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Further Results on Structural Assignment of Linear Systems

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Abstract— **This paper considers the problem of assigning structural properties of a linear system through sensor selection. The problem is, for a given matrix pair (A, B), to find an output matrix pair (C, D) such that the resulting linear system (A, B, C, D) has the pre-specified structural properties, such as the finite and infinite zero structures and the invertibility properties. Both the assignability of certain structural properties is established and an algorithm for explicitly constructing the matrices (C, D) that result in these properties is developed. In particular, by introducing the notion of infinite zero assignable sets for the pair (A, B), we establish necessary conditions under which a given set of structural properties can be assigned. Motivated by these necessary conditions, we establish a set of necessary and sufficient conditions for the assignability of a set of structural properties which includes left invertibility property. These necessary and sufficient conditions indicate the conservativeness of the existing conditions. In establishing these conditions, we develop a numerical algorithm for the construction of the required output matrix pair (C, D).**

Keywords: Linear systems, structural properties, invariant zeros, infinite zeros, structural assignment.

I. INTRODUCTION

Structural properties of linear systems, such as the finite and infinite zero structures and the invertibility properties, have played a very important role in many linear systems and control areas, including, robust and H_∞ control (see, *e.g.*, [2], [9]), H_2 optimal control [15], and control with saturation [9]. One of the major obstacles to successful applications of multivariable control synthesis techniques, such as H_2 and H_∞ control techniques, to practical control problems is the lack of adequate understanding of the linkage between achievable control performances and hardware implementation such as the selection and locations of sensors and actuators. Indeed, this linkage provides a foundation upon which trade-offs can be incorporated in the preliminary design stage. Thus, one can introduce careful control design considerations into the overall engineering design process in an early stage. For example, it is well understood in the literature that nonminimum-phase zeros are always troublesome to deal with. However, simple examples show that such troublesome nonminimum-phase zeros can be removed by properly adding, removing or relocating sensors and actuators. This is exactly what motivated our interest in the problem of structural assignment. This problem is, for a linear system characterized by a matrix pair (A, B),

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{1}$$

to find a matrix pair (C, D) that define a system output

$$y = Cx + Du, \tag{2}$$

such that the resulting linear system characterized by the matrix quadruple (A, B, C, D) has the pre-specified structural properties, such as the finite and infinite zero structures and the invertibility properties.

It is appropriate to trace a short history in the study of the problem of structural assignment for linear systems. Most results in the literature pertain to the assignment of finite zero (invariant zero or transmission zero) structures (see, *i.e.*, [6], [7], [8], [13], [14], [20], [18], [19]). In 1995, [4] proposed a technique which is capable of simultaneously assigning finite and infinite zero structures. Recently, we successfully attempted to deal with the assignment of complete system structures, including finite and infinite zero structures and invertibility structures [11]. In particular, in [11], we identified a set of sufficient conditions, and under these conditions, an algorithm that leads to the assignment of a set of complete structural properties is developed. In 2004, by using the similar technique of [14], the authors of [1] presented the necessary and sufficient conditions under which an infinite zero structure can be assigned. Other structural properties, such as finite zero structure and invertibility properties were not considered. Moreover, the tool they used to established these necessary and sufficient conditions is rational function matrix, which, though mathematically elegant, does not lead to computational algorithms to construct the required output matrix pair (C, D).

In this paper, we present some further results on the problem of structural assignment. We will first introduce the notion of infinite zero assignable sets. With this notion, we establish necessary conditions under which a given set of structural properties can be assigned. Motivated by these necessary condition, we establish a set of necessary and sufficient conditions for the assignability of a set of structural properties which includes left invertibility property. These necessary and sufficient conditions indicate the conservativeness of the existing conditions. In establishing these conditions, we develop a numerical algorithm for the construction of the required output matrix pair (C, D).

The remainder of the paper is organized as follows. Section II includes some background materials that are needed in our work. Section III presents our preliminary results which will lead to our main results in Section IV. Section V concludes the paper.

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II. BACKGROUND MATERIALS

We recall in this section the special coordinate basis decomposition of linear systems, which was introduced in [17], [16]. A toolkit [10] in the MATLAB environment containing such a structural decomposition is currently available online at the website `http://linearsystemskit.net`. The special coordinate basis, implemented in the toolkit [10], is based on a numerically stable algorithm recently reported in [5], together with an enhanced procedure reported in [3].

Let us consider a linear system Σ,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{3}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the input and the output, respectively. Without loss of generality, we assume that both $[\, B^{\mathrm{T}} \;\; D^{\mathrm{T}} \,]$ and $[\, C \;\; D \,]$ are of full rank.

