the norm

$$\|f\|_{\mathcal{C}_\omega(\Omega)} := |f(0)| + \sup_{z_1\ne z_2}\frac{|f(z_1)-f(z_2)|}{\omega(|z_1-z_2|)}.$$

It should be mentioned that with this norm and with pointwise multiplication, $\mathcal{C}_\omega(\Omega)$ forms a Banach algebra.

Example: Consider the set of functions f which satisfy a Hölder condition of order α with $0<\alpha\le1$ on a domain Ω: $|f(z_1)-f(z_2)|<C|z_1-z_2|^\alpha$ for all $z_1,z_2\in\Omega$. The modulus of continuity for this functions is $\omega(t)=t^\alpha$, and it is easily verified that ω is a regular majorant for $0<\alpha<1$ but only a weak regular majorant of type 1 for $\alpha=1$.

III. The Poisson Integral

Let ω be a majorant and let $f\in\mathcal{C}_\omega[-\pi,\pi)$. In this section, we ask which condition has the majorant ω to fulfill such that Pf has the same modulus of continuity ω as f in the closure of the unit disk, i.e. such that $(Pf)\in\mathcal{C}_\omega(\overline{\mathcal{D}})$ whenever $f\in\mathcal{C}_\omega[-\pi,\pi)$.

Remark 1: Let $R<1$ and $D_R := \{z : |z|<R\}$ an open disk inside $\mathcal{D}$, and let ω be a majorant. Then it holds $Pf\in\mathcal{C}_\omega(D_R)$ and $Qf\in\mathcal{C}_\omega(D_R)$ whenever $f\in\mathcal{C}_\omega[-\pi,\pi)$. Thus strictly inside the unit disk the Poisson- and conjugate Poisson integral have always the same modulus of continuity as the given data on $\partial\mathcal{D}$. A proof of this statement for Pf can be found in [11, Lemma 4]. Only if we require that this property holds in the closure of the unit disk $\overline{\mathcal{D}}=\mathcal{D}\cup\partial\mathcal{D}$, the majorant ω has to satisfy additional requirements.

Next, the behavior of $(Pf)(re^{it})$ as $r\to1$ is studied. We start with two preliminary lemmas which investigate the smoothness of Pf in tangential direction (i.e. along a circle around the origin) and in radial direction, respectively.

Lemma 1: *Let ω be a majorant, and let $z_1=re^{it_1}$ and $z_2=re^{it_2}$ be two points in the unit disk with the same radial* *distance $r \leq 1$ from the origin. If $f \in \mathcal{C}_\omega[-\pi, \pi)$, then there exists a constant C which depends only on ω such that*

$$|(Pf)(z_1) - (Pf)(z_2)| \leq C \ \|f\|_{\mathcal{C}_\omega} \omega(|z_1 - z_2|) \ .$$

Proof: Because of Remark 1, we only need to prove this lemma for $r \geq R_0$ with some $R_0 < 1$. Starting with the definition of the Poisson integral (5) gives for $r < 1$

$$\begin{aligned} &\left|(Pf)(re^{it_1}) - (Pf)(re^{it_2})\right| \\ &\leq \tfrac{1}{2\pi} \int_{-\pi}^{\pi} |f(\tau + t_1) - f(\tau + t_2)| \mathcal{P}_r(\tau) \ d\tau \\ &\leq \|f\|_{\mathcal{C}_\omega} \cdot \omega(|t_1 - t_2|) \ . \end{aligned}$$

Obviously, it holds $\left|e^{it_1} - e^{it_2}\right| \leq |t_1 - t_2|$ and using that $\omega(t)/t$ is a non-increasing function, the inequality $\omega(|t_1 - t_2|) \leq \frac{\pi}{2}\omega(\left|e^{it_1} - e^{it_2}\right|)$ is obtained. Moreover, since $|z_1 - z_2| \leq \left|e^{it_1} - e^{it_2}\right|$ and again, because $\omega(t)/t$ is non-increasing, it follows that $\omega(\left|e^{it_1} - e^{it_2}\right|) \leq \frac{1}{r}\omega(|z_1 - z_2|)$ such that altogether

$$\omega(|t_1 - t_2|) \leq \tfrac{\pi}{2r}\omega(|z_1 - z_2|).$$

This proves finally the lemma for $r < 1$ with $C = \pi/(2R_0)$. The statement for $r = 1$ is obvious, since the Poisson integral of a continuous function is continuous in $\overline{\mathcal{D}}$ with $\lim_{r\to 1} (Pf)(re^{it}) = f(t)$ for all $t \in [-\pi, \pi)$. ■

REMARK 2: From this lemma follows in particular that the function $g_r(z) := (Pf)(rz)$ is an element of $\mathcal{C}_\omega(\partial\mathcal{D})$ for any fixed $0 < r \leq 1$, i.e. the restriction of $g_r(z)$ to the unit circle has modulus of continuity ω, and $g_r(e^{it}) \in \mathcal{C}_\omega[-\pi, \pi)$, and there exists a C such that $\|g_r\|_{\mathcal{C}_\omega[-\pi,\pi)} \leq C \|f\|_{\mathcal{C}_\omega}$.

