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Nondegenerate Necessary Conditions for Linear Optimal Control Problems with Higher Index State Constraints

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***Abstract*— There are certain optimal control problems with state constraints for which the standard versions of the necessary conditions of optimality are unable to provide information to select minimizers. In the recent years, there has been some literature on stronger forms of the maximum principle, the so-called nondegenerate necessary conditions, that are informative for those problems. These conditions can be applied when certain constraint qualifications are satisfied. However, when the state constraints have higher index (i.e. their first derivative with respect to time does not depend on the control) the nondegenerate necessary conditions existent in the literature cannot be applied. This is because their constraint qualifications are never satisfied for higher index state constraints. Here, we provide a nondegenerate form of the necessary conditions that can be applied to problems with higher index state constraints. We note that control problems with higher index state constraints arise frequently in mechanical systems, when there is a constraint on the position (an obstacle in the path, for example) and the control acts as a force or acceleration.**

***Index Terms*— optimal control; maximum principle; degeneracy phenomenon; higher order state constraints.**

I. INTRODUCTION

Consider the following optimal control problem with linear dynamics and an inequality state constraint enforced along the trajectory.

$$
\begin{aligned}
(P)\quad & \text{Minimise} && \int_0^1 L(x(t),u(t))dt + W(x(1)) \\
& \text{subject to} && \\
& && \dot{x}(t) = Ax(t) + Bu(t) \quad \text{a.e. } t \in [0,1] \\
& && x(0) = x_0 \\
& && x(1) \in S \\
& && u(t) \in U(t) \quad \text{a.e. } t \in [0,1] \\
& && c^T x(t) \le d \quad \forall\, t \in [0,1]
\end{aligned}
$$

The data for this problem comprises: an $n \times n$ matrix A , n vectors B , c , and x_0, a scalar d, continuously differentiable functions $W : \mathbb{R}^n \mapsto \mathbb{R}$ and $L : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, a closed set $S \subset \mathbb{R}^n$ and a compact set $U \subset \mathbb{R}^m$.

Necessary conditions of optimality (NCO), in the form of a maximum principle, for this problem are well-known (see e.g. [3], [4], [8], [10]). Let the process $(\bar{x}, \bar{u})$ be a local minimizer. Then, there exist an absolutely continuous function $p : [0,1] \to \mathbb{R}^n$, a nonnegative measure $\mu \in C^*([0,1];\mathbb{R})$ and a scalar $\lambda \ge 0$ such that

$$\mu\{[0,1]\} + ||p||_{L^\infty} + \lambda > 0, \tag{1}$$

$$-\dot{p}(t) = A^T(p(t) + c\int_{[0,t)} \mu(ds)) - \lambda L_x(\bar{x}(t), \bar{u}(t)) \quad \text{a.e. }, \tag{2}$$

$$-(p(1) + c\int_{[0,1]} \mu(ds)) \in \xi + \lambda W_x(\bar{x}(1)), \tag{3}$$

$$\xi \in N_C(\bar{x}(1)) \tag{4}$$

$$\text{supp}\{\mu\} \subset \left\{t \in [0,1] : c^T(\bar{x}(t)) - d = 0\right\}, \tag{5}$$

and for almost every $t \in [0,1]$, $\bar{u}(t)$ maximizes over U

$$u \mapsto \left(p(t) + c\int_{[0,t)} \mu(ds)\right) Bu - \lambda L(\bar{x}(t), u). \tag{6}$$

However, in the case when the state constraint is active at the initial instant of time, these standard versions of the maximum principle may fail to provide the desired information to identify minimizers. Note that if

$$c^T x_0 - d = 0,$$

then the set of multipliers (called degenerate multipliers)

$$\lambda = 0, \quad \mu = \delta_{\{0\}}, \quad p = -c, \tag{7}$$

satisfies the NCO for *any* feasible process, local minimizer or not. (Here, $\delta_{\{0\}}$ denotes the Dirac unit measure concentrated at $t = 0$.)

This phenomenon is known as the degeneracy phenomenon. There has been, in the recent years, some literature on how to strengthen the NCO to avoid the degeneracy phenomenon for problems satisfying a certain constraint qualification: see e.g. [6], [2], [5], [9] and [1].

To study ways of avoiding degeneracy is important since this phenomenon might jeopardize optimizations algorithms based on the NCO. This is particularly relevant in applications using Model-Predictive / Receding-Horizon control methods (see e.g. [7]), which must solve several optimal control problems for different initial conditions along the trajectory. If, as one should expect, the trajectory hits the boundary of the state constraint, then we must solve an optimal control problem where the state constraint is active at the initial time.

The results in [6], [5] are presented in a general context, for problems with nonlinear dynamics, and in [5] for nonsmooth and nonconvex problems. In the context of our linear problem (P), these nondegeneracy results, that will be of use later on, can be written as follows:

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Proposition 1: Assume that if $c^T x_0 - d = 0$, then the following constraint qualification is satisfied

CQ There exists positive scalars δ and ϵ such that

$$\inf_{u \in U} c^T B(u - \bar{u}(t)) < -\delta$$

for all $t \in [0, \epsilon)$.

Then, the NCO (equations (1) to (6)) can be strengthened with the condition

$$\mu\{(0,1]\} + \|q\|_{L_\infty} + \lambda > 0.$$

Note that this last condition eliminates degenerate multipliers like the ones in (7).