Theorem 2.1 (SCB): Given the system Σ of (3), there exist nonsingular state, output and input transformations $\Gamma_{\rm s}$, $\Gamma_{\rm o}$ and $\Gamma_{\rm i}$, such that

$$\begin{aligned} \tilde{A} &= \Gamma_{\rm s}^{-1} A \Gamma_{\rm s} = A_{\rm s} + B_0 C_0 \\ &= \begin{bmatrix} A_{\rm aa} & L_{\rm ab} C_{\rm b} & 0 & L_{\rm ad} C_{\rm d} \\ 0 & A_{\rm bb} & 0 & L_{\rm bd} C_{\rm d} \\ E_{\rm ca} & L_{\rm cb} C_{\rm b} & A_{\rm cc} & L_{\rm cd} C_{\rm d} \\ B_{\rm d} E_{\rm da} & B_{\rm d} E_{\rm db} & B_{\rm d} E_{\rm dc} & A_{\rm dd} \end{bmatrix} \\ &\quad + \begin{bmatrix} B_{\rm 0a} \\ B_{\rm 0b} \\ B_{\rm 0c} \\ B_{\rm 0d} \end{bmatrix} [\, C_{\rm 0a} \;\; C_{\rm 0b} \;\; C_{\rm 0c} \;\; C_{\rm 0d} \,], \end{aligned} \tag{4}$$

$$\tilde{B} = \Gamma_{\rm s}^{-1} B \Gamma_{\rm i} = [\, B_0 \;\; B_{\rm s} \,] = \begin{bmatrix} B_{\rm 0a} & 0 & 0 \\ B_{\rm 0b} & 0 & 0 \\ B_{\rm 0c} & 0 & B_{\rm c} \\ B_{\rm 0d} & B_{\rm d} & 0 \end{bmatrix}, \tag{5}$$

$$\tilde{C} = \Gamma_{\rm o}^{-1} C \Gamma_{\rm s} = \begin{bmatrix} C_0 \\ C_{\rm s} \end{bmatrix} = \begin{bmatrix} C_{\rm 0a} & C_{\rm 0b} & C_{\rm 0c} & C_{\rm 0d} \\ 0 & 0 & 0 & C_{\rm d} \\ 0 & C_{\rm b} & 0 & 0 \end{bmatrix}, \tag{6}$$

$$\tilde{D} = \Gamma_{\rm o}^{-1} D \Gamma_{\rm i} = D_{\rm s} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

where

$$A_{\rm dd} = A_{\rm dd}^* + B_{\rm d} E_{\rm dd} + L_{\rm dd} C_{\rm d}$$

for some constant matrices $L_{\rm dd}$ and $E_{\rm dd}$ of appropriate dimensions, and

$$A_{\rm dd}^* = \text{blkdiag}\left\{ A_{q_1}, A_{q_2}, \ldots, A_{q_{m_{\rm d}}} \right\},$$

$$B_{\rm d} = \text{blkdiag}\left\{ B_{q_1}, B_{q_2}, \ldots, B_{q_{m_{\rm d}}} \right\},$$

$$C_{\rm d} = \text{blkdiag}\left\{ C_{q_1}, C_{q_2}, \ldots, C_{q_{m_{\rm d}}} \right\},$$

with $(A_{q_i}, B_{q_i}, C_{q_i})$ being defined as

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i - 1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1, \; 0, \; \ldots, \; 0].$$

Remark 2.1: The structural decomposition shows explicitly the finite zero and infinite zero structure, as well as left and right invertibility structures of the system Σ.

- The finite zero sturcture of Σ is characterized by the eigenstructure of $A_{\rm aa}$;
- Left invertibility structure $S_{\rm L}^\star(\Sigma)$ is the observability indices of $(A_{\rm bb}, C_{\rm b})$, and right invertibility structure of $S_{\rm R}^\star(\Sigma)$ is the controllability indices of $(A_{\rm cc}, B_{\rm c})$;
- Σ has $m_0 = \text{rank}(D)$ infinite zeros of order 0. The infinite zero structure (of order greater than 0) of Σ is given by $S_\infty^\star(\Sigma) = \{q_1, q_2, \cdots, q_{m_{\rm d}}\}$. That is, each q_i corresponds to an infinite zero of Σ of order q_i.
- The finite zero structure corresponds to Morse invariant index list $\mathcal{I}_1$[12]. Furthermore, $S_{\rm R}^\star$. $S_{\rm L}^\star$ and $S_\infty^\star$ are corresponding to lists $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ of Morse, respectively. Also, Σ is left invertible if $S_{\rm R}^\star$ is empty, right invertible if $S_{\rm L}^\star$ is empty, invertible if both $S_{\rm R}^\star$ and $S_{\rm L}^\star$ are empty, and degenerate if both $S_{\rm R}^\star$ and $S_{\rm L}^\star$ are present.

III. PRELIMINARY RESULTS

Definition 3.1: Consider a matrix pair (A, B) with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Then, a set $\{\eta_1, \eta_2, \ldots, \eta_\varpi\}$ is called a restricted infinite zero assignable set of (A, B), if $\eta_1, \eta_2, \ldots, \eta_\varpi$ are positive integers, $\varpi \le m$, and there exist a nonsingular matrix $T_I \in \mathbb{R}^{m \times m}$, such that

$$\begin{aligned} \mathbb{P} = [\; & b_1 \;\; Ab_1 \;\; \ldots \;\; A^{\eta_1 - 1} b_1 | \;\; b_2 \;\; Ab_2 \;\; \ldots \;\; A^{\eta_2 - 1} b_2 | \\ & \ldots | \;\; b_\varpi \;\; Ab_\varpi \;\; \ldots \;\; A^{\eta_\varpi - 1} b_\varpi \;] \end{aligned}$$

is of full column rank, where b_k is the kth column of BT_I.