Note that in this lemma, it was only assumed that ω is a majorant but it needs not to be weak regular or even regular.

The next lemma investigates the smoothness of the Poisson integral in radial direction.

LEMMA 2: *If ω is a weak regular majorant of type 2, and if $f \in \mathcal{C}_\omega[-\pi, \pi)$, then there exists a constant C such that*

$$\left|(Pf)(re^{it}) - f(t)\right| \leq C \|f\|_{\mathcal{C}_\omega} \omega(1 - r)$$

for all $t \in [-\pi, \pi)$ and for all $0 \leq r \leq 1$.

Proof: For the Poisson kernel holds $\int_{-\pi}^{\pi} \mathcal{P}_r(\tau) d\tau = 2\pi$ for all $0 \leq r \leq 1$. Therefore it is

$$\begin{aligned} \left|(Pf)(re^{it}) - f(t)\right| &= \left|\tfrac{1}{2\pi} \int_{-\pi}^{\pi} [f(\tau) - f(t)] \mathcal{P}_r(t - \tau) \ d\tau\right| \\ &\leq \tfrac{1}{2\pi} \|f\|_{\mathcal{C}_\omega} \int_{-\pi}^{\pi} \omega(|\tau - t|) \mathcal{P}_r(t - \tau) \ d\tau \\ &\leq \tfrac{1}{4} \ \|f\|_{\mathcal{C}_\omega} \ C \ \omega(1 - r) \end{aligned}$$

wherein for the first inequality, it was used that $\mathcal{P}_r$ is non-negative and that $f \in \mathcal{C}_\omega[-\pi, \pi)$. The second inequality follows from the auxiliary Lemma 11 in the appendix. ■

COROLLARY 3: *Let ω be a weak regular majorant of type 2, and let $z_1 = r_1 e^{it}$ and $z_2 = r_2 e^{it}$ be two points inside the unit disk with $0 \leq r_1, r_2 \leq 1$. If $f \in \mathcal{C}_\omega[-\pi, \pi)$, then there exists a constant C such that*

$$|(Pf)(z_1) - (Pf)(z_2)| \leq C \ \|f\|_{\mathcal{C}_\omega} \omega(|z_1 - z_2|) \qquad (11)$$

for all $t \in [-\pi, \pi)$.

Proof: Without loss of generality, it is assumed that $r_2 > r_1$, and because of Remark 1 it is sufficient to consider the case that $r_2 \geq R_0$ with some $R_0 < 1$. Consider the function g defined by $g(t) := (Pf)(r_2 e^{it})$. Lemma 1 shows that $g \in \mathcal{C}_\omega[-\pi, \pi)$ with $\|g\|_{\mathcal{C}_\omega} \leq C_1 \|f\|_{\mathcal{C}_\omega}$. Therewith the left-hand side of (11) can be written as

$$\left|(Pg)\left(\tfrac{r_1}{r_2} e^{it}\right) - g(t)\right| \leq C_2 \|g\|_{\mathcal{C}_\omega} \omega\left(1 - \tfrac{r_1}{r_2}\right)$$

wherein the right hand side follows from Lemma 2 with a constant C_2. Since it is assumed that $r_2 \geq R_0$ and because $\omega(t)$ is an increasing function, it is

$$\omega\left(\tfrac{r_2 - r_1}{r_2}\right) \leq \omega\left(\tfrac{r_2 - r_1}{R_0}\right) \leq \tfrac{1}{R_0} \ \omega(r_2 - r_1)$$

whereas the last inequality follows from the fact that $\omega(t)/t$ is a non-increasing function and that $R_0 < 1$. Therewith statement (11) is immediately obtained with $C = C_1/R_0$. ■

Lemma 1 makes a statement on the behavior of Pf along the arc of a circle around the origin. Lemma 2 and Corollary 3 make statements on its radial behavior. Both lemmas are used now to prove the main result of this section on the smoothness of the Poisson integral in the closure of the unit circle. We formulate it in the following two theorems.