There are, however, some problems with interest in practice for which the constraint qualification CQ cannot be satisfied.

We note that control problems with higher index state constraints arise frequently in mechanical systems, when there is a constraint on the position (an obstacle in the path, for example) and the control acts as a second derivative of the position (a force or acceleration). This is illustrated in the following example:

Example 2: Consider a second order linear system modelling a mass $(1/b)$ moving along a line by action of a force (u) and in which the position (x_1) is constrained to a certain half-space $(\leq d/c_1)$.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t), \qquad (8)$$
$$[c_1, 0]x(t) - d \leq 0.$$

We note that the quantity

$$c^T B = 0.$$

and hence CQ can never be satisfied. The constraint qualifications in other nondegenerate results (e.g. [2], [9]) are also not satisfied for these kind for problems with *higher index state constraints.*

In order to remedy this problem we will develop results based on a new constraint qualification dependent on the *index of the state constraint*.

II. Higher Index State Constrained Problems

We start by defining the index of a state constraint as in [4]. This index can be seen as a measure of how many times we have to differentiate the state constraint to have an explicit dependence on the control.

Definition 1 (Index of the State Constraint): The state constraint is said to have index k if k is a non-negative integer such that $c^T A^j b = 0$ for $j = 0, .., k-1$ and $c^T A^k b \neq 0$.

If $c^T A^j b = 0$ for all j , the state constraint is said to have index $k = \infty$.

If the index is greater than or equal to one, then as in the example, CQ cannot be satisfied. In this case, we call the problem *higher index state constrained* and the following constraint qualification is needed

CQ′ (Higher Index Constraint Qualification)
Let the state constraint have index k. If $c^T x_0 - d = 0$, then there exists positive scalars δ and ϵ such that

$$\inf_{u \in U} c^T A^k B(u - \bar{u}(t)) < -\delta$$

for all $t \in [0, \epsilon)$.

For technical reasons, the main result must assume that an initial part of the optimal trajectory does not enter and leave the boundary of the state constraint an infinite number of times. That is, the initial part of the optimal trajectory either stays on the boundary of the state constraint for some time or leaves the boundary immediately.

Assumption 1: Either

(A) $\exists_{\tau \in (0,1)}$ such that $c^T \bar{x}(t) = 0$ for all $t \in [0, \tau]$

or

(B) $\exists_{\tau \in (0,1)}$ such that $c^T \bar{x}(t) < 0$ for all $t \in (0, \tau]$.

A result analogous to the previous one, but assuming CQ' instead of CQ follows.

Theorem 3: Assume that the constraint qualification CQ' and Assumption 1 are satisfied. Then, the NCO (equations (1) to (6)) can be strengthened with the condition

$$\mu\{(0,1]\} + \|q\|_{L_\infty} + \lambda > 0.$$

III. Proof of Theorem 2

We will consider separately the cases (A) and (B) in Assumption 1.

In case (B), we are in the condition to apply directly Proposition 2.2 of [6], yielding the result immediately. In case (A), we distinguish the cases when $k = 0$, when $k = \infty$, and when k is positive and finite. When $k = 0$, the state constraint is not of higher index, so the results in [5] apply. When $k = \infty$, the state constraint can be dropped for reasons explained in [4]. We may assume, then, that k is positive and finite.

Define

$$h(t) := c^T \bar{x}(t) - d$$

and

$$h^{(i)}(t) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i h(t) \qquad i = 1, 2, \ldots, k.$$

We have that

$$h^{(1)}(t) = c^T(A\bar{x}(t) + B\bar{u}(t)) = c^T A\bar{x}(t)$$

and for $i = 1, 2, \ldots, k$

$$h^{(i)}(t) = c^T A^i \bar{x}(t) + c^T A^{i-1} B\bar{u}(t) = c^T A^i \bar{x}(t).$$

Case (A) of Assumption 1 implies that

$$h^{(i)}(0) = 0 \qquad i = 1, 2, \ldots, k.$$

Therefore the minimizer $(\bar{x}, \bar{u})$ for problem (P) is also a minimizer for the same problem with the additional constraint

$$h^{(k)}(0) = c^T A^k x_0 \leq 0.$$

We can rewrite the new state constraint(s) of the problem as

$$\tilde{h}(t, x) = \begin{cases} \max\{c^T x - d, c^T A^k x\} & \text{if } t = 0 \\ c^T x - d & \text{if } t > 0 \end{cases}$$

This function is upper semi-continuous and the nondegenerate NCO in [5] apply to this problem provided the following constraint qualification is satisfied:

If $\tilde{h}(t, x_0)$ then there exists positive constants δ and ϵ, and a control value $\tilde{u}$ such that

$$\xi B(\tilde{u} - \bar{u}(t)) < -\delta$$

for all $\xi \in \partial_x^{>} \tilde{h}(s, x), s \in (0, \epsilon)$.

Knowing that (see [3])

$$\xi \in \{c^T(\alpha I + (1-\alpha)A^k) : \alpha \in [0, 1]\}$$

and

$$c^T B = 0,$$

We have

$$c^T(\alpha I + (1-\alpha)A^k)B(\tilde{u} - \bar{u}(t)) < -\delta$$

provided we have

$$c^T A^k B(\tilde{u} - \bar{u}(t)) < -\delta'.$$

Therefore, if CQ' is satisfied, the constraint qualification in [5] is satisfied and the corresponding NCO can be applied yielding the result.

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