Definition 3.2: A positive integers set $\{\tau_1, \tau_2, \ldots, \tau_\varpi\}$ is called an infinite zero assignable set of (A, B), if there exist a feedback gain $K \in \mathbb{R}^{m \times n}$, such that $\{\tau_1, \tau_2, \ldots, \tau_\varpi\}$ is a restricted infinite zero assignable set of $(A - BK, B)$.

We next establish the following lemmas, which are crucial to the establishment of the main results in Section IV.

Lemma 3.1: Consider (A, B) with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$. Let the set of integers $\{\eta_1, \eta_2, \ldots, \eta_\varpi\}$ be a restricted infinite zero assignable set of (A, B). Then, there exist nonsingular $T_{\rm s} \in \mathbb{R}^{n \times n}$ and $T_{\rm I} \in \mathbb{R}^{m \times m}$ such that

$$T_{\rm s}^{-1} A T_{\rm s} = \begin{bmatrix} A_0 & \star & 0 & \cdots & \star & 0 \\ \star & \star & I_{\eta_1 - 1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & 0 & \cdots & \star & I_{\eta_\varpi - 1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix},$$

$$T_{\rm s}^{-1} B T_{\rm I} = \begin{bmatrix} 0 & \cdots & 0 & B_0 \\ 0 & \cdots & 0 & \star \\ 1 & \cdots & 0 & \star \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \star \\ 0 & \cdots & 1 & \star \end{bmatrix},$$

where $A_0 \in \mathbb{R}^{n_o \times n_o}$ and $B_0 \in \mathbb{R}^{n_o \times (m - \varpi)}$ with $n_o = n - \Sigma_{k=1}^{\varpi} \eta_k$.

Proof. By Definition 3.1, there exists a nonsingular $T_{\rm I} \in \mathbb{R}^{m \times m}$,

$$BT_{\rm I} = [\, b_1 \;\; b_2 \;\; \ldots \;\; b_m \,],$$

such that

$$T_1 = \begin{bmatrix} A^{\eta_1-1}b_1 & A^{\eta_1-2}b_1 & \dots & b_1 | A^{\eta_2-1}b_2 & A^{\eta_2-2}b_2 \\ \dots & b_2 | \dots | A^{\eta_\varpi-1}b_\varpi & A^{\eta_\varpi-2}b_\varpi & \dots & b_\varpi \end{bmatrix} \quad (8)$$

is of full column rank. Let T_0 be a constant matrix such that

$$T_{\rm s} := [\,T_0 \quad T_1\,]$$

is nonsingular. From (8), we have

$$\begin{aligned} BT_{\rm I} &= [T_{\rm s}e_{g_2} \;\; T_{\rm s}e_{g_3} \dots T_{\rm s}e_n \mid \; b_{\varpi+1} \;\; \dots \;\; b_m] \\ &= T_{\rm s}[\,e_{g_2}\; e_{g_3} \dots e_n \mid T_{\rm s}^{-1}b_{\varpi+1} \dots T_{\rm s}^{-1}b_m], \quad (9) \\ AT_s &= [\; AT_0 | \;\; A^{\eta_1}b_1 \;\; T_{\rm s}e_{g_1+1} \;\; \dots \;\; T_{\rm s}e_{g_2} | \; \dots \\ &\quad \dots \;\; | \;\; A^{\eta_\varpi}b_\varpi \;\; T_{\rm s}e_{g_\varpi+1} \;\; \dots \;\; T_{\rm s}e_{n-1} \;] \\ &= T_{\rm s}\,[\,T_{\rm s}^{-1}AT_0 | \;\; T_{\rm s}^{-1}A^{\eta_1}b_1 \;\; e_{g_1+1} \;\; \dots \;\; e_{g_2} | \\ &\quad \dots | \; T_{\rm s}^{-1}A^{\eta_\varpi}b_\varpi \;\; e_{g_\varpi+1} \;\; \dots \;\; e_{n-1} \;], \quad (10) \end{aligned}$$

where $g_k = n_o + \Sigma_{l=1}^{k-1}\eta_l, \;\; k = 1, \dots, \varpi$.