THEOREM 4: *If ω is a weak regular majorant of type 2 and if $f \in \mathcal{C}_\omega[-\pi, \pi)$, then $Pf \in \mathcal{C}_\omega(\overline{\mathcal{D}})$ and there exists a constant C, dependent only on ω, such that*

$$\|Pf\|_{\mathcal{C}_\omega(\overline{\mathcal{D}})} \leq C \|f\|_{\mathcal{C}_\omega} \ .$$

Let ω be a majorant, we define the auxiliary function

$$g_1(t) := \begin{cases} \omega(-t) & \text{for} \quad -\pi \leq t < 0 \\ \omega(t) & \text{for} \quad 0 \leq t < \pi \end{cases} \ . \qquad (12)$$

If $\omega \in \mathcal{C}_\omega[0, \pi]$, it is easy to see that $g_1 \in \mathcal{C}_\omega[-\pi, \pi)$.

THEOREM 5: *Let ω be a majorant and let $g_1 \in \mathcal{C}_\omega[-\pi, \pi)$ be the function defined by* (12)*. If for all $f \in \mathcal{C}_\omega[-\pi, \pi)$ always $Pf \in \mathcal{C}_\omega(\overline{\mathcal{D}})$ then ω is weak regular of type 2.*

REMARK 3: [11, Theorem 4] proved that it is sufficient that ω is a *regular majorant* in order that from $f \in \mathcal{C}_\omega[-\pi, \pi)$ always $Pf \in \mathcal{C}_\omega(\overline{\mathcal{D}})$ follows. Theorem 4 shows now that it is already sufficient that ω is only a *weak regular majorant of type 2*. Moreover, Theorem 5 shows that this requirement on ω is also necessary. In other words: From $f \in \mathcal{C}_\omega[-\pi, \pi)$ follows always that $Pf \in \mathcal{C}_\omega(\overline{\mathcal{D}})$ if and only if ω is a weak regular majorant of type 2.

Proof: (of Theorem 4) Consider two points $z_1 = r_1 e^{it_1}$ and $z_2 = r_2 e^{it_2}$ with $r_1, r_2 \leq 1$. Additionally let $z = r_2 e^{it_1}$. Now, the statement of the theorem follows with the triangular inequality from Lemma 1 and Corollary 3:

$$\begin{aligned} &|(Pf)(z_1) - (Pf)(z_2)| \\ &\leq |(Pf)(z_1) - (Pf)(z)| + |(Pf)(z) - (Pf)(z_2)| \\ &\leq C_1 \|f\|_{\mathcal{C}_\omega} \cdot \omega(|z_1 - z|) + C_2 \|f\|_{\mathcal{C}_\omega} \cdot \omega(|z - z_2|) \end{aligned}$$

Since $|z_1 - z| \leq |z_1 - z_2|$ and $|z - z_2| \leq |z_1 - z_2|$, the statement of Theorem 4 follows with $C = C_1 + C_2$. ∎

Proof: (of Theorem 5) We have to show that condition (10) is necessary in order that from $f \in \mathcal{C}_\omega[-\pi, \pi)$ always follows that $Pf \in \mathcal{C}_\omega(\overline{\mathcal{D}})$. To this end, it is sufficient to find one function in $\mathcal{C}_\omega[-\pi,\pi)$ for which (10) is indeed necessary. Because $g_1 \in \mathcal{C}_\omega[-\pi,\pi)$ and from the assumptions of the theorem, the function $g_P(z) := (Pg_1)(z)$ is in $\mathcal{C}_\omega(\overline{\mathcal{D}})$ and there exists a constant C_1 such that $\|g_P\|_{\mathcal{C}_\omega(\overline{\mathcal{D}})} \leq C_1 \|\omega\|_{\mathcal{C}_\omega}$. Since the Poisson integral is continuous in $\overline{\mathcal{D}}$ and $\omega(t)$ is a majorant, it holds $g_P(1) = \omega(0) = 0$. Moreover, because $g_P \in \mathcal{C}_\omega(\overline{\mathcal{D}})$ it is

$$|g_P(r)| = |g_P(r) - g_P(1)| \leq \|g_P\|_{\mathcal{C}_\omega(\overline{\mathcal{D}})}\, \omega(1-r) \qquad (13)$$

Substitute the Poisson integral and using that g and the kernel $\mathcal{P}_r$ are non-negative and even functions on $[-\pi,\pi)$ gives

$$\begin{aligned} |g_P(r) - g_P(1)| &= \tfrac{1}{\pi}\int_0^\pi \tfrac{\omega(\tau)\left(1-r^2\right)}{1-2r\cos(\tau)+r^2}\, d\tau \qquad (14)\\ &\geq \tfrac{1-r}{\pi}\int_{1-r}^\pi \tfrac{\omega(\tau)}{1-2r\cos(\tau)+r^2}\, d\tau\ . \end{aligned}$$

For the denominator in the integrals holds $1 - 2r\cos(\tau) + r^2 \leq (1-r)^2 + \tau^2 \leq 2\tau^2$ using that $\tau \geq (1-r)$. Therewith, from (14) and (13) the inequality

$$(1-r)\int_{1-r}^\pi \tfrac{\omega(\tau)}{\tau^2}\, d\tau \leq 2\pi\ C_\omega\ \ \|\omega\|_{\mathcal{C}_\omega}\ \ \omega(1-r)$$

is obtained, which shows that ω is indeed a weak regular majorant of type 2. ∎

In this section it was shown that if ω is a weak regular majorant of type 2 and $f \in \mathcal{C}_\omega[-\pi,\pi)$ is a given function, the Poisson integral Pf has the same modulus of continuity ω in the closure of $\mathcal{D}$ as the function f itself, i.e. $Pf \in \mathcal{C}_\omega(\overline{\mathcal{D}})$.

IV. The Hilbert Transform

Now, the behavior of the conjugate Poisson integral Qf is studied. However, since $Qf = P\widetilde{f}$, in which $\widetilde{f} = Hf$ is the conjugate of f, we only need to study the Hilbert transform. Compared to the investigations of the Poisson integral, the Hilbert transform is slightly more complicated since its kernel is singular.

We start this section with a lemma which gives a sufficient condition on the function f in order that its Hilbert transform Hf exists and is continuous.

Lemma 6: *If ω is a weak regular majorant of type 1 and if $f \in \mathcal{C}_\omega[-\pi,\pi)$ then the Hilbert transform $\widetilde{f}(t) = (Hf)(t)$ exists for all $t \in [-\pi,\pi)$ and is continuous.*

Proof: Let $\epsilon > 0$ and consider the truncated Hilbert transform $H_\epsilon f$ (7). Since $\tan(\tau/2)$ is an odd function $H_\epsilon f$ can also be written as

$$(H_\epsilon f)(t) = \frac{1}{2\pi}\int_{\epsilon\leq|\tau|\leq\pi} \frac{f(t+\tau) - f(t)}{\tan(\tau/2)}\, d\tau\ .$$

With the assumption that $f \in \mathcal{C}_\omega[-\pi,\pi)$, the following upper bound for the modulus of $H_\epsilon f$ is obtained

$$\begin{aligned} |(H_\epsilon f)(t)| &\leq \frac{1}{2\pi}\int_{\epsilon\leq|\tau|\leq\pi} \frac{|f(t+\tau) - f(t)|}{|\tan(\tau/2)|}\, d\tau \\ &\leq \|f\|_{\mathcal{C}_\omega}\ \frac{1}{\pi}\int_\epsilon^\pi \frac{\omega(\tau)}{\tan(\tau/2)}\, d\tau \end{aligned}$$

and finally with $\tan(\tau/2) \geq \tau/2$ for all $0 \leq \tau \leq \pi$, the upper bound becomes

$$|(H_\epsilon f)(t)| \leq \|f\|_{\mathcal{C}_\omega}\ \frac{2}{\pi}\int_0^\pi \frac{\omega(\tau)}{\tau}\, d\tau\ \leq \|f\|_{\mathcal{C}_\omega}\, \frac{2}{\pi}\ C\ \omega(\pi).$$

The last integral always exists, since ω is a weak regular majorant of type 1. This result shows that $|(H_\epsilon f)(t)|$ is uniformly bounded for all t. Therefore, $H_\epsilon f$ converges for $\epsilon \to 0$ to the Hilbert transform Hf. ∎

Next, the smoothness behavior of the Hilbert transform is investigated. We look for a sufficient condition on the smoothness of f such that Hf has the same modulus of continuity as f. Lemma 6 already shows that if ω is a weak regular majorant of type 1 and if $f \in \mathcal{C}_\omega[-\pi,\pi)$ then the Hilbert transform $\widetilde{f} = Hf$ always exists and is continuous. However, in order that $\widetilde{f}$ has the same modulus of continuity as f, ω has to be regular. To prove this, the following two lemmas, which investigate the singularity of the Hilbert transform, are needed. They show how smooth the truncated Hilbert transform $(H_\epsilon f)(t)$ converges to $\widetilde{f}$ as ϵ approaches zero.

Lemma 7: *If ω is a weak regular majorant of type 1, and if $f \in \mathcal{C}_\omega[-\pi,\pi)$, then there exists a constant C, dependent only on ω, such that*

$$\left|\widetilde{f}(t) - (H_\epsilon f)(t)\right| \leq C\ \omega(\epsilon)\ . \qquad (15)$$

for all $t \in [-\pi,\pi)$ and $\epsilon \geq 0$.