Left multiplying both sides of (9) and (10) by $T_{\rm s}^{-1}$, we obtain the results of the lemma. ∎

Lemma 3.2: Consider a linear system characterized by a matrix triple (A, B, C) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$,

$$A = \begin{bmatrix} A_0 & \alpha_1 & 0 & \cdots & \alpha_m & 0 \\ \star & \star & I_{\tau_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & 0 & \cdots & \star & I_{\tau_m-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where $A_0 \in \mathbb{R}^{n_o\times n_o}$ and $\alpha_1, \dots, \alpha_m \in \mathbb{R}^{n_o\times 1}$, with $n_o = n - \Sigma_{k=1}^{m}\tau_k$. Then, there exists a state transformation $T_{\rm s}$, such that

$$A_1 = T_{\rm s}^{-1}AT_{\rm s} = \begin{bmatrix} A_0 & \alpha_1 & 0 & \cdots & \alpha_m & 0 \\ 0 & \star & I_{\tau_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{\tau_m-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix} \quad (11)$$

$$B_1 = T_{\rm s}^{-1}B = B, \quad (12)$$

$$C_1 = CT_{\rm s} = C, \quad (13)$$

i.e., the triple (A, B, C) is invertible, with its invariant zeros given by the eigenvalues of A_0 and its infinite zeros being $\{\tau_1, \tau_2, \dots, \tau_m\}$.

Proof. Let

$$x = \begin{pmatrix} x_0 \\ x_d \end{pmatrix}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix},$$

$$x_i = \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,\tau_i} \end{pmatrix}, \quad i = 1, 2, \dots, m.$$

Then the system (A, B, C) can be written as

$$\begin{aligned} \dot{x}_0 &= A_0x_0 + \Sigma_{k=1}^{m}\alpha_k x_{i,1}, \\ \dot{x}_{i,j} &= A_{i,j}x_0 + x_{i,j+1} + \Sigma_{k=1}^{m}a_{i,j,k}x_{i,1}, \\ &i = 1, 2, \dots, m, \quad j = 1, 2, \dots, \tau_i, \\ \dot{x}_{i,\tau_i} &= A_{i,\tau_i}x_0 + \Sigma_{k=1}^{m}a_{i,j,k}x_{i,1} + u_i, \quad i = 1, 2, \dots, m. \end{aligned}$$

Define

$$x_{i,2}^1 := -A_{i,1}x_0 + x_{i,2},$$

we have

$$\dot{x}_{i,1} = x_{i,2}^1 + \Sigma_{k=1}^{m}a_{i,1,k}x_{i,1},$$

and

$$\begin{aligned} \dot{x}_{i,2}^1 &= -A_{i,1}\dot{x}_0 + \dot{x}_{i,2} = (A_{i,2} - A_{i,1}A_0)x_0 \\ &\quad + x_{i,3} + \Sigma_{k=1}^{m}(a_{i,2,k} - A_{i,1}\alpha_k)x_{i,1} \\ &:= A_{i,2}^1x_0 + x_{i,3} + \Sigma_{k=1}^{m}a_{i,2,k}^1x_{i,1}, \end{aligned}$$

Similarly, defining

$$x_{i,3}^1 := -A_{i,2}^1x_0 + x_{i,3},$$

we have

$$\dot{x}_{i,2} = x_{i,3}^1 + \Sigma_{k=1}^{m}a_{i,2,k}x_{i,1},$$

$$\begin{aligned} \dot{x}_{i,3}^1 &= -A_{i,2}^1\dot{x}_0 + \dot{x}_{i,3} \\ &:= A_{i,3}^1x_0 + x_{i,4} + \Sigma_{k=1}^{m}a_{i,3,k}^1x_{i,1}. \end{aligned}$$

Proceeding recursively, we finally obtain

$$\begin{aligned} \dot{x}_{i,1} &= x_{i,2}^1 + \Sigma_{k=1}^{m}a_{i,1,k}x_{i,1}, \\ \dot{x}_{i,j}^1 &= x_{i,j+1}^1 + \Sigma_{k=1}^{m}a_{i,j,k}^1x_{i,1}, \\ &i = 1, 2, \dots, m, \quad j = 2, 3, \dots, \tau_i, \\ \dot{x}_{i,\tau_i} &= A_{i,\tau_i}^1x_0 + \Sigma_{k=1}^{m}a_{i,j,k}^1x_{i,1} + u_i, \quad i = 1, 2, \dots, m. \end{aligned}$$

Thus, the desired matrix $T_{\rm s}$ can be defined as follows,

$$T_{\rm s} = \begin{bmatrix} I_{n_o} & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \Phi_1 & 0 & I_{\tau_1-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & \dots & 1 & 0 \\ \Phi_m & 0 & 0 & \dots & 0 & I_{\tau_m-1} \end{bmatrix}, \quad (14)$$

with some appropriate matrices $\Phi_k \in \mathbb{R}^{(\tau_k-1)\times n_o}$, $k = 1, 2, \dots, m$. ∎

Lemma 3.3: Consider a linear system characterized by a matrix triple $(A_1, B_1, \hat{C}_1)$, where A_1 and B_1 are in the form of (11) and (12), and $\hat{C}_1$ is given by

$$\hat{C}_1 = \begin{bmatrix} \beta_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_m & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

with $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{R}^{1\times n_o}$, $n_o = n - \Sigma_{k=1}^m \tau_k$. Then, $(A_1, B_1, \hat{C}_1)$ is invertible, with its invariant zeros given by the eigenvalues of $A_0 - \Sigma_{k=1}^m \beta_k \alpha_k$ and its infinite zeros being $\{\tau_1, \tau_2, \ldots, \tau_m\}$.