Let ω be a majorant, we define the following function $g_2(t) := g(t)\cdot\varphi(t)$ in which $g(t)$ is defined by

$$g(t) := \begin{cases} -\omega(-t) & \text{for} \quad -\pi \leq t < 0 \\ \omega(t) & \text{for} \quad 0 \leq t < \pi \end{cases}$$

and $\varphi(t)$ is a function which 1) is constant 1 close to zero, i.e. $\varphi(t) = 1$ for $|t| < \epsilon$ for a certain $\epsilon > 0$, which 2) becomes zero at $t = \pm\pi$, and 3) which is infinity times differentiable in $[-\pi,\pi)$. By this definition, $g_2(t)$ is an odd function with respect to $t = 0$, with $g_2(-\pi) = g_2(\pi) = 0$, and with $|g_2(|t|)| = \omega(|t|)$ for $|t| < \epsilon$. Moreover, if it is assumed that $\omega \in \mathcal{C}_\omega[0,\pi]$ and because of the properties of the function $\varphi(t)$, it is easy to see that also $g_2(t)$ belongs to $\mathcal{C}_\omega[-\pi,\pi)$.

Lemma 8: *Let ω be a majorant and let $g_2 \in \mathcal{C}_\omega[-\pi,\pi)$ be a function as defined above. If for all $f \in \mathcal{C}_\omega[-\pi,\pi)$ there exists a constant C such that* (15) *is fulfilled for all $t \in [-\pi,\pi)$, then ω is a weak regular majorant of type 1.*

Proof: (of Lemma 7) Lemma 6 shows that the Hilbert transform $\widetilde{f}(t)$ always exist under the hypothesis of this lemma. It follows with (7)

$$\widetilde{f}(t) - (H_\epsilon f)(t) = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} \frac{f(t+\tau) - f(t)}{\tan(\tau/2)}\, d\tau\ .$$

Using similar arguments as in the proof of Lemma 6, the statement of this lemma is easily obtained:

$$\left|\widetilde{f}(t) - (H_\epsilon f)(t)\right| \leq \|f\|_{\mathcal{C}_\omega}\, \frac{2}{\pi}\int_0^\epsilon \frac{\omega(\tau)}{\tau}\, d\tau \leq \|f\|_{\mathcal{C}_\omega}\, \frac{2}{\pi}\, C\, \omega(\tau)$$

which concludes the proof. ∎

Proof: (of Lemma 8) The conditions of the lemma implies that the Hilbert transform $\widetilde{g}_2(t)$ of $g_2(t)$ exists. Therefore for this special function g_2 holds

$$|\widetilde{g}_2(t) - (H_\epsilon g_2)(t)| = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} \frac{g_2(t+\tau)}{\tan(\tau/2)}\, d\tau \geq \frac{1}{\pi}\int_0^{\epsilon} \frac{\omega(\tau)}{\tan(\tau/2)}\, d\tau \geq \frac{1}{C_1}\frac{2}{\pi}\int_0^{\epsilon}\frac{\omega(\tau)}{\tau}\, d\tau$$

wherein for the first inequality it was used that ω is a monotone increasing function and that the kernel of the integral is an even function with respect to zero. The second inequality follows from the properties of the tangent-function: to any $0 \leq \epsilon < \pi$ there exists a constant C_1 such that $\tan(\tau) \leq C_1 \tau$ for all $0 \leq \tau \leq \epsilon$. This lower bound for $|\widetilde{g}_2(t) - (H_\epsilon g_2)(t)|$ and the assumption (15) of the lemma proves that ω is a weak regular majorant of type 1. ∎

After this preparations, we are now able to prove the main result of this section. It gives a sufficient condition on the majorant ω such that the conjugate $\widetilde{f}$ is in the same class $\mathcal{C}_\omega[-\pi,\pi)$ as the function f itself.

THEOREM 9: *Let ω be a regular majorant and let $f \in \mathcal{C}_\omega[-\pi,\pi)$, then it holds $\widetilde{f} = (Hf) \in \mathcal{C}_\omega[-\pi,\pi)$, and there exists a constant C such that $\|Hf\|_{\mathcal{C}_\omega} \leq C\,\|f\|_{\mathcal{C}_\omega}$.*

REMARK 4: It can also be proved that it is indeed necessary that ω is weak regular of type 1 and 2 in order that $\widetilde{f}$ has the same modulus of continuity as f. However, since this proof is somewhat more elaborate and because of lack of space this result is not proved here.

Proof: It has to be shown that there exist a constant C such that

$$\left|\widetilde{f}(t_1) - \widetilde{f}(t_2)\right| = |(Hf)(t_1) - (Hf)(t_2)| \leq C\cdot\omega(|t_1 - t_2|) \tag{16}$$

for all t_1, $t_2 \in [-\pi,\pi)$. For any arbitrary ϵ, it is obviously

$$\left|\widetilde{f}(t_1) - \widetilde{f}(t_2)\right| \leq \left|\widetilde{f}(t_1) - (H_\epsilon f)(t_1)\right| + |(H_\epsilon f)(t_1) - (H_\epsilon f)(t_2)| + \left|(H_\epsilon f)(t_2) - \widetilde{f}(t_2)\right| \tag{17}$$

Now, ϵ is chosen as $\epsilon = |t_1 - t_2|/2$. Therewith and with Lemma 7 an upper bound of the form $C_i \cdot \omega(|t_1 - t_2|)$ is obtained for the first and the third term on the right hand side of (17) with some constants C_1 and C_3, respectively, and with the auxiliary Lemma 12 in the Appendix an upper bound $C_2 \cdot \omega(|t_1 - t_2|)$ for the second term on the right hand side of (17). Altogether, this shows that indeed (16) holds for for all t_1, t_2 in $[-\pi,\pi)$. ∎

Note that in contrast to the preceding two lemmas, it is assumed in this theorem that ω is a regular majorant, i.e. it is sufficient that ω satisfies both conditions (10) and (9) in order that from $f \in \mathcal{C}_\omega[-\pi,\pi)$ always follows that also $\widetilde{f} \in \mathcal{C}_\omega[-\pi,\pi)$.

The conjugate Poisson integral $v(z) = (Qf)(z)$ of $F(z)$ can be determined as the Poisson integral of the conjugate function $\widetilde{f}$: $v = (P\widetilde{f})$. Knowing this, the following corollary on the smoothness of Qf in the closure of the unit disk $\overline{\mathcal{D}}$ is obtained directly from Theorem 4 and Theorem 9.

COROLLARY 10: *If ω is a regular majorant and if $f \in \mathcal{C}_\omega[-\pi,\pi)$, then $Qf \in \mathcal{C}_\omega(\overline{\mathcal{D}})$, and there exists a constant C such that $\|Qf\|_{\mathcal{C}_\omega(\overline{\mathcal{D}})} \leq C\,\|f\|_{\mathcal{C}_\omega}$.*

V. CONTINUITY PROPERTIES OF $H(z)$ AND $F_+(z)$

At the end, we ascertain the consequences of our results on the smoothness of the filters $F_+(z)$ and $H(z)$ from section I. Thereby, it is assumed that $\Phi(t) > 0$ for all $t \in [-\pi,\pi)$. This assumption is no limitation if $H(z)$ is considered, because if $\Phi(t)$ would have any zero, the resulting filter $H(z)$ is no longer stable, in general.

Let ω be a regular majorant and $\Phi \in \mathcal{C}_\omega[-\pi,\pi)$. We consider first the outer function $F_+(z)$ given by (3). Since it is assumed that $\Phi(t) > 0$, it is easily shown that also $\log\Phi \in \mathcal{C}_\omega[-\pi,\pi)$. Now from Theorem 4 and Corollary 10 follows that in this case the integral transform (1) of $\log\Phi$ is in $\mathcal{C}_\omega(\overline{\mathcal{D}})$. And since the exponential function of a $\mathcal{C}_\omega(\overline{\mathcal{D}})$-function is again in $\mathcal{C}_\omega(\overline{\mathcal{D}})$, (3) shows that the outer function $F_+(z)$ is in $\mathcal{C}_\omega(\overline{\mathcal{D}})$ if the given spectrum Φ is in $\mathcal{C}_\omega[-\pi,\pi)$.

A similar reasoning holds also for the Wiener filter $H(z)$. Let Φ and Ψ elements of $\mathcal{C}_\omega[-\pi,\pi)$, then we already showed that the spectral factors Φ_+ and Φ_- are elements of $\mathcal{C}_\omega(\overline{\mathcal{D}})$, and since $\mathcal{C}_\omega(\overline{\mathcal{D}})$ is an algebra, $\Phi_+^{-1} \in \mathcal{C}_\omega(\overline{\mathcal{D}})$ and also $\Psi/\Phi_- \in \mathcal{C}_\omega[-\pi,\pi)$. From Theorem 4 and Corollary10 follows that $[\Psi/\Phi_-]^+ \in \mathcal{C}_\omega(\overline{\mathcal{D}})$ and again because $\mathcal{C}_\omega(\overline{\mathcal{D}})$ is an algebra, (2) shows that the Wiener filter $H(z)$ is an element of $\mathcal{C}_\omega(\overline{\mathcal{D}})$.