Proof. Let

$$T_\mathrm{s} = \begin{bmatrix} I_{n_o} & 0 & 0 & \ldots & 0 & 0 \\ -\beta_1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_{\tau_1 - 1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\beta_m & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & I_{\tau_m - 1} \end{bmatrix},$$

where $n_o = n - \Sigma_{k=1}^m \tau_k$. We have

$$\begin{aligned} A_2 &= T_\mathrm{s}^{-1} A_1 T_\mathrm{s} \\ &= \begin{bmatrix} A_0 - \Sigma_{k=1}^m \beta_k \alpha_k & \alpha_1 & 0 & \cdots & \alpha_m & 0 \\ \star & \star & I_{\tau_1 - 1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & 0 & \cdots & \star & I_{\tau_m - 1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix}, \end{aligned}$$

$$B_2 = T_\mathrm{s}^{-1} B_1 = B_1,$$

$$C_2 = \hat{C}_1 T_\mathrm{s} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then, the result of the lemma follows from Lemma 3.2. ∎

IV. MAIN RESULTS

Our first main result gives a necessary conditions for the assignability of a set of structural properties.

Theorem 4.1: Consider a linear system (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let n_a be a non-negative integer, and

$$\Lambda_2 := \{\ell_1, \ell_2, \ldots, \ell_{m_\mathrm{c}}\}, \quad \Lambda_3 := \{\mu_1, \mu_2, \ldots, \mu_{p_\mathrm{b}}\},$$

$$\Lambda_4 := \{q_1, q_2, \cdots, q_{m_\mathrm{d}}\}$$

be three sets of positive integers. Then, there exist $C \in \mathbb{R}^{(m+p_\mathrm{b}-m_\mathrm{c})\times n}$ and $D \in \mathbb{R}^{(m+p_\mathrm{b}-m_\mathrm{c})\times m}$ such that the resulting system characterized by the matrix quadruple (A, B, C, D) has n_a invariant zeros, $m - m_\mathrm{d}$ infinite zeros of order 0, and the Morse invariant index lists $\mathcal{I}_2 = \Lambda_2$, $\mathcal{I}_3 = \Lambda_3$ and $\mathcal{I}_4 = \Lambda_4$, only if

1) $\{\ell_1, \ell_2, \ldots, \ell_{m_\mathrm{c}}, q_1, q_2, \cdots, q_{m_\mathrm{d}}\}$ is an infinite zero assignable set of (A, B);
2) $n_\mathrm{a} + \Sigma_{i=1}^{p_\mathrm{b}} \mu_i + \Sigma_{i=1}^{m_\mathrm{c}} \ell_i + \Sigma_{i=1}^{m_\mathrm{d}} q_i = n$.

Proof: There exist $T_\mathrm{s} \in \mathbb{R}^{n\times n}$ and $T_\mathrm{I} \in \mathbb{R}^{m\times m}$ such that $T_\mathrm{s}^{-1} A T_\mathrm{s}$ and $T_\mathrm{s}^{-1} B T_\mathrm{I}$ are in the form of (4) and (5), respectively. Moreover, $(A_\mathrm{cc}, B_\mathrm{c})$ is in the form of observability canonical form, *i.e.*,

$$A_\mathrm{cc} = A_\mathrm{cc}^* + B_\mathrm{c} E_\mathrm{cc},$$

for constant matrices E_cc of appropriate dimensions, and

$$A_\mathrm{cc}^* = \mathrm{blkdiag}\Big\{A_{\ell_1}, A_{\ell_2}, \ldots, A_{\ell_{m_\mathrm{c}}}\Big\},$$

$$B_\mathrm{c} = \mathrm{blkdiag}\Big\{B_{\ell_1}, B_{\ell_2}, \ldots, B_{\ell_{m_\mathrm{c}}}\Big\},$$

with

$$A_{\ell_i} = \begin{bmatrix} 0 & I_{\ell_i - 1} \\ 0 & 0 \end{bmatrix}, \quad B_{\ell_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Define a state feedback gain matrix K as

$$K = T_\mathrm{I} \begin{bmatrix} C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \\ 0 & 0 & E_\mathrm{dc} & E_\mathrm{dd} \\ 0 & 0 & E_\mathrm{cc} & 0 \end{bmatrix} T_\mathrm{s}^{-1}.$$

Then, we have

$$\begin{aligned} \tilde{A} :&= A - BK \\ &= T_\mathrm{s} \begin{bmatrix} A_\mathrm{aa} & L_\mathrm{ab} C_\mathrm{b} & 0 & L_\mathrm{ad} C_\mathrm{d} \\ 0 & A_\mathrm{bb} & 0 & L_\mathrm{bd} C_\mathrm{d} \\ B_\mathrm{c} E_\mathrm{ca} & L_\mathrm{cb} C_\mathrm{b} & A_\mathrm{cc}^* & L_\mathrm{cd} C_\mathrm{d} \\ B_\mathrm{d} E_\mathrm{da} & B_\mathrm{d} E_\mathrm{db} & 0 & A_\mathrm{dd}^* + L_\mathrm{dd} C_\mathrm{d} \end{bmatrix} T_\mathrm{s}^{-1}. \end{aligned}$$