VI. CONCLUSIONS AND OUTLOOK

This article gave sufficient conditions on the smoothness of the data such that the Poisson and Hilbert integral has the same modulus of continuity as the given data. For the Poisson integral, the majorant of the data has to be *weak regular of type 2*. For the Hilbert integral the majorant has to be *additionally weak regular of type 1*. The results where applied to investigate the smoothness of the outer function and the Wiener filter.

It can be proofed that the regularity of the majorant ω is not only sufficient (as shown in this paper) but also necessary in order that the integral operators (1) is continuous, and the same holds for (3) with coercive Φ.

VII. ACKNOLEDGMENT

We would like to thank the anonymous reviewer for the notice of reference [5].

APPENDIX

LEMMA 11: *Let ω be a weak regular majorant of type 2. There exists a constant C such that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\omega(\left|e^{i\tau} - e^{it}\right|)(1-r^2)}{1 - 2r\cos(\tau - t) + r^2}\, d\tau \leq C\,\omega(1-r) \tag{18}$$

for $1/2 \leq r \leq 1$ and for all $t \in [-\pi,\pi)$.

Proof: It is not hard to see that $\left|e^{i\tau} - e^{it}\right| \leq |\tau - t|$, and that there exists a positive constant C_1 such that

$$1 - 2r\cos(\tau - t) + r^2 \geq (1-r)^2 + C_1(\tau - t)^2 \ .$$

Therewith, the following upper bound for the left-hand side of (18), denoted by L, is obtained

$$\begin{aligned} L \quad &\leq \tfrac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\tfrac{\omega(|\tau-t|)}{(1-r)^2+C_1(\tau-t)^2}\ d\tau \\ &\leq \tfrac{1-r^2}{\pi}\int_{0}^{\pi}\tfrac{\omega(\phi)}{(1-r)^2+C_1\phi^2}\ d\phi \end{aligned} \qquad (19)$$

for all $t \in [-\pi, \pi)$. The last inequality was obtain by the substitution $\phi = \tau - t$ and using that the integrand is a positive function. Now, the right hand side of (19) is split up into a sum of an integral from 0 to $1-r$, denoted by L_1, and an integral from $1-r$ to π, denoted by L_2. To obtain an upper bound for L_1, it is used that $\omega(\phi)$ is a mononon increasing function and that $C_1\phi^2 \geq 0$:

$$L_1 \leq \tfrac{(1-r^2)\omega(1-r)}{\pi(1-r)^2}\int_0^{1-r}d\phi \ \leq \tfrac{1}{\pi}\ \omega(1-r)$$

Similarly, the following upper bound for L_2 is obtained

$$L_2 \leq \tfrac{1+r}{\pi C_1}(1-r)\int_{1-r}^{\pi}\tfrac{\omega(\phi)}{\phi^2}\ d\phi \leq \tfrac{1}{\pi}\tfrac{C_2}{C_1}\ \omega(1-r)$$

using the assumption that ω is a weak regular majorant of type 2. This two upper bounds together with (19) prove the statement (18) of the lemma. ■

LEMMA 12: *If ω is a regular majorant and if $f \in \mathcal{C}_\omega[-\pi,\pi)$ then there exists a constant C_H such that*

$$|(H_\epsilon f)(t_1) - (H_\epsilon f)(t_2)| \leq C_H \cdot \omega(\epsilon)\ .$$

Proof: The Hilbert transform $(H_\epsilon f)(t)$ is written with the two separate kernels as in (8). Therewith it is

$$\begin{aligned} &|(H_\epsilon f)(t_1) - (H_\epsilon f)(t_2)| \\ &\leq \left|\tfrac{1}{\pi}\int_{\epsilon<|\tau|\leq\pi}\tfrac{f(\tau+t_1)-f(\tau+t_2)}{\tau}\ d\tau\right| \\ &+\left|\tfrac{1}{\pi}\int_{\epsilon<|\tau|\leq\pi}[f(\tau+t_1)-f(\tau+t_2)]\,\mathcal{K}(\tau)\ d\tau\ \right|. \end{aligned}$$

The first term and the second term on the right hand side of this inequality is denoted by $|T_1|$ and $|T_2|$, respectively. For $|T_2|$ an upper bound is immediately found, using that $f \in \mathcal{C}_\omega[-\pi,\pi)$ and that $\mathcal{K}(\tau)$ is regular at $\tau = 0$:

$$|T_2| \leq \tfrac{2}{\pi}\ \omega(|t_1-t_2|)\int_0^{\pi}|\mathcal{K}(\tau)|\ d\tau = C_{T_2}\ \omega(|t_1-t_2|)\,.$$