Let b_k be the $(m_o + k)$th column of BT_I. It can be verified that

$$\begin{aligned} \hat{\mathbb{P}} &= [\, b_1 \;\; \tilde{A} b_1 \;\; \ldots \;\; \tilde{A}^{\ell_1 - 1} b_1 \;|\; \ldots |\; b_{m_\mathrm{c}} \;\; \tilde{A} b_{m_\mathrm{c}} \;\; \ldots \\ &\quad \tilde{A}^{\ell_{m_\mathrm{c}} - 1} b_{m_\mathrm{c}} |\;\; b_{m_\mathrm{c}+1} \;\; \tilde{A} b_{m_\mathrm{c}+1} \;\; \ldots \;\; \tilde{A}^{q_1 - 1} b_{m_\mathrm{c}+1} | \\ &\quad \ldots |\;\; b_{m_\mathrm{c}+m_\mathrm{d}} \;\; \tilde{A} b_{m_\mathrm{c}+m_\mathrm{d}} \;\; \ldots \;\; \tilde{A}^{q_{m_\mathrm{d}} - 1} b_{m_\mathrm{c}+m_\mathrm{d}} \,] \\ &= T_\mathrm{s} \begin{bmatrix} 0 & 0 \\ 0 & \Delta_c \\ \Delta_d & 0 \end{bmatrix}, \end{aligned}$$

where

$$\Delta_c = \mathrm{blkdiag}\{\delta_{\ell_1}, \ldots, \delta_{\ell_{m_\mathrm{c}}}\},$$

$$\Delta_d = \mathrm{blkdiag}\{\delta_{q_1}, \ldots, \delta_{q_{m_\mathrm{d}}}\},$$

and where δ_k is the $k\times k$ matrix with the elements in the inverse diagonal being 1s, and all the other elements being 0s. Clearly, $\hat{\mathbb{P}}$ is of full column rank. Thus, $\{\ell_1, \ell_2, \ldots, \ell_{m_\mathrm{c}}, q_1, q_2, \cdots, q_{m_\mathrm{d}}\}$ is an infinite zero assignable set of (A, B).

Condition *2* is needed only for the compatibility of dimensions. ∎

In what follows, we present the necessary and sufficiency conditions for the assignabilility of a set of structural properties which includes the left invertibility property.

Theorem 4.2: Consider a linear system (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that its uncontrollable modes are distinct. Let n_a be a non-negative integer, and

$$\Lambda_3 := \{\mu_1, \mu_2, \ldots, \mu_{p_\mathrm{b}}\}, \quad \Lambda_4 := \{q_1, q_2, \cdots, q_{m_\mathrm{d}}\},$$

are two sets of positive integers. Then, there exist $C \in \mathbb{R}^{(p_\mathrm{b}+m)\times n}$ and $D \in \mathbb{R}^{(p_\mathrm{b}+m)\times m}$ such that the resulting system characterized by the matrix quadruple (A, B, C, D) is left invertible, has n_a invariant zeros, $m - m_\mathrm{d}$ infinite zeros of order 0, and the Morse invariant index lists $\mathcal{I}_3 = \Lambda_3$ and $\mathcal{I}_4 = \Lambda_4$, if and only if

1) $\{q_1, q_2, \cdots, q_{m_\mathrm{d}}\}$ is an infinite zero assignable set of (A, B);
2) $n_\mathrm{a} + \Sigma_{i=1}^{p_\mathrm{b}} \mu_i + \Sigma_{i=1}^{m_\mathrm{d}} q_i = n$.

Proof. Necessity: It is the special case of Theorem 4.1 with Λ_2 being an empty set.

Sufficiency: We will give a constructive proof that would yield the desired matrices C and D.

Since $\{q_1, q_2, \dots, q_{m_{\rm d}}\}$ is an infinite zero assignable set of (A, B), by Lemma 3.1, there exist nonsingular $T_1 \in \mathbb{R}^{n\times n}$, $T_{\rm i} \in \mathbb{R}^{m\times m}$, $K_1 \in \mathbb{R}^{m\times n}$, such that

$$
\begin{aligned}
A_1 &= T_1^{-1}(A - BK_1)T_1 \\
&= \begin{bmatrix} A_0 & \alpha_1 & 0 & \cdots & \alpha_{m_{\rm d}} & 0 \\ \star & \star & I_{q_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & 0 & \cdots & \star & I_{q_{m_{\rm d}}-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix},
\end{aligned}
$$

$$
B_1 = T_1^{-1} B T_{\rm i} = \left[\ \hat{B}_1 \ \middle|\ B_{1*}\ \right] = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & B_{10} \\ 0 & \cdots & 0 & \star \\ 1 & \cdots & 0 & \star \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \star \\ 0 & \cdots & 1 & \star \end{array}\right],
$$

where $A_0 \in \mathbb{R}^{n_o\times n_o}$, $B_{10} \in \mathbb{R}^{n_o\times m_o}$, $m_o = m - m_{\rm d}$ and $n_o = n - \Sigma_{k=1}^{m_{\rm d}} q_k$.

By Lemma 3.2, for $(A_1, \hat{B}_1)$, there exists a state transformation T_2, such that

$$
A_2 = T_2^{-1} A_1 T_2 = \begin{bmatrix} A_0 & \alpha_1 & 0 & \cdots & \alpha_{m_{\rm d}} & 0 \\ 0 & \star & I_{q_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{q_{m_{\rm d}}-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix}, \quad (15)
$$

$$
\hat{B}_2 = T_2^{-1}\hat{B}_1 = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}.
$$

Thus, there exists a state feedback gain $K_\star$ such that

$$
A_2 - \hat{B}_2 K_\star = \begin{bmatrix} A_0 & \alpha_1 & 0 & \cdots & \alpha_{m_{\rm d}} & 0 \\ 0 & \star & I_{q_1-1} & \cdots & \star & 0 \\ 0 & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{q_{m_{\rm d}}-1} \\ 0 & \star & 0 & \cdots & \star & 0 \end{bmatrix}.
$$

Let

$$
L_\alpha = [\,\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_{m_{\rm d}}\,].
$$

By the structure of $(A_2 - \hat{B}_2 K_\star,\ \hat{B}_2)$, it is clear that all uncontrollable modes of (A, B) are the uncontrollable modes of (A_0, L_α), and the remaining modes of A_0 are all controllable. The controllable modes of (A_0, L_α) can be freely relocated by state feedback with gain

$$
K_\beta = [\,\beta_1' \quad \beta_2' \quad \dots \quad \beta_{m_{\rm d}}'\,]',
$$

where $\beta_1, \beta_2, \dots, \beta_{m_{\rm d}} \in \mathbb{R}^{1\times n_o}$. That is, the eigenvalues of $A_\gamma = A_0 - L_\alpha K_\beta = A_0 - \Sigma_{k=1}^{m_{\rm d}} \beta_k \alpha_k$ include uncontrollable modes of (A, B) and any other desired modes.

Let us define

$$
\hat{C}_2 = \begin{bmatrix} \beta_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{m_{\rm d}} & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},
$$

which is in conformity with the structures of A_2 and $\hat{B}_2$ in (15). We further define the state transformation T_3

$$
T_3 = \begin{bmatrix} I_{n_o} & 0 & 0 & \dots & 0 & 0 \\ -\beta_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & I_{q_1-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\beta_{m_{\rm d}} & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & I_{q_{m_{\rm d}}-1} \end{bmatrix}.
$$

By Lemmas 3.3 and 3.2, we can find a state transformation $\tilde{T}_3$, which is in the form of (14), such that

$$
\begin{aligned}
A_3 &= (T_3\tilde{T}_3)^{-1} A_2 (T_3\tilde{T}_3) \\
&= \begin{bmatrix} A_\gamma & \alpha_1 & 0 & \cdots & \alpha_{m_{\rm d}} & 0 \\ 0 & \star & I_{q_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{q_{m_{\rm d}}-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix}, \\
\hat{B}_3 &= (T_3\tilde{T}_3)^{-1}\hat{B}_2 = \hat{B}_2,
\end{aligned}
$$

and

$$
\tilde{C}_3 = \hat{C}_2(T_3\tilde{T}_3) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.
$$

It is straightforward to see that the triple $(A_3, \hat{B}_3, \tilde{C}_3)$ is in the form of the special coordinate basis, and by the properties of the special coordinate basis, its invariant zeros are the eigenvalues of A_γ, its infinite zero structure is given by $\{q_1, q_2, \dots, q_{m_{\rm d}}\}$, and it is invertible.

The uncontrollable modes of (A_γ, K_β) are distinct, and the remaining modes of A_γ could be relocated freely by selecting K_β. For such A_γ, we can select $C_{\rm b} \in \mathbb{R}^{p_{\rm b}\times n_o}$, such that $(A_\gamma, C_{\rm b})$ has $n_{\rm a}$ unobservable modes, and observability indices $\{\mu_1, \mu_2, \dots, \mu_{p_{\rm b}}\}$. To do this, we find a transformation T_{ab} such that

$$
T_{ab}^{-1} A_\gamma T_{ab} = \begin{bmatrix} A_{\rm aa} & * \\ 0 & A_{\rm bb} \end{bmatrix},
$$

where $A_{\rm aa} \in \mathbb{R}^{n_{\rm a}\times n_{\rm a}}$ contains the desired finite zeros, and $A_{\rm bb} \in \mathbb{R}^{n_b\times n_b}$ is a diagonal matrix. We thus select

$$
C_{\rm b} = \begin{bmatrix} 0 & \xi_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_{p_{\rm b}} \end{bmatrix} T_{ab}^{-1},
$$

where ξ_i, $i = 1, 2, \dots, p_{\rm b}$, is a $1 \times \mu_i$ vector with all its entries being nonzero.