In the following, ϵ is chosen to be $\epsilon = |t_1 - t_2|/2$. Therewith the previous bound becomes

$$|T_2| \leq 2\ C_{T_2}\ \omega(\epsilon)\ . \qquad (20)$$

The first term $|T_1|$ is written as the difference of two integrals. After a variable substitution in both integrals, $|T_1|$ becomes

$$\begin{aligned} |T_1| \leq &\left|\tfrac{1}{\pi}\int_{I_0}[f(\tau)-f(t_1)]\left(\tfrac{1}{t_1-\tau}-\tfrac{1}{t_2-\tau}\right)d\tau\right| + \\ &\left|\tfrac{1}{\pi}\int_{I_\epsilon(t_2)}\tfrac{f(\tau)-f(t_1)}{t_1-\tau}\ d\tau\right| + \left|\tfrac{1}{\pi}\int_{I_\epsilon(t_1)}\tfrac{f(\tau)-f(t_1)}{t_2-\tau}\ d\tau\right| \end{aligned}$$

The three terms on the right hand side of the last inequality are denoted by $|L_1|$, $|L_2|$ and $|L_3|$, respectively. The integration intervals in these three integrals are defined as $I_\epsilon(t_i) := \{\tau : t_i - \epsilon \leq \tau \leq t_i + \epsilon\}$ and $I_0 = \{\tau \in [-\pi,\pi) : \tau \notin I_\epsilon(t_1), \tau \notin I_\epsilon(t_2)$. Now upper bounds are derived for all three terms separately. First, $|L_1|$ is considered. Because of the special choice for ϵ, it is $|t_1 - t_2| = 2\epsilon$ and it holds $|\tau - t_2| \geq |\tau - t_1|/3$ for all $\tau \in I_0$. Therewith, the following upper bound for $|L_1|$ is obtained

$$\begin{aligned} |L_1| \qquad &\leq \tfrac{2\epsilon}{\pi}\int_{I_0}\tfrac{|f(\tau)-f(t_1)|}{|(t_1-\tau)(t_2-\tau)|}\ d\tau \\ &\leq \tfrac{6\epsilon}{\pi}\,\|f\|_{\mathcal{C}_\omega}\int_{\epsilon\leq|\tau-t_1|\leq\pi}\tfrac{\omega(|\tau-t_1|)}{|\tau-t_1|^2}\ d\tau \end{aligned} \qquad (21)$$

using the assumption that $f \in \mathcal{C}_\omega[-\pi,\pi)$, and after the variable substitution $s := \tau - t_1$ this bound becomes

$$\begin{aligned} |L_1| &\leq \tfrac{12}{\pi}\,\|f\|_{\mathcal{C}_\omega}\cdot\epsilon\int_\epsilon^\pi\tfrac{\omega(s)}{s^2}\ ds \\ &\leq \tfrac{12}{\pi}\,\|f\|_{\mathcal{C}_\omega}\,C\cdot\omega(\epsilon)\ \leq\ C_{L_1}\cdot\omega(\epsilon) \end{aligned} \qquad (22)$$

using the fact that ω is a regular majorant and in particular that ω satisfies (10). For the second term $|L_2|$ the following upper bound is obtained using again that $f \in \mathcal{C}_\omega$

$$|L_2| \leq \tfrac{1}{\pi}\,\|f\|_{\mathcal{C}_\omega}\int_{t_2-\epsilon}^{t_2+\epsilon}\tfrac{\omega(|\tau-t_1|)}{|\tau-t_1|}\ d\tau \leq \tfrac{1}{\pi}\,\|f\|_{\mathcal{C}_\omega}\int_\epsilon^{3\epsilon}\tfrac{\omega(s)}{s}ds$$

where it was used that $|t_1 - t_2| = 2\epsilon$. Now it is used that ω is a regular majorant, i.e. $\omega(3\epsilon)/3\epsilon \leq \omega(\epsilon)/\epsilon$ and there exist a constant C such that (9) is fulfilled. Therewith the upper bound becomes

$$|L_2| \leq \tfrac{3}{\pi}\,\|f\|_{\mathcal{C}_\omega}\,C\cdot\omega(\epsilon) \leq C_{L_2}\cdot\omega(\epsilon)\ .$$

With similar arguments and using again that $|\tau - t_2| \geq |\tau - t_1|/3$, an upper bound for the last therm $|L_3|$ is obtained

$$|L_3| \leq \tfrac{6}{\pi}\,\|f\|_{\mathcal{C}_\omega}\,C\cdot\omega(\epsilon) \leq C_{L_3}\cdot\omega(\epsilon)\ .$$

All the three single bounds (22),(23),(23) together give an upper bound for $|T_1|$: $|T_1| \leq C_{T_1}\cdot\omega(\epsilon)$ in which $C_{T_1} = C_{L_1} + C_{L_2} + C_{L_3}$. Together with (20) this proves the statement of the theorem with $C_H = C_{T_1} + C_{T_2}$. ■

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