Let

$$
\hat{C}_3 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ C_{\rm b} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.
$$

It can be easily verified that $(A_3, \hat{B}_3, \hat{C}_3)$ is left invertible, has n_a invariant zeros, $\mathcal{I}_3 = \Lambda_3$ and $\mathcal{I}_4 = \Lambda_4$.

From the above process, we have

$$(T_2 T_3 \tilde{T}_3)^{-1} A_1 (T_2 T_3 \tilde{T}_3) = A_3,$$
$$(T_2 T_3 \tilde{T}_3)^{-1} \hat{B}_1 = \hat{B}_3.$$

Thus, $(A_1, \hat{B}_1, \hat{C}_3 (T_2 T_3 \tilde{T}_3)^{-1})$ has the same Morse invariant index lists as $(A_3, \hat{B}_3, \hat{C}_3)$.

Letting

$$T_\mathrm{s} = T_1 T_2 T_3 \tilde{T}_3,$$

we have

$$A_3 = T_\mathrm{s}^{-1}(A - BK_1)T_\mathrm{s},$$

$$B_3 = T_\mathrm{s}^{-1} B T_\mathrm{I} = \begin{bmatrix} 0 & \cdots & 0 & B_{10} \\ 0 & \cdots & 0 & \star \\ 1 & \cdots & 0 & \star \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \star \\ 0 & \cdots & 1 & \star \end{bmatrix}.$$

Let

$$C_3 = \begin{bmatrix} \hat{C}_3 \\ 0_{(m-m_\mathrm{d})\times n} \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0_{(p_\mathrm{b}+m_\mathrm{d})\times m_\mathrm{d}} & 0 \\ 0 & D_o \end{bmatrix},$$

where D_o is an arbitrary $(m - m_\mathrm{d}) \times (m - m_\mathrm{d})$ nonsingular matrix.

We can find a state transformation such that (A_γ, C_b) is transformed into its observability canonical form. Thus, it is easy to see that (A_3, B_3, C_3, D_3) has n_a finite zeros and $m - m_\mathrm{d}$ infinite zeros of order 0, its infinite zeros of order greater than 0 are $\{q_1, q_2, \ldots, q_{m_\mathrm{d}}\}$ (*i.e.*, $\mathcal{I}_4 = \Lambda_4$) and left invertibility structure is $\{\mu_1, \mu_2, \ldots, \mu_{p_\mathrm{b}}\}$ (*i.e.*, $\mathcal{I}_3 = \Lambda_3$).

Applying a state feedback with the gain $T_\mathrm{I}^{-1} K_1 T_\mathrm{s}$ to the system (A_3, B_3, C_3, D_3) results in the system $(T_\mathrm{s}^{-1} A T_\mathrm{s},\ T_\mathrm{s}^{-1} B T_\mathrm{I},\ C_3 + D_3 T_\mathrm{I}^{-1} K_1 T_\mathrm{s},\ D_3)$. Thus, we obtain the desired C and D as

$$\begin{aligned} C &= (C_3 + D_3 T_\mathrm{I}^{-1} K_1 T_\mathrm{s}) T_\mathrm{s}^{-1} = C_3 T_\mathrm{s}^{-1} + DK_1, \\ D &= D_3 T_\mathrm{I}^{-1}. \end{aligned}$$

Note that state feedback does not change structural indices of a linear system. Thus, structural indices of (A_3, B_3, C_3, D_3) and (A, B, C, D) are the same.

Noting the special form of T_2 and $\tilde{T}_3$, we have

$$C = \begin{bmatrix} \hat{C}_3 \\ 0 \end{bmatrix} (T_1 T_3)^{-1} + DK_1, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & D_o \end{bmatrix} T_\mathrm{I}^{-1}.$$

We finally obtain a set of the desired (C, D) as

$$\Omega = \left\{ (\Gamma_\mathrm{o} C,\ \Gamma_\mathrm{o} D) \,|\, \Gamma_\mathrm{o} \in \mathbb{R}^{(p_b+m)\times(p_b+m)} \text{ is nonsingular} \right\}.$$

This completes the proof of Theorem 4.2. ■

Remark 4.1: If the uncontrollable modes of (A, B) are not distinct, then the assignment of $\mathcal{I}_3$ will be slightly more complicated. It will be subject to more constraint imposed by the Jordan canonical form of uncontrollable modes of the given (A, B).

Remark 4.2: In our earlier algorithm[11], in order to be assignable, the desired orders of infinite zeros must be equal to or less than the elements in controllability indices of (A, B). In the current algorithm, no such a constraint is imposed.

V. CONCLUSIONS

In this paper, we have revisited the the problem of structural assignment for linear systems. We first established necessary conditions under which a given set of structural properties can be assigned. Motivated by these necessary conditions, we established a set of necessary and sufficient conditions for the assignability of a set of structural properties which includes the left invertibility property. These results significantly improve the existing results on the topic.

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